Research Article

Refinements of Some Integral Inequalities for φ -Convex Functions

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In this paper, we are interested to deal with unified integral operators for strongly φ -convex function. We will present refinements of bounds of these unified integral operators and use them to get associated results for fractional integral operators. Several known results are connected with particular assumptions.

1. Introduction and Preliminaries

Convex functions play an important role in the formation of new definitions of related functions which help to give the generalization of classical results. Therefore, in recent years, many generalizations of convex functions are defined and utilized to study the Hadamard and other well-known inequalities (see [1–9]). In this paper, we deal with the strongly φ -convex functions to study the bounds of unified integral operators. The obtained results are compared with already known results.

First, we give some definitions of functions which are necessary for the findings of this paper.

Definition 1 (see [7]). A function $f: I \rightarrow \mathbb{R}$ is said to be convex on I if

$$f(\zeta u + (1 - \zeta)v) \leq \zeta f(u) + (1 - \zeta)f(v), \quad (1)$$

holds for all $u, v \in I$ and $\zeta \in [0, 1]$, where $I \subseteq \mathbb{R}$ is an interval. Reverse of inequality (1) defines f as concave function.

Definition 2 (see [10]). A function $f: I \rightarrow \mathbb{R}$ is said to be strongly convex with modulus $\lambda > 0$ if

$$f(\zeta u + (1 - \zeta)v) \leq \zeta f(u) + (1 - \zeta)f(v) - \lambda \zeta(1 - \zeta)(v - u)^2, \quad (2)$$

holds for all $u, v \in I$ and $\zeta \in [0, 1]$.

Definition 3 (see [3]). A function $f: I \rightarrow \mathbb{R}$ is said to be φ -convex on I if

$$f(\zeta u + (1 - \zeta)v) \leq f(v) + \zeta \varphi(f(u), f(v)), \quad (3)$$

holds for all $u, v \in I$ and $\zeta \in [0, 1]$, where φ is a bifunction.

Definition 4 (see [2]). A function $f: I \rightarrow \mathbb{R}$ is said to be strongly φ -convex on I if

$$f(\zeta u + (1 - \zeta)v) \leq f(v) + \zeta \varphi(f(u), f(v)) - \lambda \zeta(1 - \zeta)(v - u)^2, \quad (4)$$

holds for all $u, v \in I$ and $\zeta \in [0, 1]$, $\lambda \geq 0$, where φ is a bifunction.

It is to be noted that for $\varphi(x, y) = x - y$, strongly φ -convex function reduces to strongly convex function. Farid in [11] defined the unified integral operators (5) and (6) and has proved the continuity and the boundedness of these integral operators. The aim of this paper is the study of integral inequalities for strongly φ -convex functions via

unified integral operators. Next, we give definition of the unified integral operators.

Definition 5. Let $f, g: [u, v] \longrightarrow \mathbb{R}$ where $0 < u < v$ be the function such that f is positive and integrable over $[u, v]$ and

$$\left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f \right) (x, \eta; p) = \int_u^x J_x^\gamma \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(y) f(y) dy, \quad (5)$$

$$\left({}_g F_{\mu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f \right) (x, \eta; p) = \int_x^v J_v^\gamma \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(y) f(y) dy, \quad (6)$$

where

$$J_x^\gamma \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) = \frac{\Psi(g(x) - g(y))}{g(x) - g(y)} E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(x) - g(y))^\mu; p). \quad (7)$$

By choosing specific functions Ψ and g and fixing parameters involved in the Mittag-Leffler function $E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(x) - g(y))^\mu; p)$, various known fractional integrals can be reproduced (see [5], Remarks 6 and 7). In [4], by using unified integral operators, we have obtained integral inequalities for φ -convex functions. In the following, we give these inequalities in the form of Theorems 1–3.

Theorem 1. Let $f: [u, v] \longrightarrow \mathbb{R}$ be a positive φ -convex function and $g: [u, v] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function. Also, let Ψ/x be an increasing function on $[u, v]$, $\eta, \alpha, \xi, \gamma, \zeta \in \mathbb{C}$, $p, \mu, \nu, \delta \geq 0$, $0 < k \leq \delta + \mu$, and $0 < k \leq \delta + \nu$. Then, for $x \in [u, v]$, we have

$$\begin{aligned} & \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f \right) (x, \eta; p) + \left({}_g F_{\nu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f \right) (x, \eta; p) \\ & \leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(x) - g(u))^\mu; p) \Psi(g(x) - g(u)) f(x) \\ & \quad + J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \varphi(f(u), f(x)) (I(u, x; g) - g(u)) \\ & \quad + E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(v) - g(x))^\nu; p) \Psi(g(v) - g(x)) f(v) \\ & \quad + J_v^\nu \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \varphi(f(x), f(v)) (I(x, v; g) - g(x)). \end{aligned} \quad (8)$$

Theorem 2. Along with the assumptions of Theorem 1, if $f(u + v - x) = f(x)$ and $\varphi(x, y) = x + y$, then the following result holds:

g is differentiable and strictly increasing. Also, let Ψ/x be an increasing function on $[u, \infty)$ and $\alpha, \xi, \gamma, \zeta \in \mathbb{C}$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then, for $x \in [u, v]$, the left and right integral operators are defined as follows:

$$\begin{aligned} & \frac{1}{2} f\left(\frac{u+v}{2}\right) \left(\left({}_g F_{\mu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} 1 \right) (u, \eta; p) + \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} 1 \right) (v, \eta; p) \right) \\ & \leq \left({}_g F_{\mu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f \right) (u, \eta; p) + \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f \right) (v, \eta; p) \\ & \leq 2\Psi(g(v) - g(u)) E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(v) - g(u))^\mu; p) f(v) \\ & \quad + 2(f(u) + f(v)) J_v^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) (I(u, v; g) - g(u)). \end{aligned} \quad (9)$$

Also, the following result holds for the convolution of functions f and g .

Theorem 3. Let $f, g: [u, v] \longrightarrow \mathbb{R}$ be two differentiable functions such that $|f'|$ is φ -convex and g be strictly increasing for $0 < u < v$. Also, Ψ/x be an increasing function on $[u, v]$ and $\alpha, \xi, \gamma, \zeta \in \mathbb{C}$, $p, \mu, \nu, \delta \geq 0$ and $0 < k \leq \delta + \mu$ and $0 < k \leq \delta + \nu$. Then, for $x \in (u, v)$, we have

$$\begin{aligned} & \left| \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) + \left({}_g F_{\nu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) \right| \\ & \leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(x) - g(u))^\mu; p) \Psi(g(x) - g(u)) |f'(x)| \\ & \quad + J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \varphi(|f'(u)|, |f'(x)|) (I(u, x; g) - g(u)) \\ & \quad + E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(v) - g(x))^\nu; p) \Psi(g(v) - g(x)) |f'(v)| \\ & \quad + J_v^\nu \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \varphi(|f'(x)|, |f'(v)|) (I(x, v; g) - g(x)). \end{aligned} \quad (10)$$

Although we follow the same method which was adopted to prove the results of [4], here we will get refinements of these results by using strongly φ -convex functions. In Section 2, we give the refinements of bounds of unified integral operators given in Definition 5. In Section 3, we will

present refinements of bounds of fractional integral operators.

2. Main Results

Throughout this section, we have adopted the following notations:

$$I(u, v; g) := \frac{1}{v-u} \int_u^v g(t) dt, \quad (11)$$

$$S(\mu, v, u^+, v^-) = \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f \right)(x, \eta; p) + \left({}_g F_{v, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f \right)(x, \eta; p).$$

Theorem 4. If f is positive strongly φ -convex function with modulus $\lambda \geq 0$, along with other assumptions of Theorem 1, then we have

$$\begin{aligned} S(\mu, v, u^+, v^-) &\leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}(\eta(g(x) - g(u))^\mu; p) \Psi(g(x) - g(u)) f(x) \\ &\quad + J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(f(u), f(x)) (I(u, x; g) - g(u)) - \lambda(x-u) (2I(u, x; I_d g) - (x+u)I(u, x; g)) \} \\ &\quad + E_{v, \alpha, \xi}^{\gamma, \delta, k, \zeta}(\eta(g(v) - g(x))^\nu; p) \Psi(g(v) - g(x)) f(v) \\ &\quad + J_v^x \left(E_{v, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(f(x), f(v)) (I(x, v; g) - g(x)) - \lambda(v-x) (2I(x, v; I_d g) - (v+x)I(x, v; g)) \}, \end{aligned} \quad (12)$$

where I_d is the identity function.

Proof. For the kernel defined in (7) and the strongly φ -convexity of the function f on $[u, x]$, the following inequalities hold, respectively:

$$J_x^\varsigma \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\varsigma) \leq J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\varsigma), \quad \varsigma \in [u, x], x \in (u, v), \quad (13)$$

$$f(\varsigma) \leq f(x) + \frac{x-\varsigma}{x-u} \varphi(f(u), f(x)) - \lambda \left(\frac{x-\varsigma}{x-u} \right) \left(\frac{\varsigma-u}{x-u} \right) (u-x)^2. \quad (14)$$

The aforementioned inequalities are used to obtain the following integral inequality:

$$\begin{aligned} \int_u^x J_x^\varsigma \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\varsigma) f(\varsigma) d\varsigma &\leq f(x) J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \int_u^x g'(\varsigma) d\varsigma \\ &\quad + \frac{\varphi(f(u), f(x))}{x-u} J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \int_u^x (x-\varsigma) g'(\varsigma) d\varsigma \\ &\quad - \lambda J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \int_u^x (x-\varsigma)(\varsigma-u) g'(\varsigma) d\varsigma. \end{aligned} \quad (15)$$

In view of Definition 5 and applying integration by parts, from inequality (15), we get the following upper bound of the right-sided unified integral operator:

$$\begin{aligned} \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f \right) (x, \eta; p) &\leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta (g(x) - g(u))^\mu; p) \Psi(g(x) - g(u)) f(x) \\ &\quad + J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(f(u), f(x)) (I(u, x; g) - g(u)) - \lambda(x - u) (2I(u, x; I_d g) \\ &\quad - (x + u) (I(u, x; g))) \}. \end{aligned} \quad (16)$$

Again for the kernel defined in (7) and the strongly φ -convexity of the function f on $(x, v]$, the following inequalities hold, respectively:

$$J_\zeta^x \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\zeta) \leq J_\nu^x \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\zeta), \quad (17)$$

$$f(\zeta) \leq f(v) + \frac{\nu - \zeta}{\nu - x} \varphi(f(x), f(v)) - \lambda \left(\frac{\nu - \zeta}{\nu - x} \right) \left(\frac{\zeta - x}{\nu - x} \right) (x - v)^2. \quad (18)$$

The aforementioned inequalities (17) and (18) are used to obtain the following integral inequality:

$$\begin{aligned} &\int_x^v J_\zeta^x \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\zeta) f(\zeta) d\zeta \\ &\leq J_x^\nu \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \left(f(v) \int_x^v g'(\zeta) d\zeta + \frac{\varphi(f(x), f(v))}{\nu - x} \int_x^v (\nu - \zeta) g'(\zeta) d\zeta \right) \\ &\quad - \lambda J_x^\nu \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \int_x^v (\nu - \zeta) (\zeta - x) g'(\zeta) d\zeta. \end{aligned} \quad (19)$$

In view of Definition 5 and applying integration by parts, from inequality (19), we get the following upper bound of the left-sided unified integral operator:

$$\begin{aligned} \left({}_g F_{\nu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f \right) (x, \eta; p) &\leq E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta (g(v) - g(x))^\nu; p) \Psi(g(v) - g(x)) f(v) \\ &\quad + J_v^x \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(f(x), f(v)) (I(x, v; g) - g(x)) - \lambda(v - x) (2I(x, v; I_d g) - (v + x) I(x, v; g)) \}. \end{aligned} \quad (20)$$

Inequality (12) will be obtained by combining (16) and (20). $\square$

Corollary 1. By setting $\mu = \nu$ in (12), we get

$$\begin{aligned}
S(\mu, \mu, u^+, v^-) &\leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}(\eta(g(x) - g(u))^\mu; p) \Psi(g(x) - g(u)) f(x) \\
&\quad + J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(f(u), f(x)) (I(u, x; g) - g(u)) - \lambda(x - u) (2I(u, x; I_d g) - (x + u) (I(u, x; g))) \} \\
&\quad + E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}(\eta(g(v) - g(x))^\mu; p) \Psi(g(v) - g(x)) f(v) \\
&\quad + J_v^x \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(f(x), f(v)) (I(x, v; g) - g(x)) - \lambda(v - x) (2I(x, v; I_d g) - (v + x) I(x, v; g)) \}.
\end{aligned} \tag{21}$$

Remark 1. For $\lambda = 0$ in (12), we get inequality (8) of Theorem 1; if $2I(u, x; I_d g) > (x + u) (I(u, x; g))$ and $2I(x, v; I_d g) > (v + x) I(x, v; g)$, then we will get the refinement of (8).

For $\varphi(x, y) = x - y$ in (21), we get the result for strongly convex function.

For $\varphi(x, y) = x - y$ and $\lambda = 0$ in (21), we get the result of Theorem 8 in [5].

We will use the following lemma for our next result.

Lemma 1. Let f be strongly φ -convex function with modulus $\lambda \geq 0$. If $f(x) = f(u + v - x)$, then

$$f\left(\frac{u+v}{2}\right) \leq f(x) + \frac{1}{2} \varphi(f(x), f(x)) - \frac{\lambda(u-v)^2}{4}, \tag{22}$$

holds for $x \in [u, v]$.

Proof. Strongly φ -convexity of f implies

$$\begin{aligned}
f\left(\frac{u+v}{2}\right) &\leq f\left(\frac{x-u}{v-u}u + \frac{v-x}{v-u}v\right) + \frac{1}{2} \varphi\left(f\left(\frac{x-u}{v-u}u + \frac{v-x}{v-u}v\right), f\left(\frac{x-u}{v-u}u + \frac{v-x}{v-u}v\right)\right) - \frac{\lambda(u-v)^2}{4} \\
&= f(u + v - x) + \frac{1}{2} \varphi(f(x), f(u + v - x)) - \frac{\lambda(u-v)^2}{4}.
\end{aligned} \tag{23}$$

Using the condition $f(x) = f(u + v - x)$ in the above inequality, we get (22). $\square$

Remark 2. For $\lambda = 0$, Lemma 1 reduces to Lemma 1 of [4]. For $\lambda > 0$, we get its refinement.

For $\varphi(x, y) = x - y$ and $\lambda = 0$, Lemma 1 reduces to Lemma 21 of [5].

Theorem 5. Let $f(u + v - x) = f(x)$ and $\varphi(x, y) = x + y$ in addition with the assumptions of Theorem 4. Then, the following inequality holds:

$$\begin{aligned}
&\frac{1}{2} \left(f\left(\frac{u+v}{2}\right) + \frac{\lambda(u-v)^2}{4} \right) \left(\left({}_g F_{\mu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} 1 \right) (u, \eta; p) + \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} 1 \right) (v, \eta; p) \right) \\
&\leq S(\mu, \nu, u^+, v^-) \leq 2 \Psi(g(v) - g(u)) E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}(\eta(g(v) - g(u))^\mu; p) f(v) \\
&\quad + 2(f(u) + f(v)) J_v^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) (I(u, v; g) - g(u)) \\
&\quad - 2 \lambda J_v^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) (v - u) (2I(u, v; I_d g) - (u + v) I(u, v; g)).
\end{aligned} \tag{24}$$

Proof. For the kernel defined in equation (7) and the strongly φ -convexity of the function f on $[u, v]$, the following inequalities hold, respectively:

$$J_x^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)g'(x) \leq J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)g'(x), \quad x \in (u, v), \quad (25)$$

$$f(x) \leq f(v) + \frac{v-x}{v-u}\varphi(f(u), f(v)) - \lambda(v-x)(x-u). \quad (26)$$

The aforementioned inequalities are used to obtain the following integral inequality:

$$\begin{aligned} \int_u^v J_x^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)f(x)g'(x)dx &\leq f(v)J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)\int_u^v g'(x)dx \\ &+ \frac{\varphi(f(u), f(v))}{v-u}J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)\int_u^v (v-x)g'(x)dx \\ &- \lambda J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)\int_u^v (v-x)(x-u)g'(x)dx. \end{aligned} \quad (27)$$

In view of Definition 5, applying integration by parts, and using $\varphi(x, y) = x + y$, from inequality (27), we get the

following upper bound of the left-sided unified integral operator:

$$\begin{aligned} ({}_gF_{\mu,\alpha,\xi,\nu}^{\Psi,\gamma,\delta,k,\zeta}f)(u, \eta; p) &\leq E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}(\eta(g(v) - g(u))^\mu; p)\Psi(g(v) - g(u))f(v) \\ &+ J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)(f(u) + f(v))(I(u, v; g) - g(u)) \\ &- \lambda J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)(v-u)(2I(u, v, I_d g) - (v+u)I(u, v, g)). \end{aligned} \quad (28)$$

Also, the following inequality holds:

$$J_v^x(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)g'(x) \leq J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)g'(x), \quad x \in (u, v). \quad (29)$$

The aforementioned inequalities (26) and (29) are used to obtain the following integral inequality:

$$\begin{aligned} \int_u^v J_v^x(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)g'(x)f(x)dx \\ \leq J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)\left(f(v)\int_u^v g'(x)dx + \frac{\varphi(f(u), f(v))}{v-u}\int_u^v g'(x)(v-x)dx\right) \\ - \lambda J_v^u(E_{\mu,\alpha,\xi}^{\gamma,\delta,k,\zeta}, g; \Psi)\int_u^v (v-x)(x-u)g'(x)dx. \end{aligned} \quad (30)$$

In view of Definition 5 and applying integration by parts, from inequality (30), we get the following upper bound of the right-sided unified integral operator:

$$\begin{aligned} \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f \right) (v, \eta; p) &\leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta (g(v) - g(u))^\mu; p) \Psi(g(v) - g(u)) f(v) \\ &\quad + J_v^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) (f(u) + f(v)) (I(u, v; g) - g(u)) \\ &\quad - \lambda J_v^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) (v - u) (2I(u, v, I_d g) - (v + u)I(u, v, g)). \end{aligned} \quad (31)$$

Now, using Lemma 1, we can write

$$\begin{aligned} &\int_u^v f\left(\frac{u+v}{2}\right) J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(x) dx \\ &\leq \int_u^v J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(x) f(x) dx + \frac{1}{2} \int_u^v J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(x) \varphi(f(x), f(x)) dx \\ &\quad - \frac{\lambda(u-v)^2}{4} \int_u^v J_x^u \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(x) dx. \end{aligned} \quad (32)$$

In view of Definition 5 and $\varphi(x, y) = x + y$, from (32), we get the following upper bound of the left-sided unified integral operator:

$$\begin{aligned} f\left(\frac{u+v}{2}\right) \left({}_g F_{\mu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} 1 \right) (u, \eta; p) &\leq 2 \left({}_g F_{\mu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f \right) (u, \eta; p) \\ &\quad - \frac{\lambda(u-v)^2}{4} \left({}_g F_{\mu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} 1 \right) (u, \eta; p). \end{aligned} \quad (33)$$

Also, from Lemma 1, we can write

$$\begin{aligned} &\int_u^v f\left(\frac{u+v}{2}\right) J_v^x \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(x) dx \\ &\leq \int_u^v J_v^x \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(x) f(x) dx + \frac{1}{2} \int_u^v J_v^x \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(x) \varphi(f(x), f(x)) dx \\ &\quad - \frac{\lambda(u-v)^2}{4} \int_u^v J_v^x \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(x) dx. \end{aligned} \quad (34)$$

In view of Definition 5 and $\varphi(x, y) = x + y$, from (34), we get the following upper bound of the right-sided unified integral operator:

$$f\left(\frac{u+v}{2}\right) \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} 1 \right) (v, \eta; p) \leq 2 \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f \right) (v, \eta; p) - \frac{\lambda(u-v)^2}{4} \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} 1 \right) (v, \eta; p). \quad (35)$$

Inequality (24) will be obtained by using (28), (31), (33), and (35). $\square$

Remark 3. For $\lambda = 0$ in (24), we get (9) of Theorem 2; if $2I(u, v; I_d g) > (u + v)I(u, v; g)$, then we will get refinement of (9).

For $\varphi(x, y) = x - y$ in (24), we get the result for strongly convex function.

For $\varphi(x, y) = x - y$ and $\lambda = 0$ in (24), we get the result of Theorem 22 in [5].

Theorem 6. If $|f'|$ is strongly φ -convex with modulus $\lambda \geq 0$ along with other assumptions of Theorem 3, then the inequality

$$\begin{aligned} & \left| \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) + \left({}_g F_{\nu, \alpha, \xi, \nu^-}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) \right| \\ & \leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta (g(x) - g(u))^\mu; p) \Psi(g(x) - g(u)) |f'(x)| \\ & \quad + J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(|f'(u)|, |f'(x)|) (I(u, x; g) - g(u)) \\ & \quad - \lambda(x - u) (2I(u, x; I_d g) - (x + u)I(u, x; g)) \} \\ & \quad + E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta (g(v) - g(x))^\nu; p) \Psi(g(v) - g(x)) |f'(v)| \\ & \quad + J_v^\nu \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(|f'(x)|, |f'(v)|) (I(x, v; g) - g(x)) \\ & \quad - \lambda(v - x) (2I(x, v; I_d g) - (v + x)I(x, v; g)) \} \end{aligned} \quad (36)$$

holds for $x \in (u, v)$, where

$$\left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) = \int_u^x J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\varsigma) f'(\varsigma) d\varsigma, \quad (37)$$

$$\left({}_g F_{\nu, \alpha, \xi, \nu^-}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) = \int_x^v J_v^\nu \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\varsigma) f'(\varsigma) d\varsigma, \quad (38)$$

and I_d is the identity function.

Proof. Using strongly φ -convexity of $|f'|$ over $[u, x]$ gives

$$|f'(\varsigma)| \leq |f'(x)| + \frac{x - \varsigma}{x - u} \varphi(|f'(u)|, |f'(x)|) - \lambda(x - \varsigma)(\varsigma - u), \quad \varsigma \in [u, x]. \quad (39)$$

Using absolute value property, we can write

$$\begin{aligned} & - \left(|f'(x)| + \frac{x - \varsigma}{x - u} \varphi(|f'(u)|, |f'(x)|) - \lambda(x - \varsigma)(\varsigma - u) \right) \leq f'(\varsigma) \\ & \leq \left(|f'(x)| + \frac{x - \varsigma}{x - u} \varphi(|f'(u)|, |f'(x)|) - \lambda(x - \varsigma)(\varsigma - u) \right). \end{aligned} \quad (40)$$

The aforementioned inequality (13) and second inequality of (40) are used to obtain the following integral inequality:

$$\int_u^x J_x^\zeta \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) g'(\zeta) f'(\zeta) d\zeta \leq J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \left(|f'(x)| \int_u^x g'(\zeta) d\zeta \right. \\ \left. + \frac{\varphi(|f'(u)|, |f'(x)|)}{x-u} \int_u^x (x-\zeta) g'(\zeta) d\zeta - \lambda \int_u^x (x-\zeta)(\zeta-u) g'(\zeta) d\zeta \right). \quad (41)$$

In view of (37) and applying integration by parts, from inequality (41), we get the following upper bound:

$$\left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) \\ \leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(x) - g(u))^\mu; p) \Psi(g(x) - g(u)) |f'(x)| \\ + J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(|f'(u)|, |f'(x)|) (I(u, x; g) - g(u)) \\ - \lambda(x-u)(2I(u, x; I_d g) - (x+u)I(u, x; g)) \}. \quad (42)$$

Also, inequality (13) and the first inequality of (40) are used to obtain the following integral inequality:

$$|f'(\zeta)| \leq |f'(v)| + \frac{v-\zeta}{v-x} \varphi(|f'(x)|, |f'(v)|) - \lambda(v-\zeta)(\zeta-x), \quad \zeta \in (x, v]. \quad (44)$$

Inequalities (17), (38), and (44) are used to obtain the following upper bounds:

$$\left({}_g F_{\nu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) \\ \leq E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(v) - g(x))^\nu; p) \Psi(g(v) - g(x)) |f'(v)| \\ + J_v^x \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(|f'(x)|, |f'(v)|) (I(x, v; g) - g(x)) \\ - \lambda(v-x)(2I(x, v; I_d g) - (v+x)I(x, v; g)) \}, \quad (45)$$

$$\left({}_g F_{\nu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) \\ \geq - \left[E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(v) - g(x))^\nu; p) \Psi(g(v) - g(x)) |f'(v)| \right. \\ \left. + J_v^x \left(E_{\nu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(|f'(x)|, |f'(v)|) (I(x, v; g) - g(x)) \right. \\ \left. - \lambda(v-x)(2I(x, v; I_d g) - (v+x)I(x, v; g)) \} \right]. \quad (46)$$

Inequality (36) will be obtained by using (42)–(46). $\square$

$$\left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) \\ \geq - \left[E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta(g(x) - g(u))^\mu; p) \Psi(g(x) - g(u)) |f'(x)| \right. \\ \left. + J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi(|f'(u)|, |f'(x)|) (I(u, x; g) - g(u)) \right. \\ \left. - \lambda(x-u)(2I(u, x; I_d g) - (x+u)I(u, x; g)) \} \right]. \quad (43)$$

Now, using φ -convexity of $|f'|$ over $(x, v]$, we have

Corollary 2. By setting $\mu = \nu$ in (36), we get the following inequality:

$$\begin{aligned}
& \left| \left({}_g F_{\mu, \alpha, \xi, u^+}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) + \left({}_g F_{\mu, \alpha, \xi, v^-}^{\Psi, \gamma, \delta, k, \zeta} f^* g \right) (x, \eta; p) \right| \\
& \leq E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta (g(x) - g(u))^\mu; p) \Psi (g(x) - g(u)) |f'(x)| \\
& \quad + J_x^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi (|f'(u)|, |f'(x)|) (I(u, x; g) - g(u)) \\
& \quad - \lambda (x - u) (2I(u, x; I_d g) - (x + u)I(u, x; g)) \} \\
& \quad + E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta} (\eta (g(v) - g(x))^\mu; p) \Psi (g(v) - g(x)) |f'(v)| \\
& \quad + J_v^\mu \left(E_{\mu, \alpha, \xi}^{\gamma, \delta, k, \zeta}, g; \Psi \right) \{ \varphi (|f'(x)|, |f'(v)|) (I(x, v; g) - g(x)) \\
& \quad - \lambda (v - x) (2I(x, v; I_d g) - (v + x)I(x, v; g)) \}.
\end{aligned} \tag{47}$$

Remark 4. For $\lambda = 0$ in (36), we get inequality (10) of Theorem 3; if $2I(u, x; I_d g) > (x + u)$

$I(u, x; g)$ and $2I(x, v; I_d g) > (v + x)I(x, v; g)$, then we will get the refinement of (10).

For $\varphi(x, y) = x - y$ in (47), we get the result for strongly convex function.

For $\varphi(x, y) = x - y$ and $\lambda = 0$ in (47), we get the result of Theorem 25 in [5].

3. Results for Fractional Integral Operators

In this section, we give the bounds of some of the fractional integral operators which will be deduced from the results of Section 2. Throughout this section, we assume that $p = \eta = 0$.

Proposition 1. Under the assumptions of Theorem 4, the following result holds:

$$\begin{aligned}
\Gamma(\alpha) \left(({}_g I_{u^+}^\alpha f)(x) + ({}_g I_{v^-}^\alpha f)(x) \right) & \leq (g(x) - g(u))^\alpha f(x) + (g(v) - g(x))^\alpha f(v) \\
& \quad + (g(x) - g(u))^{\alpha-1} \{ \varphi(f(u), f(x)) (I(u, x; g) - g(u)) - \lambda (x - u) (2I(u, x; I_d g) - (x + u)I(u, x; g)) \} \\
& \quad + (g(v) - g(x))^{\alpha-1} \{ \varphi(f(x), f(v)) (I(x, v; g) - g(x)) - \lambda (v - x) (2I(x, v; I_d g) - (v + x)I(x, v; g)) \}.
\end{aligned} \tag{48}$$

Proof. For $\Psi(\zeta) = \zeta^\alpha$, where $\alpha > 0$, Theorem 4 gives (48). $\square$

Proposition 2. Under the assumptions of Theorem 4, the following inequality holds:

$$\begin{aligned}
& \Gamma(\alpha) \left(({}_u I_\Psi f)(x) + ({}_v I_\Psi f)(x) \right) \\
& \leq \Psi(x - u) f(x) + \frac{\Psi(x - u)}{2} \varphi(f(u), f(x)) + \Psi(v - x) f(v) \\
& \quad + \frac{\Psi(v - x)}{2} \varphi(f(x), f(v)) - \Psi(x - u) \frac{\lambda(x - u)^2}{6} - \Psi(v - x) \frac{\lambda(v - x)^2}{6}.
\end{aligned} \tag{49}$$

Proof. For g as identity function, Theorem 4 gives (49). $\square$

Corollary 3. For $\Psi(\zeta) = ((\zeta^{\alpha/k}) / (k\Gamma_k(\alpha)))$, (5) and (6) reduce to the fractional integral operators given in [5]. Further, the following bound for $\alpha \geq k$ is also satisfied:

$$\begin{aligned}
({}_g I_{u^+}^k f)(x) + ({}_g I_{v^-}^k f)(x) & \leq \frac{1}{k\Gamma_k(\alpha)} \left[(g(x) - g(u))^{\alpha/k} f(x) + (g(v) - g(x))^{\alpha/k} f(v) \right. \\
& \quad + (g(x) - g(u))^{(\alpha/k)-1} \{ \varphi(f(u), f(x)) (I(u, x; g) - g(u)) - \lambda (x - u) (2I(u, x; I_d g) \\
& \quad - (x + u)I(u, x; g)) \} + (g(v) - g(x))^{(\alpha/k)-1} \\
& \quad \cdot \{ \varphi(f(x), f(v)) (I(x, v; g) - g(x)) - \lambda (v - x) (2I(x, v; I_d g) - (v + x)I(x, v; g)) \} \}.
\end{aligned} \tag{50}$$

Corollary 4. For $\Psi(\varsigma) = \varsigma^\alpha$, where $\alpha \geq 1$, and g as identity function, (5) and (6) give fractional integrals defined in [12]. Further, the following bound is also satisfied:

$$\begin{aligned} & \Gamma(\alpha)((^{\alpha}I_{u^+}f)(x) + (^{\alpha}I_{v^-}f)(x)) \\ & \leq (x-u)^{\alpha}f(x) + (v-x)^{\alpha}f(v) + \frac{(x-u)^{\alpha}}{2}\varphi(f(u), f(x)) \\ & \quad + \frac{(v-x)^{\alpha}}{2}\varphi(f(x), f(v)) - \lambda \frac{(x-u)^{\alpha+2}}{6} - \lambda \frac{(v-x)^{\alpha+2}}{6}. \end{aligned} \quad (51)$$

Corollary 5. Using $\Psi(\varsigma) = ((\varsigma^{\alpha/k})/(k\Gamma_k(\alpha)))$ and g as identity functions, (5) and (6) reduce to the fractional integral

operators given in [13]. Further, the following bound is also satisfied:

$$\begin{aligned} k\Gamma_k(\alpha)((^{\alpha}I_{u^+}^k f)(x) + (^{\alpha}I_{v^-}^k f)(x)) & \leq (x-u)^{\alpha/k}f(x) + (v-x)^{\alpha/k}f(v) + \frac{(x-u)^{\alpha/k}}{2} \\ & \quad \varphi(f(u), f(x)) + \frac{(v-x)^{\alpha/k}}{2}\varphi(f(x), f(v)) - \lambda \frac{(x-u)^{\alpha/k+2}}{6} \\ & \quad - \lambda \frac{(v-x)^{\alpha/k+2}}{6}. \end{aligned} \quad (52)$$

Corollary 6. For $\Psi(\varsigma) = \varsigma^\alpha$, where $\alpha > 0$, and $g(x) = x^\rho/\rho$, where $\rho > 0$, (5) and (6) reduce to the fractional integral

operators given in [14]. Further, the following bound is also satisfied:

$$\begin{aligned} & ({}^{\rho}I_{u^+}^{\alpha}f)(x) + ({}^{\rho}I_{v^-}^{\alpha}f)(x) \\ & \leq \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \left[(x^{\rho} - u^{\rho})^{\alpha}f(x) + (v^{\rho} - x^{\rho})^{\alpha}f(v) + (x^{\rho} - u^{\rho})^{\alpha-1}(\varphi(f(u), f(x)) \right. \\ & \quad \cdot \left(\frac{x^{\rho+1} - u^{\rho+1}}{(x-u)(\rho+1)} - u^{\rho} \right) - \lambda \left(\frac{2(x^{\rho+2} - u^{\rho+2})}{\rho+2} - \frac{(x+u)(x^{\rho+1} - u^{\rho+1})}{\rho+1} \right) \Big) \\ & \quad + (v^{\rho} - x^{\rho})^{\alpha-1} \left(\varphi(f(x), f(v)) \left(\frac{v^{\rho+1} - x^{\rho+1}}{(v-x)(\rho+1)} - x^{\rho} \right) \right. \\ & \quad \left. \left. - \lambda \left(\frac{2(v^{\rho+2} - x^{\rho+2})}{\rho+2} - \frac{(v+x)(v^{\rho+1} - x^{\rho+1})}{\rho+1} \right) \right) \right]. \end{aligned} \quad (53)$$

Corollary 7. For $\Psi(\varsigma) = \varsigma^\alpha$, where $\alpha > 0$, and $g(x) = ((x^{s+1})/(s+1))$, where $s > 0$, (5) and (6) give the following fractional integral operators:

$$\begin{aligned} (F_{u^+}^{((\varsigma^\alpha)/(\Gamma(\alpha))),g}f)(x) & = ({}^sI_{u^+}^{\alpha}f)(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_u^x (x^{s+1} - \varsigma^{s+1})^{\alpha-1} \varsigma^s f(\varsigma) d\varsigma, \\ (F_{v^-}^{((\varsigma^\alpha)/(\Gamma(\alpha))),g}f)(x) & = ({}^sI_{v^-}^{\alpha}f)(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^v (\varsigma^{s+1} - x^{s+1})^{\alpha-1} \varsigma^s f(\varsigma) d\varsigma. \end{aligned} \quad (54)$$

Further, the following bound is also satisfied:

$$\begin{aligned}
 (s+1)^{\alpha} \Gamma(\alpha) & \left(({}^S I_{u^+}^{\alpha} f)(x) + ({}^S I_{v^-}^{\alpha} f)(x) \right) \leq (x^{s+1} - u^{s+1})^{\alpha} f(x) \\
 & + (v^{s+1} - x^{s+1})^{\alpha} f(v) + (x^{s+1} - u^{s+1})^{\alpha-1} \left(\varphi \left(f(u), f(x) \left(\frac{x^{s+2} - u^{s+2}}{(x-u)(s+2)} - u^{s+1} \right) \right. \right. \\
 & \left. \left. - \lambda \left(\frac{2(x^{s+3} - u^{s+3})}{s+3} - \frac{(x+u)(x^{s+2} - u^{s+2})}{s+2} \right) \right) \right) + (v^{s+1} - x^{s+1})^{\alpha-1} \\
 & \cdot \left(\varphi(f(x), f(v)) \left(\frac{v^{s+2} - x^{s+2}}{(v-x)(s+2)} - x^{s+1} \right) \right. \\
 & \left. \left. - \lambda \left(\frac{2(v^{s+3} - x^{s+3})}{s+3} - \frac{(v+x)(v^{s+2} - x^{s+2})}{s+2} \right) \right) \right).
 \end{aligned} \tag{55}$$

Corollary 8. For $\Psi(\varsigma) = ((\varsigma^{\alpha/k})/k\Gamma_k(\alpha))$ and $g(x) = ((x^{s+1})/(s+1))$, where $s > 0$, (5) and (6) reduce to the fractional integral operators given in [15]. Further, the following bound is also satisfied:

$$\begin{aligned}
 & ({}^S I_{u^+}^{\alpha} f)(x) + ({}^S I_{v^-}^{\alpha} f)(x) \\
 & \leq \frac{1}{(s+1)^{\alpha/k} k\Gamma_k(\alpha)} (x^{s+1} - u^{s+1})^{\alpha/k} f(x) + (v^{s+1} - x^{s+1})^{\alpha/k} f(v) \\
 & + (x^{s+1} - u^{s+1})^{(\alpha/k)-1} \left(\varphi \left(f(u), f(x) \left(\frac{x^{s+2} - u^{s+2}}{(x-u)(s+2)} - u^{s+1} \right) \right) \right. \\
 & \left. - \lambda \left(\frac{2(x^{s+3} - u^{s+3})}{s+3} - \frac{(x+u)(x^{s+2} - u^{s+2})}{s+2} \right) \right) + (v^{s+1} - x^{s+1})^{(\alpha/k)-1} \\
 & \cdot \left(\varphi(f(x), f(v)) \left(\frac{v^{s+2} - x^{s+2}}{(v-x)(s+2)} - x^{s+1} \right) \right. \\
 & \left. - \lambda \left(\frac{2(v^{s+3} - x^{s+3})}{s+3} - \frac{(v+x)(v^{s+2} - x^{s+2})}{s+2} \right) \right).
 \end{aligned} \tag{56}$$

Corollary 9. For $\Psi(\varsigma) = \varsigma^{\alpha}$, where $\alpha > 0$, $g(x) = ((x^{\beta+s})/(\beta+s))$, where β and $s > 0$, (5) and (6) reduce to the fractional integral operators given in [16]. Further, the following bound is also satisfied:

$$\begin{aligned}
& \left({}^s I_{u^+}^\alpha f\right)(x) + \left({}^s I_{v^-}^\alpha f\right)(x) \\
& \leq \left[\left(x^{\beta+s} - u^{\beta+s}\right)^\alpha f(x) + \left(v^{\beta+s} - x^{\beta+s}\right)^\alpha f(v) \right. \\
& \quad + \left(x^{\beta+s} - u^{\beta+s}\right)^{\alpha-1} \left(\varphi(f(u), f(x)) \left(\frac{x^{\beta+s} + 1 - u^{\beta+s+1}}{(x-u)(\beta+s+1)} - u^{\beta+s} \right) \right. \\
& \quad \left. - \lambda \left(\frac{2(x^{\beta+s+2} - u^{\beta+s+2})}{\beta+s+2} - \frac{(x+u)(x^{\beta+s+1} - u^{\beta+s+1})}{\beta+s+1} \right) \right) \\
& \quad + \left(v^{\beta+s} - x^{\beta+s}\right)^{\alpha-1} \left(\varphi(f(x), f(v)) \left(\frac{v^{\beta+s+1} - x^{\beta+s+1}}{(v-x)(\beta+s+1)} - x^{\beta+s} \right) \right. \\
& \quad \left. \left. - \lambda \left(\frac{2(v^{\beta+s+2} - x^{\beta+s+2})}{\beta+s+2} - \frac{(v+x)(v^{\beta+s+1} - x^{\beta+s+1})}{\beta+s+1} \right) \right) \right]. \tag{57}
\end{aligned}$$

Corollary 10. Using $\Psi(\zeta) = \zeta^\alpha$ and $g(x) = ((x-u)^\rho)/\rho$ in (5) and $g(x) = (-(v-x)^\rho)/\rho$ in (6), where $\rho > 0$, fractional

integral operators given in [17] are obtained. Further, the following bound is also satisfied:

$$\begin{aligned}
\left({}^\rho I_{u^+}^\alpha f\right)(x) + \left({}^\rho I_{v^-}^\alpha f\right)(x) & \leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \left[(x-u)^{\rho\alpha} f(x) + \varphi(f(u), f(x)) \frac{(x-u)^{\rho\alpha}}{\rho+1} \right. \\
& \quad - \lambda \frac{\rho}{(\rho+1)(\rho+2)} (x-u)^{\rho\alpha+1} + (v-x)^{\rho\alpha} f(v) + \varphi(f(x), f(v)) \frac{(v-x)^{\rho\alpha}}{\rho+1} \\
& \quad \left. - \lambda \frac{\rho}{(\rho+1)(\rho+2)} (v-x)^{\rho\alpha+1} \right]. \tag{58}
\end{aligned}$$

Corollary 11. For $\Psi(\zeta) = ((\zeta^{\alpha/k})/(k\Gamma_k(\alpha)))$, where $\alpha > k$, and $g(x) = ((x-u)^\rho)/\rho$ in (5) and $g(x) = (-(v-x)^\rho)/\rho$ in

(6), where $\rho > 0$, fractional integral operators given in [18] are obtained. Further, the following bound is also satisfied:

$$\begin{aligned}
\left({}^\rho I_{u^+}^\alpha f\right)(x) + \left({}^\rho I_{v^-}^\alpha f\right)(x) & \leq \frac{1}{\rho^{\alpha/k} k\Gamma_k(\alpha)} \left[(x-u)^{\rho\alpha/k} f(x) + \varphi(f(u), f(x)) \frac{(x-u)^{\rho\alpha/k}}{\rho+1} \right. \\
& \quad - \lambda \frac{\rho}{(\rho+1)(\rho+2)} (x-u)^{\rho(\alpha/k)+2} + (v-x)^{\rho\alpha/k} f(v) + \varphi(f(x), f(v)) \frac{(v-x)^{\rho\alpha/k}}{\rho+1} \\
& \quad \left. - \lambda \frac{\rho}{(\rho+1)(\rho+2)} (v-x)^{\rho(\alpha/k)+2} \right]. \tag{59}
\end{aligned}$$

Remark 5

For $\lambda = 0$, all the results of Section 3 reduce to the results of Section 3 in [4]; if $\lambda > 0$, then all the results of Section 3 give the refinements of the results of Section 3 in [4].

For $\varphi(x, y) = x - y$ and $\lambda = 0$, all the results of Section 3 reduce to the propositions and corollaries of [5].

Further, various bounds can be obtained by applying Theorems 5 and 6 which we leave for the reader.

4. Concluding Remarks

The paper presents bounds of unified integral operators (5) and (6) for strongly φ -convex functions. These bounds are refinements of bounds obtained for unified integral

operators for φ -convex functions in [4]. The results for fractional integral operators have been deduced which provide bounds for Riemann–Liouville and other well-known fractional integral operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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