Hindawi
Journal of Mathematics
Volume 2020, Article ID 8829140, 17 pages
https://doi.org/10.1155/2020/8829140

Research Article

On Caputo–Fabrizio Fractional Integral Inequalities of Hermite–Hadamard Type for Modified h-Convex Functions

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Received 24 September 2020; Revised 25 October 2020; Accepted 16 November 2020; Published 22 December 2020

Academic Editor: Sunil Kumar

The theory of convex functions plays an important role in engineering and applied mathematics. The Caputo–Fabrizio fractional derivatives are one of the important notions of fractional calculus. The aim of this paper is to present some properties of Caputo–Fabrizio fractional integral operator in the setting of h-convex function. We present some new Caputo–Fabrizio fractional estimates from Hermite–Hadamard-type inequalities. The results of this paper can be considered as the generalization and extension of many existing results of inequalities and convex functions. Moreover, we also present some application of our results to special means of real numbers.

1. Introduction and Preliminaries

The subject of fractional calculus got rapid development because of its diverse applications, not only in mathematics but also into many other fields of sciences. Nowadays, the researchers from biology (e.g., Cesarone et al. [1] and Caputo and Cametti [2]), economy (e.g., Caputo [3]), demography (e.g., Jumarie [4]), geophysics (e.g., Iaffaldano et al. [5]), medicine (e.g., El Sahede [6]), and bioengineering (e.g., Magin [7]) and signal processing are using fractional calculus as a key tool.

Many researchers in the last three decades are studying fractional calculus [8–12]. Some researchers deduced that it is essential to define new fractional derivatives with different singular or nonsingular kernels in order to provide more sufficient area to model more real-world problems in different fields of science and engineering [13–19].

In the present research, we will restrict ourselves to Caputo–Fabrizio fractional derivative. The features that make the operators different from each other comprise singularity and locality, while kernel expression of the operator is presented with functions such as the power law, the exponential function, or a Mittag–Leffler function. The unique feature of the Caputo–Fabrizio operator is that it has a nonsingular kernel. The main feature of the Caputo–Fabrizio operator can be described as a real power turned in to the integer by means of the Laplace transformation, and consequently, the exact solution can be easily found for several problems.

Fractional calculus plays a very significant role in the development of inequality theory. To study convex functions and its generalizations, the Hermite–Hadamard-type inequality is considered as one of the fundamental inequality is given as.

Theorem 1 (see [20]). *Let ψ: $J\subseteq\mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $a_1, b_1 \in J$ with $a_1 < b_1$, then the following double inequality holds:*

$$\psi\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\mathrm{d}x \leq \frac{\psi(a_1)+\psi(b_1)}{2}. \quad (1)$$

The Hermite–Hadamard inequality has been generalized by numerous fractional integral operators [21–23]. For the

interesting readers, we refer [24–27] to study about Hermite–Hadamard inequalities.

The paper is organized as follows: first of all, we give some definitions and preliminary material related to our work. In Section 2, we will establish Hermite–Hadamard-type inequalities via Caputo–Fabrizio fractional integral operator for modified h-convex functions. Section 3 is devoted for some new inequalities via Caputo–Fabrizio fractional operator. At last, we give some application to special means and concluding remarks for our paper.

Now, we start by some necessary definitions and preliminary results which will be used and in this paper.

In [28], Toader gave the concept of modified h-convex functions as follows.

Definition 1. (see [28]). Let $\psi, h\colon J \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative function, then ψ is called modified h-convex function, if

$$\psi(\alpha a_1 + (1-\alpha)b_2) \le h(\alpha)\psi(a_1) + (1-h(\alpha))\psi(b_2), \quad (2)$$

for all $a_1, b_2 \in J$ and $\alpha \in [0,1]$ holds.

In [8, 29, 30], the concept of Caputo–Fabrizio fractional operator has been given.

Definition 2 (see [8, 29, 30]). Let $\psi \in H^1(a_1, b_1)$, $a_1 < b_1, \sigma \in [0,1]$, then the left fractional derivative in the sense of Caputo and Fabrizio is given by

$$\left({}^{\mathrm{CFC}}_{a_1}\mathrm{D}^{\sigma}\psi\right)(t) = \frac{B(\sigma)}{1-\sigma}\int_{a_1}^{t}\psi'(x)e^{(-\sigma(t-x)^{\sigma}/1-\sigma)}\mathrm{d}x, \quad (3)$$

and the associated fractional integral is

$$\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(t) = \frac{1-\sigma}{B(\sigma)}\psi(t) + \frac{\sigma}{B(\sigma)}\int_{a_1}^{t}\psi(x)\mathrm{d}x, \quad (4)$$

where $B(\sigma) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

The right fractional derivative is given as

$$\left({}^{\mathrm{CFC}}D^{\sigma}_{b_1}\psi\right)(t) = \frac{-B(\sigma)}{1-\sigma}\int_{t}^{b_1}\psi'(x)e^{(-\sigma(x-t)^{\sigma}/1-\sigma)}\mathrm{d}x, \quad (5)$$

and the associated fractional integral is

$$\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(t) = \frac{1-\sigma}{B(\sigma)}\psi(t) + \frac{\sigma}{B(\sigma)}\int_{t}^{b_1}\psi(x)\mathrm{d}x. \quad (6)$$

The following lemma is proven by Dragomir and Agarwal in [31].

Lemma 1 (see [31], Lemma 2.1). *Let $\psi\colon J = [a_1, b_1] \longrightarrow \mathbb{R}$ be a differentiable mapping on $\mathring{J}$, $a_1, b_1 \in \mathring{J}$ with $a_1 < b_1$. If $\psi \in L[a_1, b_1]$, then the following equality holds:*

$$\begin{aligned}\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\mathrm{d}x \\ = \frac{b_1-a_1}{2}\int_0^1(1-2\alpha)\psi'(\alpha a_1 - (1-\alpha)b_1)\mathrm{d}\alpha.\end{aligned} \quad (7)$$

Mustafa Gurbuz et al., in [32], generalized the kernal used in Lemma 1 with the help of Caputo–Fabrizio fractional integral operator.

Lemma 2 (see [32], Lemma 2). *Let $\psi\colon J = [a_1, b_1] \longrightarrow \mathbb{R}$ be a differentiable mapping on $\mathring{J}$, $a_1, b_1 \in \mathring{J}$ with $a_1 < b_1$. If $\psi \in L_1[a_1, b_1]$ and $\sigma \in [0,1]$, then the following equality holds"*

$$\begin{aligned}&\frac{b_1-a_1}{2}\int_0^1(1-2\alpha)\psi'(\alpha a_1-(1-\alpha)b_1)\mathrm{d}\alpha - \frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k) \\ &= \frac{\psi(a_1)+\psi(b_1)}{2} - \frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k) + \left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right],\end{aligned} \quad (8)$$

where $k \in [a_1, b_1]$ and $B(\sigma) > 0$ is a normalization function.

Iscan gave a refinement of Hölder integral inequality in [33], which is given in the following theorem.

Theorem 2 (Hölder–Iscan integral inequality [33]). *Let $p > 1$ and $(1/p) + (1/q) = 1$. If ψ and ϕ are real functions defined on interval $[a_1, b_1]$ and $|\psi|^q, |\phi|^q$ are integrable functions on $[a_1, b_1]$, then*

$$\begin{aligned}&\int_{a_1}^{b_1}|\psi(x)\phi(x)|\mathrm{d}x \\ &\le \frac{1}{b_1-a_1}\left[\left(\int_{a_1}^{b_1}(b_1-x)|\psi(x)|^p\mathrm{d}x\right)^{(1/p)}\left(\int_{a_1}^{b_1}(b_1-x)|\phi(x)|^q\mathrm{d}x\right)^{(1/q)}\right. \\ &\quad \left.+\left(\int_{a_1}^{b_1}(x-a_1)|\psi(x)|^p\mathrm{d}x\right)^{(1/p)}\left(\int_{a_1}^{b_1}(x-a_1)|\phi(x)|^q\mathrm{d}x\right)^{(1/q)}\right].\end{aligned} \quad (9)$$

A refinement of power-mean integral inequality is given in the following theorem.

Theorem 3 (improved power-mean integral inequality [34]). *Let $q \ge 1$. If ψ and ϕ are real functions defined on*

interval $[a_1, b_1]$ *and* $|\psi|, |\psi||g|^q$ *are integrable functions on* $[a_1, b_1]$, *then*

$$\begin{aligned}
&\int_{a_1}^{b_1} |\psi(x)\phi(x)|\mathrm{d}x \\
&\leq \frac{1}{b_1 - a_1}\left[\left(\int_{a_1}^{b_1}(b_1 - x)|\psi(x)|\mathrm{d}x\right)^{1-(1/q)}\left(\int_{a_1}^{b_1}(b_1 - x)|\psi(x)||\phi(x)|^q\mathrm{d}x\right)^{(1/q)}\right. \\
&\quad \left.+\left(\int_{a_1}^{b_1}(x - a_1)|\psi(x)|\mathrm{d}x\right)^{1-(1/q)}\left(\int_{a_1}^{b_1}(x - a_1)|\psi(x)||\phi(x)|^q\mathrm{d}x\right)^{(1/q)}\right].
\end{aligned} \tag{10}$$

2. Generalization of Hermite–Hadamard Inequality via the Caputo–Fabrizio Fractional Operator

The following theorem is a variant of Hermite–Hadamard inequality for modified h-convex functions.

Theorem 4. *Suppose that* $\psi: J = [a_1, b_1] \longrightarrow \mathbb{R}$ *is a modified h-convex function and* $\psi \in L_1[a_1, b_1]$. *If* $\sigma \in [0, 1]$, *then the following double inequality holds:*

$$\begin{aligned}
\psi\left(\frac{a_1 + b_1}{2}\right) &\leq \frac{B(\sigma)}{\sigma(b_1 - a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k) + \left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k) - \frac{2(1-\sigma)}{B(\sigma)}\psi(k)\right] \\
&\leq \psi(a_1) + [\psi(b_1) - \psi(a_1)]\int_0^1 h(\alpha)\mathrm{d}\alpha,
\end{aligned} \tag{11}$$

where $k \in [a_1, b_1]$ *and* $B(\sigma) > 0$ *is a normalization function.*

Proof. Since ψ is modified h-convex function on $[a_1, b_1]$, we can write

$$\begin{aligned}
2\psi\left(\frac{a_1 + b_1}{2}\right) &\leq \frac{2}{b_1 - a_1}\int_{a_1}^{b_1}\psi(x)\mathrm{d}x \\
&= \frac{2}{b_1 - a_1}\left[\int_{a_1}^{k}\psi(x)\mathrm{d}x + \int_{k}^{b_1}\psi(x)\mathrm{d}x\right].
\end{aligned} \tag{12}$$

Multiplying both sides of (12) by $(\sigma(b_1 - a_1)/2B(\sigma))$ and adding $(2(1-\sigma)/B(\sigma))\psi(k)$, we get

$$\begin{aligned}
&\frac{2(1-\sigma)}{B(\sigma)}\psi(k) + \frac{\sigma(b_1 - a_1)}{B(\sigma)}\psi\left(\frac{a_1 + b_1}{2}\right) \\
&\leq \frac{2(1-\sigma)}{B(\sigma)}\psi(k) + \frac{\sigma}{B(\sigma)}\left[\int_{a_1}^{k}\psi(x)\mathrm{d}x + \int_{k}^{b_1}\psi(x)\mathrm{d}x\right] \\
&= \left[\frac{(1-\sigma)}{B(\sigma)}\psi(k) + \frac{\sigma}{B(\sigma)}\int_{a_1}^{k}\psi(x)\mathrm{d}x\right] + \left[\frac{(1-\sigma)}{B(\sigma)}\psi(k) + \frac{\sigma}{B(\sigma)}\int_{k}^{b_1}\psi(x)\mathrm{d}x\right] \\
&= \left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k) + \left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k).
\end{aligned} \tag{13}$$

After suitable rearrangement of (13), we get the required left-hand side of (11).

For the right-hand side, we will use the right-hand side of Hermite–Hadamard inequality for modified h-convex functions:

$$\frac{2}{b_1 - a_1}\int_{a_1}^{b_1}\psi(x)\mathrm{d}x \le 2\psi(a_1) + 2[\psi(b_1) - \psi(a_1)]\int_0^1 h(\alpha)\mathrm{d}\alpha. \tag{14}$$

By using the same operator with (12) in (14), we have

$$\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k) + \left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k) \le \frac{2(1-\sigma)}{B(\sigma)}\psi(k) + \frac{\sigma(b_1 - a_1)}{B(\sigma)}\left[\psi(a_1) + [\psi(b_1) - \psi(a_1)]\int_0^1 h(\alpha)\mathrm{d}\alpha\right]. \tag{15}$$

After suitable rearrangement of (15), we get the required right-hand side of (11), which completes the proof. □

Remark 1. If we take $h(\alpha) = \alpha$, then inequality (11) reduces to the Hermite–Hadamard inequality for convex functions via Caputo–Fabrizio fractional operator [32].

Theorem 5. *Let ψ and ϕ are modified h-convex functions on J. If $\psi\phi \in L([a_1, b_1])$, then we have the following inequality:*

$$\begin{aligned}
&\frac{2B(\sigma)}{\sigma(b_1 - a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\phi\right)(k) + \left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\phi\right)(k) - \frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)\right]\\
&\le 2[M(a_1, b_1) - N(a_1, b_1)]\int_0^1 [h(\alpha)]^2\mathrm{d}\alpha\\
&\quad + 2\psi(b_1)\phi(b_1) + 2[N(a_1, b_1) - 2\psi(b_1)\phi(b_1)]\int_0^1 h(\alpha)\mathrm{d}\alpha,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
M(a_1, b_1) &= \psi(a_1)\phi(a_1) + \psi(b_1)\phi(b_1),\\
N(a_1, b_1) &= \psi(a_1)\phi(b_1) + \psi(b_1)\phi(a_1),
\end{aligned} \tag{17}$$

$k \in [a_1, b_1]$, and $B(\sigma) > 0$ is a normalization function.

Proof. Since ψ and ϕ are convex on $[a_1, b_1]$, we have

$$\psi(\alpha a_1 + (1-\alpha)b_1) \le h(\alpha)\psi(a_1) + (1 - h(\alpha))\psi(b_1), \quad \forall\alpha \in [0, 1], \tag{18}$$

$$\phi(\alpha a_1 + (1-\alpha)b_1) \le h(\alpha)\phi(a_1) + (1 - h(\alpha))\phi(b_1), \quad \forall\alpha \in [0, 1]. \tag{19}$$

Multiplying both sides of (18) and (19), we have

$$\begin{aligned}
&\psi(\alpha a_1 + (1-\alpha)b_1)\phi(\alpha a_1 + (1-\alpha)b_1)\\
&\le [h(\alpha)]^2\psi(a_1)\phi(a_1) + [1 - h(\alpha)]^2\psi(b_1)\phi(b_1) + h(\alpha)[1 - h(\alpha)][\psi(a_1)\phi(b_1) + \psi(b_1)\phi(a_1)].
\end{aligned} \tag{20}$$

Integrating (20) with "α" over $[0,1]$, and using the change of variable technique, we obtain

$$\begin{aligned}
&\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\phi(x)\mathrm{d}x\\
&\le \int_0^1\left[[h(\alpha)]^2\psi(a_1)\phi(a_1)+[1-h(\alpha)]^2\psi(b_1)\phi(b_1)\right]\mathrm{d}\alpha\\
&\quad+\int_0^1\left[h(\alpha)[1-h(\alpha)]\left[\psi(a_1)\phi(b_1)+\psi(b_1)\phi(a_1)\right]\right]\mathrm{d}\alpha\\
&\le M(a_1,b_1)\int_0^1[h(\alpha)]^2\mathrm{d}\alpha+\psi(b_1)\phi(b_1)\int_0^1[1-2h(\alpha)]\mathrm{d}\alpha\\
&\quad+N(a_1,b_1)\int_0^1 h(\alpha)[1-h(\alpha)]\mathrm{d}\alpha.
\end{aligned}\tag{21}$$

So,

$$\begin{aligned}
&\frac{2}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\phi(x)\mathrm{d}x\\
&\le 2\left[M(a_1,b_1)-N(a_1,b_1)\right]\int_0^1[h(\alpha)]^2\mathrm{d}\alpha+2\psi(b_1)\phi(b_1)\\
&\quad+2\left[N(a_1,b_1)-2\psi(b_1)\phi(b_1)\right]\int_0^1 h(\alpha)\mathrm{d}\alpha.
\end{aligned}\tag{22}$$

Multiplying both sides of (22) by $(\sigma(b_1-a_1)/2B(\sigma))$ and adding $(2(1-\sigma)/B(\sigma))\psi(k)\phi(k)$, we get

$$\begin{aligned}
&\frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)+\frac{\sigma}{B(\sigma)}\left[\int_{a_1}^{k}\psi(x)\phi(x)\mathrm{d}x+\int_k^{b_1}\psi(x)\phi(x)\mathrm{d}x\right]\\
&\le\frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)+\frac{\sigma(b_1-a_1)}{2B(\sigma)}\left[2\left[M(a_1,b_1)-N(a_1,b_1)\right]\int_0^1[h(\alpha)]^2\mathrm{d}\alpha\right.\\
&\quad\left.+2\psi(b_1)\phi(b_1)+2\left[N(a_1,b_1)-2\psi(b_1)\phi(b_1)\right]\int_0^1 h(\alpha)\mathrm{d}\alpha\right].
\end{aligned}\tag{23}$$

So,

$$\begin{aligned}
&\left[\frac{(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)+\frac{\sigma}{B(\sigma)}\int_{a_1}^{k}\psi(x)\phi(x)\mathrm{d}x\right]+\left[\frac{(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)+\frac{\sigma}{B(\sigma)}\int_k^{b_1}\psi(x)\phi(x)\mathrm{d}x\right]\\
&\le\frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)+\frac{\sigma(b_1-a_1)}{2B(\sigma)}\left[2\left[M(a_1,b_1)-N(a_1,b_1)\right]\int_0^1[h(\alpha)]^2\mathrm{d}\alpha\right.\\
&\quad\left.+2\psi(b_1)\phi(b_1)+2\left[N(a_1,b_1)-2\psi(b_1)\phi(b_1)\right]\int_0^1 h(\alpha)\mathrm{d}\alpha\right].
\end{aligned}\tag{24}$$

Thus,

$$\begin{aligned}&\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\phi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\phi\right)(k)\\&\leq\frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)+\frac{\sigma(b_1-a_1)}{2B(\sigma)}\left[2\left[M(a_1,b_1)-N(a_1,b_1)\right]\int_0^1[h(\alpha)]^2\mathrm{d}\alpha\right.\\&\left.+2\psi(b_1)\phi(b_1)+2\left[N(a_1,b_1)-2\psi(b_1)\phi(b_1)\right]\int_0^1h(\alpha)\mathrm{d}\alpha\right],\end{aligned}\tag{25}$$

with suitable rearrangements, and the proof is completed. □

Remark 2. If we take $h(\alpha)=\alpha$ in Theorem 5, we obtain ([32], Theorem 3).

Theorem 6. *Let ψ and ϕ are modified h-convex functions on J. If $\psi\phi\in L([a_1,b_1])$, the set of integrable functions and $h(1/2)[1-h(1/2)]\neq 0$, then we have the following inequality:*

$$\begin{aligned}&\frac{1}{h(1/2)[1-h(1/2)]}\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right)-\frac{B(\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}{\sigma(b_1-a_1)h(1/2).[1-h(1/2)]}\\&\cdot\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\phi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\phi\right)(k)\right]+\frac{2(1-\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}{\sigma(b_1-a_1)h(1/2).[1-h(1/2)]}\psi(k)\phi(k)\\&\leq 2M(a_1,b_1)\int_0^1h(\alpha)[1-h(\alpha)]\mathrm{d}\alpha+N(a_1,b_1)\int_0^1\left[[h(\alpha)]^2+[1-h(\alpha)]^2\right]\mathrm{d}\alpha,\end{aligned}\tag{26}$$

where

$$\begin{aligned}M(a_1,b_1)&=\psi(a_1)\phi(a_1)+\psi(b_1)\phi(b_1),\\N(a_1,b_1)&=\psi(a_1)\phi(b_1)+\psi(b_1)\phi(a_1),\end{aligned}\tag{27}$$

and $k\in[a_1,b_1]$, and $B(\sigma)>0$ is a normalization function.

Proof. Since ψ and ϕ are modified h-convex functions on J, then for $\alpha=(1/2)$, we have

$$\begin{aligned}\psi\left(\frac{a_1+b_1}{2}\right)&\leq h\left(\frac{1}{2}\right)\psi((1-\alpha)a_1+\alpha b_1)+\left[1-h\left(\frac{1}{2}\right)\right]\psi(\alpha a_1+(1-\alpha)b_1),\\\phi\left(\frac{a_1+b_1}{2}\right)&\leq h\left(\frac{1}{2}\right)\phi((1-\alpha)a_1+\alpha b_1)+\left[1-h\left(\frac{1}{2}\right)\right]\phi(\alpha a_1+(1-\alpha)b_1).\end{aligned}\tag{28}$$

Multiplying the above inequalities at both sides, we have

$$
\begin{aligned}
&\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right)\\
&\leq\left[h\left(\frac{1}{2}\right)\right]^2\psi\left((1-\alpha)a_1+\alpha b_1\right)\phi\left((1-\alpha)a_1+\alpha b_1\right)\\
&\quad+\left[1-h\left(\frac{1}{2}\right)\right]^2\psi\left(\alpha a_1+(1-\alpha)b_1\right)\phi\left(\alpha a_1+(1-\alpha)b_1\right)\\
&\quad+\left[h\left(\frac{1}{2}\right)\right].\left[1-h\left(\frac{1}{2}\right)\right]\left[\psi\left((1-\alpha)a_1+\alpha b_1\right)\phi\left(\alpha a_1+(1-\alpha)b_1\right)+\psi\left(\alpha a_1+(1-\alpha)b_1\right)\phi\left((1-\alpha)a_1+\alpha b_1\right)\right]\\
&\leq\left[h\left(\frac{1}{2}\right)\right]^2\psi\left((1-\alpha)a_1+\alpha b_1\right)\phi\left((1-\alpha)a_1+\alpha b_1\right)\\
&\quad+\left[1-h\left(\frac{1}{2}\right)\right]^2\psi\left(\alpha a_1+(1-\alpha)b_1\right)\phi\left(\alpha a_1+(1-\alpha)b_1\right)\\
&\quad+\left[h\left(\frac{1}{2}\right)\right].\left[1-h\left(\frac{1}{2}\right)\right]\left[\left(\left[1-h(\alpha)\right]\psi\left(a_1\right)+h(\alpha)\psi\left(b_1\right)\right)\right)\left(h(\alpha)\phi\left(a_1\right)+\left[1-h[\alpha]\right]\phi\left(b_1\right)\right)\\
&\quad+\left(h(\alpha)\psi\left(a_1\right)+\left[1-h(\alpha)\right]\psi\left(b_1\right)\right)\left(\left[1-h(\alpha)\right]\phi\left(a_1\right)+h(\alpha)\phi\left(b_1\right)\right)\right]\\
&\leq\left[h\left(\frac{1}{2}\right)\right]^2\psi\left((1-\alpha)a_1+\alpha b_1\right)\phi\left((1-\alpha)a_1+\alpha b_1\right)\\
&\quad+\left[1-h\left(\frac{1}{2}\right)\right]^2\psi\left(\alpha a_1+(1-\alpha)b_1\right)\phi\left(\alpha a_1+(1-\alpha)b_1\right)\\
&\quad+\left[h\left(\frac{1}{2}\right)\right].\left[1-h\left(\frac{1}{2}\right)\right]\left[\left(\left[h(\alpha)\right]\left[1-h(\alpha)\right]\left(\psi\left(a_1\right)\phi\left(a_1\right)+\psi\left(b_1\right)\psi\left(b_1\right)\right)\right)\right]\\
&\quad+\left[h(\alpha)\right]^2\psi\left(b_1\right)\phi\left(a_1\right)+\left[1-h(\alpha)\right]^2\psi\left(a_1\right)\phi\left(b_1\right)\Big)\\
&\quad+\left(\left[h(\alpha)\right]\left[1-h(\alpha)\right]\left(\psi\left(a_1\right)\phi\left(a_1\right)+\psi\left(b_1\right)\psi\left(b_1\right)\right)\right)\\
&\quad+\left[h(\alpha)\right]^2\psi\left(a_1\right)\phi\left(b_1\right)+\left[1-h(\alpha)\right]^2\psi\left(b_1\right)\phi\left(a_1\right)\Big)\Big]\\
&\leq\left[h\left(\frac{1}{2}\right)\right]^2\psi\left((1-\alpha)a_1+\alpha b_1\right)\phi\left((1-\alpha)a_1+\alpha b_1\right)\\
&\quad+\left[1-h\left(\frac{1}{2}\right)\right]^2\psi\left(\alpha a_1+(1-\alpha)b_1\right)\phi\left(\alpha a_1+(1-\alpha)b_1\right)\\
&\quad+\left[h\left(\frac{1}{2}\right)\right].\left[1-h\left(\frac{1}{2}\right)\right]\\
&\left[\left[h(\alpha)\right]\left[1-h(\alpha)\right]\left(2\psi\left(a_1\right)\phi\left(a_1\right)+2\psi\left(b_1\right)\psi\left(b_1\right)\right)\right.\\
&\quad\left.+\left(\left[1-h(\alpha)\right]^2+\left[h(\alpha)\right]^2\right)\left(\psi\left(a_1\right)\phi\left(b_1\right)\right)+\psi\left(b_1\right)\phi\left(a_1\right)\right)\Big)\Big].
\end{aligned}
\tag{29}
$$

Integrating (29) with respect to α over $[0,1]$ and using change of variable technique, we obtain

$$2\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right)\leq\left[\left[h\left(\frac{1}{2}\right)\right]^2+\left[1-h\left(\frac{1}{2}\right)\right]^2\right]$$
$$\left(\frac{2}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\phi(x)\mathrm{d}x\right)+2h\left(\frac{1}{2}\right).\left[1-h\left(\frac{1}{2}\right)\right]\left[2M(a_1,b_1)\int_0^1 h(\alpha)[1-h(\alpha)]\mathrm{d}\alpha\right.$$
$$\left.+N(a_1,b_1)\int_0^1\left[[h(\alpha)]^2+[1-h(\alpha)]^2\right]\mathrm{d}\alpha\right]. \tag{30}$$

So,

$$\frac{2}{\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right)\leq\frac{2}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\phi(x)\mathrm{d}x$$
$$+\frac{2h(1/2).[1-h(1/2)]}{\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\left[2M(a_1,b_1)\int_0^1 h(\alpha)[1-h(\alpha)]\mathrm{d}\alpha\right.$$
$$\left.+N(a_1,b_1)\int_0^1\left[[h(\alpha)]^2+[1-h(\alpha)]^2\right]\mathrm{d}\alpha\right]. \tag{31}$$

Multiplying both sides of (31) by $(\sigma(b_1-a_1)/2B(\sigma))$ and adding $(2(1-\sigma)/B(\sigma))\psi(k)\phi(k)$, we get

$$\frac{\sigma(b_1-a_1)}{B(\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right)+\frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)$$
$$\leq\frac{\sigma}{B(\sigma)}\left[\int_{a_1}^{k}\psi(x)\phi(x)\mathrm{d}x+\int_k^{b_1}\psi(x)\phi(x)\mathrm{d}x\right]$$
$$+\frac{\sigma(b_1-a_1)h(1/2).[1-h(1/2)]}{B(\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\left[2M(a_1,b_1)\int_0^1 h(\alpha)[1-h(\alpha)]\mathrm{d}\alpha\right.$$
$$\left.+N(a_1,b_1)\int_0^1\left[[h(\alpha)]^2+[1-h(\alpha)]^2\right]\mathrm{d}\alpha\right]+\frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k) \tag{32}$$
$$\leq\left[\frac{(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)+\frac{\sigma}{B(\sigma)}\int_{a_1}^{k}\psi(x)\phi(x)\mathrm{d}x\right]+\left[\frac{(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)+\frac{\sigma}{B(\sigma)}\int_k^{b_1}\psi(x)\phi(x)\mathrm{d}x\right]$$
$$+\frac{\sigma(b_1-a_1)h(1/2).[1-h(1/2)]}{B(\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\left[2M(a_1,b_1)\int_0^1 h(\alpha)[1-h(\alpha)]\mathrm{d}\alpha\right.$$
$$\left.+N(a_1,b_1)\int_0^1\left[[h(\alpha)]^2+[1-h(\alpha)]^2\right]\mathrm{d}\alpha\right].$$

Thus,

$$\begin{aligned}
&\frac{\sigma(b_1-a_1)}{B(\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right)+\frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)\\
&\leq\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\phi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\phi\right)(k)\\
&\quad+\frac{\sigma(b_1-a_1)h(1/2).[1-h(1/2)]}{B(\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\left[2M(a_1,b_1)\int_0^1 h(\alpha)[1-h(\alpha)]\mathrm{d}\alpha\right.\\
&\quad\left.+N(a_1,b_1)\int_0^1\left[[h(\alpha)]^2+[1-h(\alpha)]^2\right]\mathrm{d}\alpha\right].
\end{aligned}\tag{33}$$

This implies that

$$\begin{aligned}
&\frac{\sigma(b_1-a_1)}{B(\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right)\\
&\quad-\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\phi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\phi\right)(k)\right]+\frac{2(1-\sigma)}{B(\sigma)}\psi(k)\phi(k)\\
&\leq\frac{\sigma(b_1-a_1)h(1/2).[1-h(1/2)]}{B(\sigma)\left[[h(1/2)]^2+[1-h(1/2)]^2\right]}\left[2M(a_1,b_1)\int_0^1 h(\alpha)[1-h(\alpha)]\mathrm{d}\alpha\right.\\
&\quad\left.+N(a_1,b_1)\int_0^1\left[[h(\alpha)]^2+[1-h(\alpha)]^2\right]\mathrm{d}\alpha\right].
\end{aligned}\tag{34}$$

Multiplying both sides of the above inequality by $((B(\sigma)[[h(1/2)]^2+[1-h(1/2)]^2])/(\sigma(b_1-a_1)h(1/2).[1-h(1/2)]))$, we obtain our required result. □

Remark 3. If we take $h(\alpha)=\alpha$ in Theorem 6, we obtain ([32], Theorem 4).

3. Some New Results Related with Caputo–Fabrizio Fractional Operator

In this section, we establish some new inequalities for modified h-convex functions via Caputo–Fabrizio fractional operator.

Theorem 7. *Let $\psi\colon J\longrightarrow\mathbb{R}$ be a differentiable function on $\mathring{J}$. If $|\psi'|$ is a modified h-convex function on interval $[a_1,b_1]$, where $a_1,b_1\in J$ with $a_1<b_1$, $\psi'\in L_1[a_1,b_1]$ and $\sigma\in[0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right|\\
&\leq\frac{b_1-a_1}{2}\left[\frac{|\psi'(b_1)|}{2}+\left[|\psi'(a_1)|-|\psi'(b_1)|\right]\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right],
\end{aligned}\tag{35}$$

where $k \in [a_1, b_1]$, and $B(\sigma) > 0$ is a normalization function.

Proof. Using Lemma 2 and the definition of modified h-convexity of $|\psi'|$, we get

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right| \\
&\leq \frac{b_1-a_1}{2}\int_0^1|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|\mathrm{d}\alpha \\
&\leq \frac{b_1-a_1}{2}\int_0^1|1-2\alpha|\left[h(\alpha)\left|\psi'(a_1)\right|+(1-h(\alpha))\left|\psi'(b_1)\right|\right]\mathrm{d}\alpha \\
&\leq \frac{b_1-a_1}{2}\left[\left|\psi'(a_1)\right|\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha+\left|\psi'(b_1)\right|\int_0^1|1-2\alpha|(1-h(\alpha))\mathrm{d}\alpha\right] \\
&\leq \frac{b_1-a_1}{2}\left[\left|\psi'(b_1)\right|\int_0^1|1-2\alpha|\mathrm{d}\alpha+\left[\left|\psi'(a_1)\right|-\left|\psi'(b_1)\right|\right]\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right] \\
&\leq \frac{b_1-a_1}{2}\left[\frac{\left|\psi'(b_1)\right|}{2}+\left[\left|\psi'(a_1)\right|-\left|\psi'(b_1)\right|\right]\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right].
\end{aligned} \tag{36}$$

which completes the proof. □

Remark 4. If we take $h(\alpha) = \alpha$ in Theorem 7, we obtain ([32], Theorem 5).

Theorem 8. *Let $\psi: J \longrightarrow \mathbb{R}$ be a positive differentiable function on $\mathring{J}$. If $|\psi'|^q$ is a modified h-convex function on interval $[a_1, b_1]$, $a_1, b_1 \in \mathring{J}$ with $a_1 < b_1, q > 1$, $(1/p) + (1/q) = 1$, where $a_1, b_1 \in J$ with $a_1 < b_1$, $\psi' \in L_1[a_1, b_1]$ and $\sigma \in [0, 1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right| \\
&\leq \frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{(1/q)}\times\left[\left|\psi'(b_1)\right|^q+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1 h(\alpha)\mathrm{d}\alpha\right]^{(1/q)},
\end{aligned} \tag{37}$$

where $k \in [a_1, b_1]$, and $B(\sigma) > 0$ is a normalization function.

Proof. Using Lemma 2, Hölder's integral inequality and modified h-convexity of $|\psi'|^q$, we get

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right| \\
&\leq \frac{b_1-a_1}{2}\int_0^1|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|\mathrm{d}\alpha \\
&\leq \frac{b_1-a_1}{2}\times\left[\left(\int_0^1|1-2\alpha|^p\mathrm{d}\alpha\right)^{(1/p)}\left(\int_0^1\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q\mathrm{d}\alpha\right)^{(1/q)}\right] \\
&\leq \frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{(1/p)}\times\left(\int_0^1\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q\mathrm{d}\alpha\right)^{(1/q)} \\
&\leq \frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{(1/p)}\times\left(\int_0^1\left[h(\alpha)\left|\psi'(a_1)\right|^q+(1-h(\alpha))\left|\psi'(b_1)\right|^q\right]\mathrm{d}\alpha\right)^{(1/q)} \\
&\leq \frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{(1/p)}\times\left[\left|\psi'(b_1)\right|^q+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1 h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}.
\end{aligned} \tag{38}$$

which completes the proof. □

Remark 5. If we take $h(\alpha)=\alpha$ in Theorem 8, we obtain ([32], Theorem 6).

Theorem 9. *Let $\psi: J \longrightarrow \mathbb{R}$ be a positive differentiable function on $\mathring{J}$. If $|\psi'|^q$ is a modified h-convex function on interval $[a_1,b_1]$, $a_1,b_1 \in \mathring{J}$ with $a_1<b_1, q\geq 1$, where $a_1,b_1\in J$ with $a_1<b_1$, $\psi' \in L_1[a_1,b_1]$ and $\sigma\in[0,1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right|\\&\leq\frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-(1/q)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)},\end{aligned}\tag{39}$$

where $k\in[a_1,b_1]$, and $B(\sigma)>0$ is a normalization function.

Proof. Assuming $q>1$, using Lemma 2 and the power mean inequality and modified h-convex of $|\psi'|^q$, we get

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right|\\&\leq\frac{b_1-a_1}{2}\int_0^1|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|\mathrm{d}\alpha\\&\leq\frac{b_1-a_1}{2}\times\left[\left(\int_0^1|1-2\alpha|\mathrm{d}\alpha\right)^{1-(1/q)}\left(\int_0^1|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q\mathrm{d}\alpha\right)^{(1/q)}\right]\\&\leq\frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-(1/q)}\times\left(\int_0^1|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q\mathrm{d}\alpha\right)^{(1/q)}\\&\leq\frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-(1/q)}\times\left(\int_0^1|1-2\alpha|\left[h(\alpha)|\psi'(a_1)|^q+(1-h(\alpha))|\psi'(b_1)|^q\right]d\alpha\right)^{(1/q)}\\&\leq\frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-(1/q)}\times\left(|\psi'(a_1)|^q\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha+|\psi'(b_1)|^q\int_0^1|1-2\alpha|(1-h(\alpha))\mathrm{d}\alpha\right)^{(1/q)}\\&\leq\frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-(1/q)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}.\end{aligned}\tag{40}$$

For $q=1$, we use the estimates of Theorem 7, which also follows step by step the above estimates. This completes the proof of theorem. □

Remark 6

(1) Under the assumptions of Theorem 9 with $q=1$, we get conclusion of Theorem 7

(2) If we take $h(\alpha)=\alpha$ and $q=1$ in Theorem 9, we obtain ([32], Theorem 5)

Now, we will prove Theorems 8 and 9 by Hölder–Iscan and improved power mean integral inequality, respectively. Then, we will show that the results we have obtained in these theorems gives better approximation of Theorems 8 and 9, respectively.

Theorem 10. *Let $\psi: J \longrightarrow \mathbb{R}$ be a positive differentiable function on $\mathring{J}$. If $|\psi'|^q$ is a modified h-convex function on interval $[a_1,b_1]$, $a_1,b_1\in\mathring{J}$ with $a_1<b_1, q>1$, $(1/p)+(1/q)=1$, where $a_1,b_1\in J$ with $a_1<b_1$, $\psi'\in L_1[a_1,b_1]$ and $\sigma\in[0,1]$, then the following inequality holds:*

$$\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right|$$
$$\leq\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1(1-\alpha)h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}$$
$$+\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1\alpha h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}, \tag{41}$$

where $k\in[a_1,b_1]$, and $B(\sigma)>0$ is a normalization function.

Proof. Using Lemma 2, Hölder–Iscan integral inequality and modified h-convexity of $|\psi'|^q$, we get

$$\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right|$$
$$\leq\frac{b_1-a_1}{2}\int_0^1|1-2\alpha||\psi'(\alpha a_1-(1-\alpha)b_1)|\mathrm{d}\alpha$$
$$\leq\frac{b_1-a_1}{2}\times\left[\left(\int_0^1(1-\alpha)|1-2\alpha|^p\mathrm{d}\alpha\right)^{(1/p)}\left(\int_0^1(1-\alpha)|\psi'(\alpha a_1-(1-\alpha)b_1)|^q\mathrm{d}\alpha\right)^{(1/q)}\right]$$
$$+\frac{b_1-a_1}{2}\times\left[\left(\int_0^1\alpha|1-2\alpha|^p\mathrm{d}\alpha\right)^{(1/p)}\left(\int_0^1\alpha|\psi'(\alpha a_1-(1-\alpha)b_1)|^q\mathrm{d}\alpha\right)^{(1/q)}\right]$$
$$\leq\frac{b_1-a_1}{2}\times\left[\left(\frac{1}{2(p+1)}\right)^{(1/p)}\left(\int_0^1(1-\alpha)|\psi'(\alpha a_1-(1-\alpha)b_1)|^q\mathrm{d}\alpha\right)^{(1/q)}\right]$$
$$+\frac{b_1-a_1}{2}\times\left[\left(\frac{1}{2(p+1)}\right)^{(1/p)}\left(\int_0^1\alpha|\psi'(\alpha a_1-(1-\alpha)b_1)|^q\mathrm{d}\alpha\right)^{(1/q)}\right]$$
$$\leq\frac{b_1-a_1}{2}\times\left[\left(\frac{1}{2(p+1)}\right)^{(1/p)}\left(\int_0^1(1-\alpha)\left[h(\alpha)|\psi'(a_1)|^q+(1-h(\alpha))|\psi'(b_1)|^q\right]\mathrm{d}\alpha\right)^{(1/q)}\right]$$
$$+\frac{b_1-a_1}{2}\times\left[\left(\frac{1}{2(p+1)}\right)^{(1/p)}\left(\int_0^1\alpha\left[h(\alpha)|\psi'(a_1)|^q+(1-h(\alpha))|\psi'(b_1|^q\right]\mathrm{d}\alpha\right)^{(1/q)}\right]$$
$$\leq\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left(|\psi'(a_1)|^q\int_0^1(1-\alpha)h(\alpha)\mathrm{d}\alpha+|\psi'(b_1)|^q\int_0^1(1-\alpha)(1-h(\alpha))\mathrm{d}\alpha\right)^{(1/q)}$$
$$+\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left(|\psi'(a_1)|^q\int_0^1\alpha h(\alpha)\mathrm{d}\alpha+|\psi'(b_1)|^q\int_0^1\alpha(1-h(\alpha))\mathrm{d}\alpha\right)^{(1/q)}$$
$$\leq\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1(1-\alpha)h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}$$
$$+\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1\alpha h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}. \tag{42}$$

which completes the proof. □

Corollary 1. *If we take $h(\alpha)=\alpha$ in inequality (41), we get the following inequality:*

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right|\\&\leq\frac{b_1-a_1}{4}\left(\frac{1}{p+1}\right)^{(1/p)}\times\left[\left(\frac{|\psi'(a_1)|^q+2|\psi(b_1)|^q}{3}\right)^{(1/q)}+\left(\frac{2|\psi'(a_1)|^q+|\psi(b_1)|^q}{3}\right)^{(1/q)}\right].\end{aligned}\tag{43}$$

Remark 7. The inequality (41) gives better results than inequality[35], and we have the following inequality:

$$\begin{aligned}&\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1(1-\alpha)h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\&+\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1\alpha h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\&\leq\frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{(1/q)}\times\left[|\psi'(b_1)|^q+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}.\end{aligned}\tag{44}$$

Proof. Using concavity of μ: $[0,\infty]\longrightarrow\mathbb{R}, \mu(x)=x^{\lambda}$, $0<\lambda\leq 1$, we get

$$\begin{aligned}&\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1(1-\alpha)h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\&+\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times\left[\frac{|\psi'(b_1)|^q}{2}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1\alpha h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\&\leq\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{(1/p)}\times 2\left[\frac{|\psi'(b_1)|^q}{2}+\frac{|\psi'(a_1)|^q-|\psi'(b_1)|^q}{2}\int_0^1h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\&=\frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{(1/q)}\times\left[|\psi'(a_1)|^q+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}.\end{aligned}\tag{45}$$

which completes the proof. □

Theorem 11. *Let ψ: $J\longrightarrow\mathbb{R}$ be a positive differentiable function on $\mathring{J}$. If $|\psi'|^q$ is a modified h-convex function on interval $[a_1,b_1]$, $a_1,b_1\in\mathring{J}$ with $a_1<b_1$, $q\geq 1$, where $a_1,b_1\in J$ with $a_1<b_1$. If $\psi'\in L_1[a_1,b_1]$ and $\sigma\in[0,1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right|\\&\leq\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{|\psi'(b_1)|^q}{4}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1(1-\alpha)|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\&+\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{|\psi'(b_1)|^q}{4}+\left[|\psi'(a_1)|^q-|\psi'(b_1)|^q\right]\int_0^1\alpha|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)},\end{aligned}\tag{46}$$

where $k \in [a_1, b_1]$, and $B(\sigma) > 0$ is a normalization function.

Proof. Assuming $q > 1$ and using Lemma 2, improved power-mean integral inequality and modified h-convexity of $|\psi'|^q$, we get

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right| \\
&\le \frac{b_1-a_1}{2}\int_0^1 |1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|\mathrm{d}\alpha \\
&\le \frac{b_1-a_1}{2}\times\left[\left(\int_0^1(1-\alpha)|1-2\alpha|\mathrm{d}\alpha\right)^{1-(1/q)}\left(\int_0^1(1-\alpha)|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q\mathrm{d}\alpha\right)^{(1/q)}\right] \\
&\quad+\frac{b_1-a_1}{2}\times\left[\left(\int_0^1\alpha|1-2\alpha|\mathrm{d}\alpha\right)^{1-(1/q)}\left(\int_0^1\alpha|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q\mathrm{d}\alpha\right)^{(1/q)}\right] \\
&\le \frac{b_1-a_1}{2}\times\left[\left(\frac{1}{4}\right)^{1-(1/q)}\left(\int_0^1(1-\alpha)|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q\mathrm{d}\alpha\right)^{(1/q)}\right] \\
&\quad+\frac{b_1-a_1}{2}\times\left[\left(\frac{1}{4}\right)^{1-(1/q)}\left(\int_0^1\alpha|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q\mathrm{d}\alpha\right)^{(1/q)}\right] \\
&\le \frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left(\int_0^1(1-\alpha)|1-2\alpha|\left[h(\alpha)\left|\psi'(a_1)\right|^q+(1-h(\alpha))\left|\psi'(b_1)\right|^q\right]\mathrm{d}\alpha\right)^{(1/q)} \\
&\quad+\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left(\int_0^1\alpha|1-2\alpha|\left[h(\alpha)\left|\psi'(a_1)\right|^q+(1-h(\alpha))\left|\psi'(b_1)\right|^q\right]\mathrm{d}\alpha\right)^{(1/q)} \\
&\le \frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(b_1)\right|^q}{4}+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1(1-\alpha)|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)} \\
&\quad+\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(b_1)\right|^q}{4}+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1\alpha|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}.
\end{aligned} \tag{47}$$

For $q = 1$, we use the estimates of Theorem 7 which also follows step by step the above estimates. This completes the proof of theorem. □

Corollary 2. *If we take $h(\alpha) = \alpha$ in inequality (41), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(b_1)}{2}+\frac{2(1-\sigma)}{\sigma(b_1-a_1)}\psi(k)-\frac{B(\sigma)}{\sigma(b_1-a_1)}\left[\left({}^{\mathrm{CF}}_{a_1}I^{\sigma}\psi\right)(k)+\left({}^{\mathrm{CF}}I^{\sigma}_{b_1}\psi\right)(k)\right]\right| \\
&\le \frac{b_1-a_1}{8}\left[\left(\frac{\left|\psi'(a_1)\right|^q+3\left|\psi(b_1)\right|^q}{4}\right)^{(1/q)}+\left(\frac{3\left|\psi'(a_1)\right|^q+\left|\psi(b_1)\right|^q}{4}\right)^{(1/q)}\right].
\end{aligned} \tag{48}$$

Remark 8. Inequality (46) gives better results than inequality (39), and we have the following inequality:

$$\begin{aligned}
&\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(b_1)\right|^q}{4}+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1(1-\alpha)|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\
&+\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(b_1)\right|^q}{4}+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1\alpha|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\
&\le\frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(b_1)\right|^q}{2}+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}.
\end{aligned}\tag{49}$$

Proof. Using concavity of ψ: $[0,\infty]\longrightarrow\mathbb{R}$, $\psi(x)=x^\lambda$, $0<\lambda\le 1$, we get

$$\begin{aligned}
&\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(b_1)\right|^q}{4}+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1(1-\alpha)|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\
&+\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(b_1)\right|^q}{4}+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1\alpha|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\
&\le\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times 2\left[\frac{\left|\psi'(b_1)\right|^q}{4}+\frac{\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q}{2}\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}\\
&=\frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(b_1)\right|^q}{2}+\left[\left|\psi'(a_1)\right|^q-\left|\psi'(b_1)\right|^q\right]\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right]^{(1/q)}.
\end{aligned}\tag{50}$$

which completes the proof. □

4. Application to Means

For two positive numbers $a_1>0$ and $b_1>0$, define

$$\begin{aligned}
&A(a_1,b_1)=\frac{a_1+b_1}{2},\\
&L_p=L_p(a_1,b_1)=\begin{cases}\left[\dfrac{b_1^{p+1}-a_1^{p+1}}{(p+1)(b_1-a_1)}\right]^{(1/p)}, & a_1\ne b_1,\ p\in\mathbb{R},\{-1,0\},\\ a_1, & a_1=b_2.\end{cases}
\end{aligned}\tag{51}$$

These means are, respectively, called the arithmetic and p-logarithmic means of two positive numbers a_1 and b_1.

Proposition 1. *Let $a_1,b_1\in[0,\infty)$ with $a_1<b_1$, then the following inequality holds:*

$$\left|A^n(a_1,b_1)-L_n^n(a_1,b_1)\right|\le\frac{n(b_1-a_1)}{2}\left[\frac{\left|b_1^{n-1}\right|}{2}+\left[\left|a_1^{n-1}\right|-\left|b_1^{n-1}\right|\right]\int_0^1|1-2\alpha|h(\alpha)\mathrm{d}\alpha\right].\tag{52}$$

Proof. In Theorem 7, if we set $\psi(x) = x^n$, where n is an even number with $\sigma = 1$ and $B(\sigma) = B(1) = 1$, we obtain the required result. □

Proposition 2. *Let $a_1, b_1 \in [0, \infty)$ with $a_1 < b_1$, then the following inequality holds:*

$$\left|A^2(a_1, b_1) - L_2^2(a_1, b_1)\right| \leq \frac{(b_1 - a_1)}{2}\left[\left|b_1\right| + 2\left[\left|a_1\right| - \left|b_1\right|\right]\int_0^1 |1 - 2\alpha| h(\alpha) \mathrm{d}\alpha\right]. \tag{53}$$

Proof. In Theorem 7, if we set $\psi(x) = x^2$, with $\sigma = 1$ and $B(\sigma) = B(1) = 1$, we obtain the required result. □

5. Conclusion

Hermite–Hadamard-type inequalities for modified h-convex functions via Caputo–Fabrizio integral operator are derived. Some new and interesting integral inequalities involving Caputo–Fabrizio fractional integral operator are also obtained for modified h-convex functions. Many existing results in literature become the particular cases for these results as mentioned in remarks.

Data Availability

All data required for this paper are included within the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally to this paper.

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