Research Article

Inequalities for Estimations of Integrals Related to Higher-Order Strongly n -Polynomial Preinvex Functions

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In this paper, we establish an integral identity associated with m -times differentiable functions. The result is then used to derive some integral estimations for higher-order strongly n -polynomial preinvex functions. Finally, we apply the obtained inequalities to construct new inequalities involving special means.

1. Introduction

The convexity properties of functions have been a powerful tool for dealing with various problems of pure and applied sciences. In recent decades, the concept of classical convexity has been extended and generalized in different directions, and the study relevant to this subject has stimulated an increased interest. In [1], Weir and Mond introduced the notion of preinvexity as follows.

Definition 1. A set $\mathcal{X} \subset \mathbb{R}$ is said to be invex with respect to $\xi: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ if

$$e + \tau \xi(f, e) \in \mathcal{X}, \quad \forall e, f \in \mathcal{X}, \tau \in [0, 1]. \quad (1)$$

Definition 2. A function $\Lambda: \mathcal{X} \longrightarrow \mathbb{R}$ is said to be preinvex with respect to $\xi: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ if

$$\Lambda(e + \tau \xi(f, e)) \leq (1 - \tau)\Lambda(e) + \tau\Lambda(f), \quad (2)$$

$$\forall e, f \in \mathcal{X}, \tau \in [0, 1].$$

For $\xi(f, e) = f - e$, the class of preinvex functions reduces to the class of convex functions. It had been shown that the preinvex functions may not be convex functions. For example, the function $\Lambda = -|x|$ is not a convex function, but it is a preinvex function with respect to ξ provided that

$$\xi(f, e) = \begin{cases} f - e, & \text{if } f \leq 0, e \leq 0 \text{ and } f \geq 0, e \geq 0, \\ e - f, & \text{otherwise.} \end{cases} \quad (3)$$

Mohan and Neogy [2] and Mititelu [3] discussed the properties of invex sets and preinvex functions. Noor [4] and Noor et al. [5] investigated integral inequalities of Hermite–Hadamard type for log-preinvex functions and h -preinvex functions, respectively. Wu et al. [6] considered harmonically (p, h, m) -preinvex functions and relevant inequalities. Peng et al. [7] established fractional Simpson's inequalities with the aid of (m, h_1, h_2) -preinvexity. Du et al. [8] dealt with integral inequalities involving generalized m -convex functions on fractal sets. Saleh [9] defined (ε, m) -preinvex function and studied the properties of (ε, m) -preinvex functions. Mohammed [10] focused on the integral inequalities for preinvex functions via the generalized beta function.

Very recently, Toplu et al. [11] introduced and investigated the class of n -polynomial convex functions as follows.

Definition 3. Let $n \in \mathbb{N}$. A nonnegative function $\Lambda: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is called an n -polynomial convex function if for any $e, f \in I$ and $\tau \in [0, 1]$,

$$\begin{aligned} \Lambda(\tau e + (1 - \tau)f) &\leq \frac{1}{n} \sum_{s=1}^n [1 - (1 - \tau)^s] \Lambda(e) \\ &\quad + \frac{1}{n} \sum_{s=1}^n [1 - \tau^s] \Lambda(f). \end{aligned} \quad (4)$$

Note that if we take $n = 1$ in the above inequality, then we have 1-polynomial convexity which is just the classical convexity. It is also necessary to mention here that every nonnegative convex function is also an n -polynomial convex function. For more properties on the n -polynomial convex function, see [11].

Kadalkal [12] and Karamardian [13] put forward the class of strongly convex functions, respectively. Strong convexity is just the strengthening version of convexity, which reads as follows.

Definition 4. A function $\Lambda: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a strongly convex function with modulus $\mu > 0$ if

$$\begin{aligned} \Lambda((1 - \tau)e + \tau f) &\leq (1 - \tau)\Lambda(e) \\ &\quad + \tau\Lambda(f) - \mu\tau(1 - \tau)|f - e|^2, \end{aligned} \quad (5)$$

$\forall e, f \in I, \tau \in [0, 1]$.

In order to deal with the problems of mathematical programs with equilibrium constraints, Lin and Fukushima [14] proposed a generalization of the strongly convex function, which is called the higher-order strongly convex functions.

Definition 5. A function $\Lambda: I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a strongly convex function with order $\sigma > 0$ and modulus $\mu > 0$ if

$$\begin{aligned} \Lambda((1 - \tau)e + \tau f) &\leq (1 - \tau)\Lambda(e) \\ &\quad + \tau\Lambda(f) - \mu\tau(1 - \tau)|f - e|^\sigma, \end{aligned} \quad (6)$$

$\forall e, f \in I, \tau \in [0, 1]$.

The higher-order strongly convex functions would reduce to the strongly convex function in the special case when $\sigma = 2$.

As an extension of classical convex functions, the strongly convex functions have been widely used in establishing new inequalities and dealing with estimations of integrals. Angulo et al. [15] and Zhang et al. [16] presented some inequalities for strongly h -convex functions and strongly (s, m) -convex functions, respectively. In order to further generalize the strongly convex functions, recently, some researchers, such as Mishra and Sharma [17], Noor and Noor [18], and Mohsen et al. [19], began to study the higher-order strongly convex functions. Wu et al. [20, 21]

gave some estimates of the upper bound for differentiable functions associated with k -fractional integrals and higher-order strongly convex functions. Khan et al. [22] and Ullah et al. [23] introduced the concept of coordinate and the technique of majorization into the study of strongly convex functions.

In this paper, we introduce the notion for a new class of strongly convex functions which are named as the strongly n -polynomial preinvex functions of order $\sigma > 0$.

Definition 6. Let $n \in \mathbb{N}$. A nonnegative function $\Lambda: \mathcal{X} \longrightarrow \mathbb{R}$ is said to be a strongly n -polynomial preinvex function of order $\sigma > 0$ with modulus $\mu > 0$ if

$$\begin{aligned} \Lambda(e + (1 - \tau)\xi(f, e)) &\leq \frac{1}{n} \sum_{s=1}^n [1 - (1 - \tau)^s] \Lambda(e) \\ &\quad + \frac{1}{n} \sum_{s=1}^n [1 - \tau^s] \Lambda(f) \\ &\quad - \mu[\tau^\sigma(1 - \tau) + \tau(1 - \tau)^\sigma]|\xi(f, e)|^\sigma \end{aligned} \quad (7)$$

holds for all $e, f \in \mathcal{X}$ and $\tau \in [0, 1]$.

Remark 1. Note that

- (i) If we take $\sigma = 2$ in Definition 6, then we have the class of strongly n -polynomial preinvex functions
- (ii) If we take $\xi(f, e) = f - e$ in Definition 6, then we have the class of strongly n -polynomial convex functions of order $\sigma > 0$
- (iii) If we take $\xi(f, e) = f - e$ and $\sigma = 2$ in Definition 6, then we have the class of strongly n -polynomial convex functions
- (iv) If we take $n = 1$ in Definition 6, then we have the class of strongly preinvex functions of order $\sigma > 0$

This shows that the class of strongly n -polynomial preinvex functions of order $\sigma > 0$ is quite unifying one as it relates several unrelated classes of convexity.

The main objective of this paper is to establish a new integral identity for m -times differentiable functions. With the help of this identity, we deduce some inequalities for estimations of integrals associated with m -times differentiable strongly preinvex functions of order $\sigma > 0$. As applications, we discuss some special cases of the main results and establish several new inequalities involving special means.

2. A Key Lemma

Here, we establish an auxiliary result which will play a significant role in the development of our main results.

Lemma 1. Let $\Lambda: [e, e + \xi(f, e)] \longrightarrow \mathbb{R}$ be an m -times differentiable function on $(e, e + \xi(f, e))$ with $\xi(f, e) > 0$. If $\Lambda^{(m)} \in L[e, e + \xi(f, e)]$, then the following integral equality holds for $m \in \mathbb{N}_+$:

$$\begin{aligned}
& \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \\
&= \frac{\xi^m(f, e)}{2m!} \int_0^1 [\tau^m + (\tau - 1)^m] \Lambda^{(m)}(e + (1 - \tau)\xi(f, e)) d\tau.
\end{aligned} \tag{8}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
& \frac{\xi^{m+1}(f, e)}{2m!} \int_0^1 \tau^m \Lambda^{(m)}(e + (1 - \tau)\xi(f, e)) d\tau \\
&= -\frac{\xi^m(f, e)}{2m!} \Lambda^{(m-1)}(e) + \frac{\xi^m(f, e)}{2(m-1)!} \int_0^1 \tau^{m-1} \Lambda^{(m-1)}(e + (1 - \tau)\xi(f, e)) d\tau \\
&= -\frac{\xi^m(f, e)}{2m!} \Lambda^{(m-1)}(e) - \frac{\xi^{m-1}(f, e)}{2(m-1)!} \Lambda^{(m-2)}(e) + \frac{\xi^{m-1}(f, e)}{2(m-2)!} \int_0^1 \tau^{m-2} \Lambda^{(m-2)}(e + (1 - \tau)\xi(f, e)) d\tau \\
&= -\sum_{k=1}^{m-1} \frac{\xi^{k+1}(f, e) \Lambda^{(k)}(e)}{2(k+1)!} + \frac{\xi^2(f, e)}{2} \int_0^1 \tau \Lambda'(e + (1 - \tau)\xi(f, e)) d\tau \\
&= -\sum_{k=1}^m \frac{\xi^k(f, e) \Lambda^{(k-1)}(e)}{2k!} + \frac{1}{2} \int_e^{e+\xi(f, e)} \Lambda(x) dx.
\end{aligned} \tag{9}$$

Similarly, we deduce that

$$\begin{aligned}
& \frac{\xi^{m+1}(f, e)}{2m!} \int_0^1 (\tau - 1)^m \Lambda^{(m)}(e + (1 - \tau)\xi(f, e)) d\tau \\
&= \frac{\xi^m(f, e)}{2m!} (-1)^m \Lambda^{(m-1)}(e + \xi(f, e)) + \frac{\xi^m(f, e)}{2(m-1)!} \int_0^1 (\tau - 1)^{m-1} \Lambda^{(m-1)}(e + (1 - \tau)\xi(f, e)) d\tau \\
&= \frac{\xi^m(f, e)}{2m!} (-1)^m \Lambda^{(m-1)}(e + \xi(f, e)) + \frac{\xi^{m-1}(f, e)}{2(m-1)!} (-1)^{m-1} \Lambda^{(m-2)}(e + \xi(f, e)) \\
&\quad + \frac{\xi^{m-1}(f, e)}{2(m-2)!} \int_0^1 (\tau - 1)^{m-2} \Lambda^{(m-2)}(e + (1 - \tau)\xi(f, e)) d\tau \\
&= \sum_{k=1}^{m-1} \frac{\xi^{k+1}(f, e) (-1)^{k+1} \Lambda^{(k)}(e + \xi(f, e))}{2(k+1)!} + \frac{\xi^2(f, e)}{2} \int_0^1 (\tau - 1) \Lambda'(e + (1 - \tau)\xi(f, e)) d\tau \\
&= \sum_{k=1}^m \frac{\xi^k(f, e) (-1)^k \Lambda^{(k-1)}(e + \xi(f, e))}{2k!} + \frac{1}{2} \int_e^{e+\xi(f, e)} \Lambda(x) dx.
\end{aligned} \tag{10}$$

Combining (9) and (10) completes the proof of Lemma 1. $\square$

3. Main Results

We are now in a position to state our main results.

Theorem 1. Let $\Lambda: [e, e + \xi(f, e)] \longrightarrow \mathbb{R}$ be an m -times differentiable function on $(e, e + \xi(f, e))$ with $\xi(f, e) > 0$ and $\Lambda^{(m)} \in L[e, e + \xi(f, e)]$. If $|\Lambda^{(m)}|$ is a strongly n -polynomial preinvex function of order $\sigma > 0$ with modulus $\mu > 0$, then

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left[\frac{|\Lambda^{(m)}(e)|}{n} \sum_{s=1}^n \mathcal{H}_1 + \frac{|\Lambda^{(m)}(f)|}{n} \sum_{s=1}^n \mathcal{H}_1 - \mu \mathcal{H}_2 \xi^\sigma(f, e) \right], \quad (11)$$

where

$$\mathcal{H}_1 = \frac{m+2s+1}{(m+1)(m+s+1)} - B(m+1, s+1),$$

$$\mathcal{H}_2 = \frac{2}{(m+\sigma+1)(m+\sigma+2)} + 2B(\sigma+1, m+2), \quad (12)$$

and $B(x, y) = \int_0^1 \tau^{x-1} (1-\tau)^{y-1} d\tau$ is the beta function.

Proof. Using Lemma 1 and the fact that $|\Lambda^{(m)}|$ is a strongly n -polynomial preinvex function of order $\sigma > 0$, we have

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \int_0^1 [\tau^m + (1-\tau)^m] |\Lambda^{(m)}(e + (1-\tau)\xi(f, e))| d\tau$$

$$\leq \frac{\xi^m(f, e)}{2m!} \int_0^1 [\tau^m + (1-\tau)^m] \left[\frac{1}{n} \sum_{s=1}^n [1 - (1-\tau)^s] |\Lambda^{(m)}(e)| + \frac{1}{n} \sum_{s=1}^n [1 - \tau^s] |\Lambda^{(m)}(f)| - \mu \xi^\sigma(f, e) [\tau^\sigma(1-\tau) + \tau(1-\tau)^\sigma] \right] d\tau$$

$$= \frac{\xi^m(f, e)}{2m!} \left[\frac{|\Lambda^{(m)}(e)|}{n} \int_0^1 [\tau^m + (1-\tau)^m] \sum_{s=1}^n [1 - (1-\tau)^s] d\tau + \frac{|\Lambda^{(m)}(f)|}{n} \int_0^1 [\tau^m + (1-\tau)^m] \sum_{s=1}^n [1 - \tau^s] d\tau - \mu \xi^\sigma(f, e) \int_0^1 [\tau^m + (1-\tau)^m] [\tau^\sigma(1-\tau) + \tau(1-\tau)^\sigma] d\tau \right]$$

$$= \frac{\xi^m(f, e)}{2m!} \left[\frac{|\Lambda^{(m)}(e)|}{n} \sum_{s=1}^n \int_0^1 [\tau^m + (1-\tau)^m] [1 - (1-\tau)^s] d\tau + \frac{|\Lambda^{(m)}(f)|}{n} \sum_{s=1}^n \int_0^1 [\tau^m + (1-\tau)^m] [1 - \tau^s] d\tau - \mu \xi^\sigma(f, e) \int_0^1 [\tau^m + (1-\tau)^m] [\tau^\sigma(1-\tau) + \tau(1-\tau)^\sigma] d\tau \right], \quad (13)$$

where

$$\begin{aligned}\int_0^1 [\tau^m + (1-\tau)^m][1 - (1-\tau)^s] d\tau &= \frac{m+2s+1}{(m+1)(m+s+1)} - B(m+1, s+1) = \mathcal{H}_1, \\ \int_0^1 [\tau^m + (1-\tau)^m][1 - \tau^s] d\tau &= \frac{m+2s+1}{(m+1)(m+s+1)} - B(s+1, m+1) = \mathcal{H}_1, \\ \int_0^1 [\tau^m + (1-\tau)^m][\tau^\sigma(1-\tau) + \tau(1-\tau)^\sigma] d\tau &= \frac{2}{(m+\sigma+1)(m+\sigma+2)} + 2B(\sigma+1, m+2) = \mathcal{H}_2.\end{aligned}\quad (14)$$

The proof of Theorem 1 is completed.

In the following, we discuss some special cases of Theorem 1.

- (1) If we take the limit $\mu \rightarrow 0$ in Theorem 1, we get a result for the n -polynomial preinvex function. $\square$

Corollary 1. Under the assumptions of Theorem 1, if $|\Lambda^{(m)}|$ is an n -polynomial preinvex function, then

$$\begin{aligned}& \left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right| \\ & \leq \frac{\xi^m(f, e)}{2m!} \left[\frac{|\Lambda^{(m)}(e)|}{n} \sum_{s=1}^n \mathcal{H}_1 + \frac{|\Lambda^{(m)}(f)|}{n} \sum_{s=1}^n \mathcal{H}_1 \right],\end{aligned}\quad (15)$$

where

$$\mathcal{H}_1 = \frac{m+2s+1}{(m+1)(m+s+1)} - B(m+1, s+1). \quad (16)$$

Corollary 2. Under the assumptions of Theorem 1, if $|\Lambda^{(m)}|$ is a strongly n -polynomial preinvex function, then

- (2) If we take $\sigma = 2$ in Theorem 1, we get a result for the strongly n -polynomial preinvex function.

$$\begin{aligned}& \left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right| \\ & \leq \frac{\xi^m(f, e)}{2m!} \left[\frac{|\Lambda^{(m)}(e)|}{n} \sum_{s=1}^n \mathcal{H}_1 + \frac{|\Lambda^{(m)}(f)|}{n} \sum_{s=1}^n \mathcal{H}_1 - \mu \mathcal{H}_3 \xi^2(f, e) \right],\end{aligned}\quad (17)$$

where

$$\begin{aligned}\mathcal{H}_1 &= \frac{m+2s+1}{(m+1)(m+s+1)} - B(m+1, s+1), \\ \mathcal{H}_3 &= \int_0^1 [\tau^m + (1-\tau)^m][\tau^2(1-\tau) + \tau(1-\tau)^2] d\tau = \frac{2}{(m+2)(m+4)}.\end{aligned}\quad (18)$$

- (3) If we take $n = 1$ in Theorem 1, we get a result for the higher-order strongly preinvex function.

Corollary 3. Under the assumptions of Theorem 1, if $|\Lambda^{(m)}|$ is a strongly preinvex function of order $\sigma > 0$, then

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left[\frac{|\Lambda^{(m)}(e)| + |\Lambda^{(m)}(f)|}{m+1} - \mu \mathcal{H}_2 \xi^\sigma(f, e) \right]. \quad (19)$$

Theorem 2. Let $\Lambda: [e, e + \xi(f, e)] \rightarrow \mathbb{R}$ be an m -times differentiable function on $(e, e + \xi(f, e))$ with $\xi(f, e) > 0$, $1/p + 1/q = 1$, $p > 1, q > 1$, and $\Lambda^{(m)} \in L[e, e + \xi(f, e)]$. If

$|\Lambda^{(m)}|^q$ is a strongly n -polynomial preinvex function of order $\sigma > 0$ with modulus $\mu > 0$, then

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{\frac{1}{p}} \left(\frac{2}{n} \sum_{s=1}^n \frac{s}{s+1} \mathcal{A} \left(|\Lambda^{(m)}(e)|^q, |\Lambda^{(m)}(f)|^q \right) - \frac{2\mu \xi^\sigma(f, e)}{(\sigma+1)(\sigma+2)} \right)^{1/q}, \quad (20)$$

where $\mathcal{A}(\cdot, \cdot)$ is the arithmetic mean.

Proof. Utilizing Lemma 1, Hölder's integral inequality, and the fact that $|\Lambda^{(m)}|^q$ is a strongly n -polynomial preinvex function of order $\sigma > 0$, one has

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \int_0^1 [\tau^m + (1-\tau)^m] |\Lambda^{(m)}(e + (1-\tau)\xi(f, e))| d\tau$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\int_0^1 [\tau^m + (1-\tau)^m]^p d\tau \right)^{1/p} \left(\int_0^1 |\Lambda^{(m)}(e + (1-\tau)\xi(f, e))|^q d\tau \right)^{1/q}$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\int_0^1 2^{p-1} [\tau^{mp} + (1-\tau)^{mp}] d\tau \right)^{1/p} \left(\int_0^1 |\Lambda^{(m)}(e + (1-\tau)\xi(f, e))|^q d\tau \right)^{1/q}$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{1/p} \left(\int_0^1 \frac{1}{n} \sum_{s=1}^n [1 - (1-\tau)^s] |\Lambda^{(m)}(e)|^q d\tau \right.$$

$$\left. + \int_0^1 \frac{1}{n} \sum_{s=1}^n [1 - \tau^s] |\Lambda^{(m)}(f)|^q d\tau - \mu \xi^\sigma(f, e) \int_0^1 [\tau^\sigma (1-\tau) + \tau(1-\tau)^\sigma] d\tau \right)^{1/q}$$

$$\begin{aligned}
&= \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{1/p} \left(\frac{|\Lambda^{(m)}(e)|^q}{n} \int_0^1 \sum_{s=1}^n [1 - (1-\tau)^s] d\tau \right. \\
&\quad \left. + \frac{|\Lambda^{(m)}(f)|^q}{n} \int_0^1 \sum_{s=1}^n [1 - \tau^s] d\tau - \mu \xi^\sigma(f, e) \int_0^1 [\tau^\sigma (1-\tau) + \tau(1-\tau)^\sigma] d\tau \right)^{1/q} \\
&= \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{1/p} \left(\frac{|\Lambda^{(m)}(e)|^q}{n} \sum_{s=1}^n \int_0^1 [1 - (1-\tau)^s] d\tau \right. \\
&\quad \left. + \frac{|\Lambda^{(m)}(f)|^q}{n} \sum_{s=1}^n \int_0^1 [1 - \tau^s] d\tau - \mu \xi^\sigma(f, e) \int_0^1 [\tau^\sigma (1-\tau) + \tau(1-\tau)^\sigma] d\tau \right) \\
&= \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{1/p} \left(\frac{2}{n} \sum_{s=1}^n \frac{s}{s+1} \mathcal{A} \left(|\Lambda^{(m)}(e)|^q, |\Lambda^{(m)}(f)|^q \right) - \frac{2\mu \xi^\sigma(f, e)}{(\sigma+1)(\sigma+2)} \right)^{1/q}.
\end{aligned} \tag{21}$$

This completes the proof of Theorem 2.

In the following, we discuss some special cases of Theorem 2.

- (1) Putting $\mu \rightarrow 0$ in Theorem 2, we obtain a result for the n -polynomial preinvex function. $\square$

Corollary 4. Under the assumptions of Theorem 2, if $|\Lambda^{(m)}|^q$ is an n -polynomial preinvex function of order $\sigma > 0$, then

$$\begin{aligned}
&\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right| \\
&\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{1/p} \left(\frac{2}{n} \sum_{s=1}^n \frac{s}{s+1} \mathcal{A} \left(|\Lambda^{(m)}(e)|^q, |\Lambda^{(m)}(f)|^q \right) \right)^{1/q}.
\end{aligned} \tag{22}$$

- (2) Putting $\sigma = 2$ in Theorem 2, we obtain a result for the strongly n -polynomial preinvex function.

Corollary 5. Under the assumptions of Theorem 2, if $|\Lambda^{(m)}|^q$ is a strongly n -polynomial preinvex function, then

$$\begin{aligned}
&\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right| \\
&\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{1/p} \left(\frac{2}{n} \sum_{s=1}^n \frac{s}{s+1} \mathcal{A} \left(|\Lambda^{(m)}(e)|^q, |\Lambda^{(m)}(f)|^q \right) - \frac{\mu |\xi(f, e)|^2}{6} \right)^{1/q}.
\end{aligned} \tag{23}$$

- (3) Putting $n = 1$ in Theorem 2, we obtain a result for the higher-order strongly preinvex function.

Corollary 6. Under the assumptions of Theorem 2, if $|\Lambda^{(m)}|^q$ is a strongly preinvex function of order $\sigma > 0$, then

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{1/p} \left(\mathcal{A} \left(|\Lambda^{(m)}(e)|^q, |\Lambda^{(m)}(f)|^q \right) - \frac{2\mu \xi^\sigma(f, e)}{(\sigma+1)(\sigma+2)} \right)^{1/q}. \quad (24)$$

Theorem 3. Let $\Lambda: [e, e + \xi(f, e)] \longrightarrow \mathbb{R}$ be an m -times differentiable function on $(e, e + \xi(f, e))$ with $\xi(f, e) > 0$, $q \geq 1$, and $\Lambda^{(m)} \in L[e, e + \xi(f, e)]$. If $|\Lambda^{(m)}|^q$ is a

strongly n -polynomial preinvex function of order $\sigma > 0$ with modulus $\mu > 0$, then

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{m+1} \right)^{1-1/q} \left(\frac{|\Lambda^{(m)}(e)|^q}{n} \sum_{s=1}^n \mathcal{H}_1 + \frac{|\Lambda^{(m)}(f)|^q}{n} \sum_{s=1}^n \mathcal{H}_1 - \mu \mathcal{H}_2 \xi^\sigma(f, e) \right)^{1/q}, \quad (25)$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the expressions as described in Theorem 1.

Proof. Applying Lemma 1, power-mean integral inequality, and the fact that $|\Lambda^{(m)}|^q$ is a strongly n -polynomial preinvex function of order $\sigma > 0$, we have

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \int_0^1 [\tau^m + (1-\tau)^m] |\Lambda^{(m)}(e + (1-\tau)\xi(f, e))| d\tau$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\int_0^1 [\tau^m + (1-\tau)^m] d\tau \right)^{1-1/q} \times \left(\int_0^1 [\tau^m + (1-\tau)^m] |\Lambda^{(m)}(e + (1-\tau)\xi(f, e))|^q d\tau \right)^{1/q}$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\int_0^1 [\tau^m + (1-\tau)^m] d\tau \right)^{1-1/q}$$

$$\times \left(\int_0^1 [\tau^m + (1-\tau)^m] \left[\frac{1}{n} \sum_{s=1}^n [1 - (1-\tau)^s] |\Lambda^{(m)}(e)|^q + \frac{1}{n} \sum_{s=1}^n \left\{ [1 - \tau^s] |\Lambda^{(m)}(f)|^q - \mu \xi^\sigma(f, e) \right\} \times [\tau^\sigma(1-\tau) + \tau(1-\tau)^\sigma] \right] d\tau \right)^{1/q}$$

$$\begin{aligned}
&= \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{m+1} \right)^{1-1/q} \left(\frac{|\Lambda^{(m)}(e)|^q}{n} \int_0^1 [\tau^m + (1-\tau)^m] \sum_{s=1}^n [1 - (1-\tau)^s] d\tau + \frac{|\Lambda^{(m)}(f)|^q}{n} \int_0^1 [\tau^m + (1-\tau)^m] \right. \\
&\quad \left. \sum_{s=1}^n [1 - \tau^s] d\tau - \mu \xi^\sigma(f, e) \int_0^1 [\tau^m + (1-\tau)^m] [\tau^\sigma(1-\tau) + \tau(1-\tau)^\sigma] d\tau \right)^{1/q} \\
&= \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{m+1} \right)^{1-1/q} \left(\frac{|\Lambda^{(m)}(e)|^q}{n} \sum_{s=1}^n \int_0^1 [\tau^m + (1-\tau)^m] [1 - (1-\tau)^s] d\tau + \frac{|\Lambda^{(m)}(f)|^q}{n} \right. \\
&\quad \left. \sum_{s=1}^n \int_0^1 [\tau^m + (1-\tau)^m] [1 - \tau^s] d\tau - \mu \xi^\sigma(f, e) \int_0^1 [\tau^m + (1-\tau)^m] [\tau^\sigma(1-\tau) + \tau(1-\tau)^\sigma] d\tau \right)^{1/q} \\
&= \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{m+1} \right)^{1-1/q} \left(\frac{|\Lambda^{(m)}(e)|^q}{n} \sum_{s=1}^n \mathcal{H}_1 + \frac{|\Lambda^{(m)}(f)|^q}{n} \sum_{s=1}^n \mathcal{H}_1 - \mu \mathcal{H}_2 \xi^\sigma(f, e) \right)^{1/q}.
\end{aligned} \tag{26}$$

Theorem 3 is proved.

Finally, we discuss some special cases of Theorem 3.

- (1) Letting $\mu \rightarrow 0$ in Theorem 3, we get a result for the n -polynomial preinvex function. $\square$

Corollary 7. Under the assumptions of Theorem 3, if $|\Lambda^{(m)}|^q$ is an n -polynomial preinvex function of order $\sigma > 0$, then

$$\begin{aligned}
&\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right| \\
&\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{m+1} \right)^{1-1/q} \left(\frac{|\Lambda^{(m)}(e)|^q}{n} \sum_{s=1}^n \mathcal{H}_1 + \frac{|\Lambda^{(m)}(f)|^q}{n} \sum_{s=1}^n \mathcal{H}_1 \right)^{1/q}.
\end{aligned} \tag{27}$$

- (2) Choosing $\sigma = 2$ in Theorem 3, we get a result for the strongly n -polynomial preinvex function.

Corollary 8. Under the assumptions of Theorem 3, if $|\Lambda^{(m)}|^q$ is a strongly n -polynomial preinvex function, then

$$\begin{aligned}
&\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right| \\
&\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{m+1} \right)^{1-1/q} \left(\frac{|\Lambda^{(m)}(e)|^q}{n} \sum_{s=1}^n \mathcal{H}_1 + \frac{|\Lambda^{(m)}(f)|^q}{n} \sum_{s=1}^n \mathcal{H}_1 - \mu \mathcal{H}_3 |\xi(f, e)|^2 \right)^{1/q}.
\end{aligned} \tag{28}$$

(3) Choosing $n = 1$ in Theorem 3, we get a result for the higher-order strongly preinvex function.

Corollary 9. Under the assumptions of Theorem 3, if $|\Lambda^{(m)}|^q$ is a strongly preinvex function of order $\sigma > 0$, then

$$\left| \frac{1}{\xi(f, e)} \int_e^{e+\xi(f, e)} \Lambda(x) dx - \sum_{k=1}^m \frac{\xi^{k-1}(f, e) [\Lambda^{(k-1)}(e) + (-1)^{k-1} \Lambda^{(k-1)}(e + \xi(f, e))]}{2k!} \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{m+1} \right)^{1-1/q} \left(\frac{|\Lambda^{(m)}(e)|^q + |\Lambda^{(m)}(f)|^q}{m+1} - \mu \mathcal{H}_2 \xi^\sigma(f, e) \right)^{1/q}. \quad (29)$$

4. Application to Special Means

We consider the following special means.

(1) The arithmetic mean:

$$\mathcal{A}(e, f) = \frac{e+f}{2}. \quad (30)$$

(2) The mean of the function Ψ on $[e, f]$

$$\mathcal{A}_\Psi(e, f) = \frac{1}{f-e} \int_e^f \Psi(x) dx. \quad (31)$$

Proposition 1. Let $\Psi: [e, e + \xi(f, e)] \rightarrow \mathbb{R}$ be m -times differentiable function on $(e, e + \xi(f, e))$ with $\xi(f, e) > 0$ and $\Psi^{(m)} \in L[e, e + \xi(f, e)]$. If $|\Psi^{(m)}|$ is a strongly n -polynomial preinvex function of order $\sigma > 0$ with modulus $\mu > 0$, then the following inequality holds:

$$\left| \mathcal{A}_\Psi(e, e + \xi(f, e)) - \sum_{k=1}^m \frac{\xi^{k-1}(f, e)}{k!} \mathcal{A}(\Psi^{(k-1)}(e), (-1)^{k-1} \Psi^{(k-1)}(e + \xi(f, e))) \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{n} \sum_{s=1}^n \mathcal{H}_1(\Psi^{(m)}(e), \Psi^{(m)}(f)) - \mu \mathcal{H}_2 \xi^\sigma(f, e) \right), \quad (32)$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are as in Theorem 1.

Proof. Taking $\Lambda(x) = \Psi(x)$, $x \in [e, e + \xi(f, e)]$ and using Theorem 1, we get the desired result (32). $\square$

Proposition 2. Let $\Psi: [e, e + \xi(f, e)] \rightarrow \mathbb{R}$ be m -times differentiable function on $(e, e + \xi(f, e))$ with $\xi(f, e) > 0$, $1/p + 1/q = 1$, $p > 1, q > 1$ and $\Psi^{(m)} \in L[e, e + \xi(f, e)]$. If $|\Psi^{(m)}|^q$ is a strongly n -polynomial preinvex function of order $\sigma > 0$ with modulus $\mu > 0$, then the following inequality holds:

$$\left| \mathcal{A}_\Psi(e, e + \xi(f, e)) - \sum_{k=1}^m \frac{\xi^{k-1}(f, e)}{k!} \mathcal{A}(\Psi^{(k-1)}(e), (-1)^{k-1} \Psi^{(k-1)}(e + \xi(f, e))) \right|$$

$$\leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2^p}{mp+1} \right)^{1/p} \left(\frac{2}{n} \sum_{s=1}^n \frac{s}{s+1} \mathcal{A}(|\Psi^{(m)}(e)|^q, |\Psi^{(m)}(f)|^q) - \frac{2\mu \xi^\sigma(f, e)}{(\sigma+1)(\sigma+2)} \right)^{1/q}. \quad (33)$$

Proof. Setting $\Lambda(x) = \Psi(x)$, $x \in [e, e + \xi(f, e)]$ and applying Theorem 2, we get the desired result (33). $\square$

Proposition 3. Let $\Psi: [e, e + \xi(f, e)] \rightarrow \mathbb{R}$ be m -times differentiable function on $(e, e + \xi(f, e))$ with $\xi(f, e) > 0$, $q \geq 1$

and $\Psi^{(m)} \in L[e, e + \xi(f, e)]$. If $|\Psi^{(m)}|^q$ is a strongly n -polynomial preinvex function of order $\sigma > 0$ with modulus $\mu > 0$, then the following inequality holds:

$$\left| \mathcal{A}_\Psi(e, e + \xi(f, e)) - \sum_{k=1}^m \frac{\xi^{k-1}(f, e)}{k!} \mathcal{A}(\Psi^{(k-1)}(e), (-1)^{k-1} \Psi^{(k-1)}(e + \xi(f, e))) \right| \leq \frac{\xi^m(f, e)}{2m!} \left(\frac{2}{m+1} \right)^{1-1/q} \left(\frac{2}{n} \sum_{s=1}^n \mathcal{H}_1 \mathcal{A}(|\Psi^{(m)}(e)|^q, |\Psi^{(m)}(f)|^q) - \mu \mathcal{H}_2 \xi^\sigma(f, e) \right)^{1/q}, \quad (34)$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are as in Theorem 1.

Proof. Taking $\Lambda(x) = \Psi(x)$, $x \in [e, e + \xi(f, e)]$ and utilizing Theorem 3, we get the desired result (34). $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly to this paper. All authors read and approved the final manuscript.

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