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Research Article

Hyperstability of the k-Cubic Functional Equation in Non-Archimedean Banach Spaces

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Using the fixed point approach, we investigate a general hyperstability results for the following k-cubic functional equations $f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2k(k^2-1)f(x)$, where k is a fixed positive integer ≥ 2, in ultrametric Banach spaces.

1. Introduction

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [1] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Ulam's problem: let G_1 be a group and let G_2 be a metric group with a metric d. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if a mapping $h\colon G_1 \longrightarrow G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta, \tag{1}$$

for all $x, y \in G_1$, then there exists a homomorphism $H\colon G_1 \longrightarrow G_2$ with

$$d(h(x), H(x)) < \varepsilon, \tag{2}$$

for all $x \in G_1$.

The first partial answer, in the case of Cauchy equation in Banach spaces, to Ulam question was given by Hyers [2]. Later, the result of Hyers was first generalized by Aoki [3], and only much later by Rassias [4] and Găvruţa [5]. Since then, the stability problems of several functional equations have been extensively investigated [6–10].

We say a functional equation is *hyperstable* if any function f satisfying the equation approximately (in some sense) must be actually a solution to it. It seems that the first hyperstability result was published in [11] and concerned the ring homomorphisms. However, the term *hyperstability* has been used for the first time in [12]. Quite often, hyperstability is confused with superstability, which also admits bounded functions. Numerous papers on this subject have been published, and we refer, for example, to [3, 12–29].

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers and N_{m_0}, the set of integers greater than or equal m_0, $\mathbb{R}_+ := [0, \infty)$, and we use the notation X_0 for the set $X \setminus \{0\}$.

Let us recall (see, for instance, [30]) some basic definitions and facts concerning non-Archimedean normed spaces.

Definition 1. By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|\colon \mathbb{K} \longrightarrow [0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$

(2) $|rs| = |r||s|$

(3) $|r + s| \leq \max\{|r|, |s|\}$

The pair $(\mathbb{K}, |.|)$ is called a valued field.

In any non-Archimedean field, we have $|1| = |-1| = 1$ and $|n| \leq 1$, for $n \in \mathbb{N}_0$. In any field $\mathbb{K}$, the function $|\cdot|\colon \mathbb{K} \longrightarrow \mathbb{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases} \tag{3}$$

is a valuation which is called *trivial*, but the most important examples of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings, and superstrings.

Definition 2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\|_*: X \longrightarrow \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

(1) $\|x\|_* = 0$ if and only if $x = 0$,

(2) $\|rx\|_* = |r|\|x\|_*$ $(r \in \mathbb{K}, x \in X)$,

(3) The strong triangle inequality (ultrametric), namely,

$$\|x + y\|_* \leq \max\{\|x\|_*, \|y\|_*\}, \quad x, y \in X. \tag{4}$$

Then, $(X, \|\cdot\|_*)$ is called a non-Archimedean normed space or an ultrametric normed space.

Definition 3. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

(1) A sequence$\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is *a Cauchy sequence* iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.

(2) The sequence $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that, for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x\|_* \leq \varepsilon$, for all $n \geq N$. Then, the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\longrightarrow\infty} x_n = x$.

(3) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space* or an *ultrametric Banach space.*

Let X and Y be normed spaces. A function $f: X \longrightarrow Y$ is called a k–cubic function provided it satisfies the functional equation:

$$\begin{aligned} f(kx + y) + f(kx - y) &= kf(x + y) + kf(x - y) \\ &\quad + 2k\left(k^2 - 1\right)f(x), \\ &\quad \text{for all } x, y \in X, \end{aligned} \tag{5}$$

and we can say that $f: X \longrightarrow Y$ is k–cubic on X_0 if it satisfies (5) for all $x, y \in X_0$.

In 2013, Bahyrycz et al. [31] used the fixed point theorem from Theorem 1 in [24] to prove the stability results for the generalization of p-Wright affine equation in ultrametric spaces. Recently, corresponding results for more general functional equations (in classical spaces) have been proved in [32–35].

In this paper, by using the fixed point method derived from [20, 21, 36], we present some hyperstability results for equation (5) in ultrametric Banach spaces. Before proceeding to the main results, we state Theorem 1 which is useful for our purpose. To present it, we introduce the following three hypotheses:

(H1): X is a nonempty set, Y is an ultrametric Banach space over a non-Archimedean field, $f_1, \ldots, f_k$: $X \longrightarrow X$, and $L_1, \ldots, L_k$: $X \longrightarrow \mathbb{R}_+$ are given.

(H2): $\mathcal{T}: Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\begin{aligned} &\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\|_* \\ &\quad \leq \max_{1\leq i\leq k}\left\{L_i(x)\left\|\xi\left(f_i(x)\right) - \mu\left(f_i(x)\right)\right\|_*\right\}, \\ &\quad \xi, \mu \in Y^X, x \in X. \end{aligned} \tag{6}$$

(H3): $\Lambda: \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \max_{1\leq i\leq k}\left\{L_i(x)\delta\left(f_i(x)\right)\right\}, \quad \delta \in \mathbb{R}_+^X, x \in X. \tag{7}$$

Thanks to a result due to Brzdęk and Ciepliński ([25], Remark 2), we state a slightly modified version of the fixed point theorem ([24], Theorem 1) in ultrametric spaces. We use it to assert the existence of a unique fixed point of operator $\mathcal{T}: Y^X \longrightarrow Y^X$.

Theorem 1. *Let hypotheses (H1)–(H3) be valid, and functions $\varepsilon: X \longrightarrow \mathbb{R}_+$ and $\varphi: X \longrightarrow Y$ fulfill the following two conditions:*

$$\begin{aligned} \|\mathcal{T}\varphi(x) - \varphi(x)\|_* &\leq \varepsilon(x), \quad x \in X, \\ \lim_{n\longrightarrow\infty} \Lambda^n\varepsilon(x) &= 0, \quad x \in X. \end{aligned} \tag{8}$$

Then, there exists a unique fixed point $\psi \in Y^X$ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\|_* \leq \sup_{n\in\mathbb{N}_0} \Lambda^n\varepsilon(x), \quad x \in X. \tag{9}$$

Moreover,

$$\psi(x) := \lim_{n\longrightarrow\infty} \mathcal{T}^n\varphi(x), \quad x \in X. \tag{10}$$

2. Main Results

In this section, we use Theorem 1 as a basic tool to prove the hyperstability results of the k-cubic functional equation (5) in ultrametric Banach spaces.

Theorem 2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and ultrametric Banach space, respectively, $c \geq 0$, $p, q \in \mathbb{R}$, and $p + q < 0$, and let $f: X \longrightarrow Y$ satisfies*

$$\begin{aligned} &\left\|f(kx + y) + f(kx - y) - kf(x + y) - kf(x - y)\right. \\ &\quad \left. - 2\left(k^3 - k\right)f(x)\right\|_* \leq c\|x\|^p\|y\|^q, \end{aligned} \tag{11}$$

for all $x, y \in X_0$ such that $kx + y \neq 0$, $kx - y \neq 0$, $x + y \neq 0$, and $x - y \neq 0$. Then, f is k-cubic on X_0.

Proof. Take $m \in \mathbb{N}$ such that

$$\alpha_m := \left(\frac{m+1}{k}\right)^{p+q} < 1, \quad m \geq k. \tag{12}$$

Since $p+q<0$, one of p,q must be negative. Assume that $q<0$ and replacing y by mx and x by $((m+1)/k)x$ in (11), we obtain

$$\begin{aligned}&\left\|2\left(k^3-k\right)f\left(\frac{m+1}{k}x\right)+kf\left(\frac{km+m+1}{k}x\right)+kf\left(\frac{m-km+1}{k}x\right)-f\left((2m+1)x\right)-f(x)\right\|_* \\ &\quad\leq cm^q\left(\frac{m+1}{k}\right)^p\|x\|^{p+q}.\end{aligned} \tag{13}$$

Define operators $\mathcal{T}_m: Y^{X_0} \longrightarrow Y^{X_0}$ and $\Lambda_m: \mathbb{R}_+^{X_0} \longrightarrow \mathbb{R}_+^{X_0}$ by

$$\begin{aligned}&\mathcal{T}_m\xi(x) := 2\left(k^3-k\right)\xi\left(\frac{m+1}{k}x\right)+k\xi\left(\frac{km+m+1}{k}x\right)+2\xi\left(\frac{m-km+1}{k}x\right)-\xi\left((2m+1)x\right), \quad \xi \in Y^{X_0},\ x \in X_0,\\ &\Lambda_m\delta(x) := \max\left\{\delta\left(\frac{m+1}{k}x\right),\delta\left(\frac{km+m+1}{k}x\right),\delta\left(\left(\frac{m-km+1}{k}\right)x\right),\delta(2m+1)\right\}, \quad \delta \in \mathbb{R}_+^{X_0},\ x \in X_0,\end{aligned} \tag{14}$$

and write

$$\varepsilon_m(x) := cm^q\left(\frac{m+1}{k}\right)^p\|x\|^{p+q}, \quad x \in X_0. \tag{15}$$

It is easily seen that Λ_m has the form described in (H3) with $k=4$, $f_1(x)=((m+1)/k)x$, $f_2(x)=((km+m+1)/k)x$, $f_3(x)=((m-km+1)/k)x$, $f_4(x)=(2m+1)x$, and $L_1(x)=L_2(x)=L_3(x)=L_4(x)=1$. Furthermore, (11) can be written in the following way:

$$\left\|\mathcal{T}_m f(x)-f(x)\right\|_* \leq \varepsilon_m(x), \quad x \in X_0. \tag{16}$$

Moreover, for every $\xi,\mu \in Y^{X_0}$, $x \in X_0$,

$$\begin{aligned}&\left\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x)\right\|_* = \left\|2\left(k^3-k\right)\xi\left(\left(\frac{m+1}{k}\right)x\right)+k\xi\left(\left(\frac{km+m+1}{k}\right)x\right)+k\xi\left(\left(\frac{m-km+1}{k}\right)x\right)-\right.\\ &\left.\xi\left((2m+1)x\right)-2\left(k^3-k\right)\mu\left(\left(\frac{m+1}{k}\right)x\right)-k\mu\left(\left(\frac{km+m+1}{k}\right)x\right)-k\mu\left(\left(\frac{m-km+1}{k}\right)x\right)+\xi\left((2m+1)x\right)\right\|_* \\ &\leq \max\left\{\left\|\xi\left(\left(\frac{m+1}{k}\right)x\right)-\mu\left(\left(\frac{m+1}{k}\right)x\right)\right\|_*,\left\|\xi\left(\frac{km+m+1}{k}x\right)-\mu\left(\frac{km+m+1}{k}x\right)\right\|_*,\right. \\ &\left.\left\|\xi\left(\frac{m-km+1}{K}x\right)-\mu\left(\frac{m-km+1}{k}x\right)\right\|_*,\|\xi(2m+1)x)-\mu\left((2m+1)x\right)\|_*\right\}.\end{aligned} \tag{17}$$

So, (H2) is valid.

By using mathematical induction, we will show that, for each $x \in X_0$, we have

$$\Lambda_m^n\varepsilon_m(x) = cm^q\left(\frac{m+1}{k}\right)^p\|x\|^{p+q}\alpha_m^n, \tag{18}$$

where $\alpha_m = ((m-1)/k)^{p+q}$. We obtain that (18) holds for $n=0$. Next, we will assume that (18) holds for $n=r$, where $r \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\Lambda_m^{r+1}\varepsilon_m(x) &= \Lambda_m\left(\Lambda_m^r\varepsilon_m(x)\right)\\
&= \max\left\{\Lambda_m^r\varepsilon_m\left(\frac{m+1}{k}x\right), \Lambda_m^r\varepsilon_m\left(\frac{km+m+1}{k}x\right), \Lambda_m^r\varepsilon_m\left(\frac{km-m-1}{k}x\right), \Lambda_m^r\varepsilon_m((2m+1)x)\right\}\\
&= \max\left\{cm^q\left(\frac{m+1}{k}\right)^p\|x\|^{p+q}\alpha_m^r\left(\frac{m+1}{k}\right)^{p+q}, cm^q\left(\frac{m+1}{k}\right)^p\|x\|^{p+q}\alpha_m^r\left(\frac{km+m+1}{k}\right)^{p+q},\right.\\
&\quad \left. cm^q\left(\frac{m+1}{K}\right)^p\|x\|^{p+q}\alpha_m^r\left(\frac{km-m-1}{k}\right)^{p+q}, cm^q\|x\|^{p+q}\alpha_m^k(2m+1)^{p+q}\right\}\\
&= cm^q\|x\|^{p+q}\alpha_m^r\max\left\{\left(\frac{m+1}{k}\right)^{p+q}, \left(\frac{km+m+1}{k}\right)^{p+q}, \left(\frac{km-m-1}{k}\right)^{p+q}, (2m+1)^{p+q}\right\}\\
&= cm^q\left(\frac{m+1}{k}\right)^p\|x\|^{p+q}\alpha_m^{r+1}, \quad x \in X_0.
\end{aligned} \tag{19}$$

This shows that (18) holds for $n = r+1$. Now, we can conclude that inequality (18) holds for all $n \in \mathbb{N}_0$. From (18), we obtain

$$\lim_{n\longrightarrow\infty}\Lambda^n\varepsilon_m(x) = 0, \tag{20}$$

for all $x \in X_0$.

Hence, according to Theorem 1, there exists a unique solution $C_m\colon X_0 \longrightarrow Y$ of the equation:

$$C_m(x) = 2\left(k^3-k\right)C_m\left(\frac{m+1}{k}x\right) + kC_m\left(\frac{km+m+1}{k}x\right) + kC_m\left(\frac{1-km+m}{k}x\right) - C_m((2m+1)x), \quad x \in X_0, \tag{21}$$

such that

$$\left\|f(x)-C_m(x)\right\|_* \le \sup_{n\in\mathbb{N}_0}\left\{cm^q\left(\frac{m+1}{k}\right)^p\|x\|^{p+q}\alpha_m^n\right\}, \quad x \in X_0. \tag{22}$$

Moreover,

$$C_m(x) \coloneqq \lim_{n\longrightarrow\infty}\mathcal{T}_m^n f(x), \tag{23}$$

for all $x \in X_0$.

Now, we show that

$$\begin{aligned}
&\left\|\mathcal{T}_m^n f(kx+y) + \mathcal{T}_m^n f(kx-y) - k\mathcal{T}_m^n f(x+y) - k\mathcal{T}_m^n f(x-y) - 2\left(k^3-k\right)\mathcal{T}_m^n f(x)\right\|_*\\
&\quad \le c\alpha_m^n\|x\|^p\|y\|^q,
\end{aligned} \tag{24}$$

for every $x, y \in X_0$ such that $x+y \neq 0$, $x-y \neq 0$. Since the case $n = 0$ is just (11), take $r \in \mathbb{N}$, and suppose the last inequality holds for $n = r$ and every $x, y \in X_0$ such that $x+y \neq 0$, $x-y \neq 0$. Then,

$$
\begin{aligned}
&\left\|\mathscr{T}_m^{r+1} f(kx+y)+\mathscr{T}_m^{r+1} f(kx-y)-k\mathscr{T}_m^{r+1} f(x+y)-k\mathscr{T}_m^{r+1} f(x-y)-2\left(k^3-k\right)\mathscr{T}_m^{r+1} f(x)\right\|_* \\
&=\Bigg\|2\left(k^3-k\right)\mathscr{T}_m^k f\left(\frac{m+1}{k}(kx+y)\right)+k\mathscr{T}_m^r f\left(\frac{km+m+1}{k}(kx+y)\right)+k\mathscr{T}_m^r f\left(\frac{1+m-km}{k}(kx+y)\right) \\
&\quad-\mathscr{T}_m^r f((2m+1)(kx+y))+2\left(k^3-k\right)\Big]\mathscr{T}_m^r f\left(\frac{m+1}{k}(kx-y)\right)+k\mathscr{T}_m^r f\left(\frac{km+m+1}{k}(kx-y)\right) \\
&\quad+k\mathscr{T}_m^r f\left(\frac{1+m-km}{k}(kx-y)\right)-\mathscr{T}_m^r f((2m+1)(kx-y))-2k\left(k^3-k\right)\mathscr{T}_m^r f\left(\frac{m+1}{k}(x+y)\right) \\
&\quad-k^2\mathscr{T}_m^r f\left(\frac{km+m+1}{k}(x+y)\right)-k^2\mathscr{T}_m^r f\left(\frac{1+m-km}{k}(x+y)\right)+k\mathscr{T}_m^r f((2m+1)(x+y)) \\
&\quad-2k\left(k^3-k\right)\mathscr{T}_m^r f\left(\frac{m+1}{k}(x-y)\right)-k^2\mathscr{T}_m^r f\left(\frac{km+m+1}{k}(x-y)\right)-k^2\mathscr{T}_m^r f\left(\frac{1+m-km}{k}(x-y)\right) \\
&\quad+\mathscr{T}_m^r f((2m+1)(x-y))-\left(2\left(k^3-k\right)\right)^2\mathscr{T}_m^r f\left(\left(\frac{m+1}{k}\right)(x)\right)-2k\left(k^3-k\right)\mathscr{T}_m^r f\left(\left(\frac{km+m+1}{k}\right)(x)\right) \\
&\quad-2k\left(k^3-k\right)\mathscr{T}_m^r f\left(\left(\frac{1+m-km}{k}\right)(x)\right)+2\left(k^3-k\right)\mathscr{T}_m^k f((2m+1)(x)\Big|_* \\
&\le\max\Bigg\{\Bigg\|\mathscr{T}_m^n f\left(\frac{m+1}{k}(kx+y)\right)+\mathscr{T}_m^n f\left(\frac{m+1}{k}(kx-y)\right)-k\mathscr{T}_m^n f\left(\frac{m+1}{k}(x+y)\right) \\
&\quad-k\mathscr{T}_m^n f\left(\frac{m+1}{k}(x-y)\right)-2\left(k^3-k\right)\mathscr{T}_m^n f\left(\frac{m+1}{k}x\right)\Bigg\|_*, \\
&\quad\Bigg\|\mathscr{T}_m^n f\left(\frac{km+m+1}{k}(kx+y)\right)+\mathscr{T}_m^n f\left(\frac{km+m+1}{k}(kx-y)\right)-k\mathscr{T}_m^n f\left(\frac{km+m+1}{k}(x+y)\right) \\
&\quad-k\mathscr{T}_m^n f\left(\frac{km+m+1}{k}(x-y)\right)-2\left(k^3-k\right)\mathscr{T}_m^n f\left(\frac{km+m+1}{k}x\right)\Bigg\|_*, \\
&\quad\Bigg\|\mathscr{T}_m^n f\left(\frac{1+m-km}{k}(kx+y)\right)+\mathscr{T}_m^n f\left(\frac{1+m-km}{k}(kx-y)\right)-k\mathscr{T}_m^n f\left(\frac{1+m-km}{k}(x+y)\right) \\
&\quad-k\mathscr{T}_m^n f\left(\frac{1+m-km}{k}(x-y)\right)-2\left(k^3-k\right)\mathscr{T}_m^n f\left(\frac{1+m-km}{k}x\right)\Bigg\|_*, \\
&\quad\left\|\mathscr{T}_m^n f((2m+1)(kx+y))+\mathscr{T}_m^n f((2m+1)(kx-y))-k\mathscr{T}_m^n f((2m+1)(x+y))\right. \\
&\quad\left.-\mathscr{T}_m^n f((2m+1)(x-y))-2\left(k^3-k\right)\mathscr{T}_m^n f((2m+1)x)\right\|_*\Bigg\} \\
&\le\max\Bigg\{c\alpha_m^k\|x\|^p\|y\|^q\left(\frac{m+1}{k}\right)^{p+q},c\alpha_m^k\|x\|^p\|y\|^q\left(\frac{km+m+1}{k}\right)^{p+q}, \\
&\quad c\alpha_m^k\|x\|^p\|y\|^q\left(\frac{1+m-km}{k}\right)^{p+q},c\alpha_m^k\|x\|^p\|y\|^q(2m+1)^{p+q}\Bigg\} \\
&=c\alpha_m^k\|x\|^p\|y\|^q\max\left\{\left(\frac{m+1}{k}\right)^{p+q},\left(\frac{km+m+1}{k}\right)^{p+q},\left(\frac{km-m-1}{k}\right)^{p+q},(km+m+1)^{p+q}\right\} \\
&\le c\alpha_m^{k+1}\|x\|^p\|y\|^q,
\end{aligned}
\tag{25}
$$

for all $x, y \in X_0$ such that $x + y \neq 0$ and $x - y \neq 0$. Thus, by induction, we have shown that suppose the last inequality holds for every $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$, we obtain that

$$C_m(kx + y) + C_m(kx - y) = kC_m(x + y) + kC_m(x - y) + 2\left(k^3 - k\right)C_m(x), \tag{26}$$

for all $x, y \in X_0$ such that $x + y \neq 0, x - y \neq 0$. In this way, we obtain a sequence $\{C_m\}_{m \geq m_0}$ of k–cubic functions on X_0 such that

$$\left\|f(x) - C_m(x)\right\|_* \leq \sup_{n \in \mathbb{N}_0}\left\{cm^q\left(\frac{m+1}{k}\right)^p \|x\|^{p+q}\alpha_m^n\right\}, \quad x \in X_0. \tag{27}$$

This implies that

$$\left\|f(x) - C_m(x)\right\|_* \leq cm^q\left(\frac{m+1}{k}\right)^p \|x\|^{p+q}, \quad x \in X_0. \tag{28}$$

It follows, with $m \longrightarrow \infty$, that f is k–cubic on X_0. In a similar way, we can prove the following theorem. □

Theorem 3. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and ultrametric Banach space, respectively, $c \geq 0$, $p, q \in \mathbb{R}$, and $p + q > 0$, and let $f: X \longrightarrow Y$ satisfies*

$$\left\|f(kx + y) + f(kx - y) - kf(x + y) - kf(x - y) - 2\left(k^3 - k\right)f(x)\right\|_* \leq c\|x\|^p\|y\|^q, \tag{29}$$

for all $x, y \in X_0$ such that $x + y \neq 0$ and $x - y \neq 0$. Then, f is k–cubic on X_0.

Proof. Take $m \in \mathbb{N}$ such that

$$\alpha_m = \left(\frac{m-4}{m}\right)^{p+q} < 1. \tag{30}$$

Since $p + q > 0$, one of p, q must be positive, and let $q > 0$ and replace y by $(-2/m)y$ and x by $((m-2)/km)y$ in (29). Thus,

$$\left\|2\left(k^3 - k\right)f\left(\frac{m-2}{km}y\right) + kf\left(\frac{m+2}{km}y\right) + kf\left(\frac{m-6}{km}y\right) - f\left(\frac{m-4}{m}y\right) - f(y)\right\|_* \leq c\left(\frac{2}{m}\right)^q\left(\frac{m-2}{km}\right)^p\|y\|^{p+q}, \quad x \in X_0. \tag{31}$$

Writing

$$\mathscr{T}_m\xi(x) \coloneqq 2\left(k^3 - k\right)\xi\left(\frac{m-2}{km}x\right) + k\xi\left(\frac{m+2}{km}x\right) + k\xi\left(\frac{m-6}{km}x\right) - \xi\left(\frac{m-4}{m}x\right), \quad \xi \in Y^{X_0}, x \in X_0, \tag{32}$$

$$\varepsilon_m(x) \coloneqq c\left(\frac{2}{m}\right)^q\left(\frac{m-1}{km}\right)^p\|x\|^{p+q}, \quad x \in X_0, \tag{33}$$

(31) takes form

$$\left\|\mathscr{T}_m f(x) - f(x)\right\|_* \leq \varepsilon_m(x), \quad x \in X_0. \tag{34}$$

The rest of the proof is similar to the proof of the last theorem. It easy to show the hyperstability of cubic equation on the set containing 0. The above theorems imply, in particular, the following corollary, which shows their simple application. □

Corollary 1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be normed space and ultrametric Banach space, respectively, $G: X^2 \longrightarrow Y$ and $G(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X$, and*

$$\|G(x, y)\|_* \leq c\|x\|^p\|y\|^q, \quad x, y \in X, \tag{35}$$

where $c \geq 0$, $p, q \in \mathbb{R}$. Assume that the numbers p and q satisfy one of the following conditions:

(1) $p + q < 0$, and (11) holds for all $x, y \in X_0$

(2) $p+q>0$, and (36) holds for all $x, y \in X_0$

Then, the functional equation

$$\begin{aligned} f(kx+y)+f(kx+y) &= kf(x+y)+kf(x-y) \\ &+2\left(k^3-k\right)f(x)+G(x,y), \quad x,y\in X, \end{aligned} \tag{36}$$

has no solution in the class of functions $g: X \longrightarrow Y$.

In the following theorem, we present a general hyperstability for the k−cubic equation where the control function is $\varphi(x)+\varphi(y)$, which corresponds to the approach introduced in [36].

Theorem 4. *Let $(X, \|\cdot\|)$ be a normed space, $(Y, \|\cdot\|_*)$ be an ultrametric Banach space over a field $\mathbb{K}$, and $\varphi: X \longrightarrow \mathbb{R}_+$ be a function such that*

$$\begin{aligned} U := \{n\in\mathbb{N}:\ \alpha_n :=\ & \max\{\lambda(n), \lambda(kn+n-1), \\ & \lambda(-kn+n+1), \lambda(2kn-1)\}<1\}, \end{aligned} \tag{37}$$

is a nonempty set, where $\lambda(a) := \inf\{t\in\mathbb{R}_+: \varphi(ax)\le t\varphi(x) \text{ for } x\in X\}$ for all $a\in\mathbb{K}_0$. Suppose that

$$\begin{aligned} \lim_{a\longrightarrow\infty}\lambda(a)&=0, \\ \lim_{a\longrightarrow\infty}\lambda(-a)&=0, \end{aligned} \tag{38}$$

and $f: X \longrightarrow Y$ satisfies the inequality

$$\left\|f(kx+y)+f(kx+y)-kf(x+y)-kf(x-y)-2\left(k^3-k\right)f(x)\right\|_* \le \varphi(x)+\varphi(y), \tag{39}$$

for all $x, y\in X_0$, such that $x+y\neq 0$ and $x-y\neq 0$. Then, f is cubic on X_0.

Proof. Replacing x by mx and y by $(-km+1)x$ for $m\in\mathbb{N}$ in (39), we get

$$\begin{aligned} &\left\|f(x)+f((2km-1)x)-kf((-km+m+1)x)-kf((km+m-1)x)-2\left(k^3-k\right)f(mx)\right\|_* \\ &\le \varphi((-km+1)x)+\varphi(mx), \end{aligned} \tag{40}$$

for all $x\in X_0$. For each $m\in U$, we define the operator $\mathcal{T}_m: Y^{X_0}\longrightarrow Y^{X_0}$ by

$$\begin{aligned} \mathcal{T}_m\xi(x) := {}& 2\left(k^3-k\right)\xi(mx)+k\xi((km+m-1)x) \\ &+k\xi((-km+m+1)x)-\xi((2km-1)x), \\ &\xi\in Y^{X_0}, x\in X_0. \end{aligned} \tag{41}$$

Furthermore, we put

$$\begin{aligned} \varepsilon_m(x) := \varphi((-km+1)x)+\varphi(mx) \le {}& (\lambda(-km+1) \\ &+\lambda(mx))\varphi(x), \quad x\in X_0. \end{aligned} \tag{42}$$

Then, inequality (40) takes the form

$$\left\|\mathcal{T}_m f(x)-f(x)\right\|_* \le \varepsilon_m(x), \quad x\in X_0. \tag{43}$$

For each $m\in U$, the operator $\Lambda_m: \mathbb{R}_+^{X_0}\longrightarrow\mathbb{R}_+^{X_0}$ which is defined by $\Lambda_m\delta(x) := \max\{\delta(mx), \delta((km+m-1)x), \delta((-km+m+1)x), \delta((2km-1)x)\}$, $\delta\in\mathbb{R}_+^{X_0}$, $x\in X_0$, has the form described in (H3) with $k=4$ and

$$\begin{aligned} f_1(x)&=mx, \\ f_2(x)&=(km+m-1)x, \\ f_3(x)&=(-km+m+1)x, \\ f_4(x)&=(2km-1)x, \\ L_1(x)&=L_2(x)=L_2(x)=L_4(x)=1, \end{aligned} \tag{44}$$

for all $x\in X_0$. Moreover, for every $\xi,\mu\in Y^{X_0}$, $x\in X_0$,

$$\begin{aligned} \left\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x)\right\|_* = {}& \Big\|2\left(k^3-k\right)\xi((m)x)+k\xi((km+m-1)x)+k\xi((-km+m+1)x) \\ &-\xi((2km-1)x)-2\left(k^3-k\right)\mu((m)x)-k\mu((km+m-1)x)-k\mu((-km+m+1)x) \\ &+\xi((2km-1)x)\Big\|_* \\ \le {}& \max\{\|\xi((m)x)-\mu((m)x)\|_*, \|\xi((km+m-1)x)-\mu((km+m-1)x)\|_*, \\ &\|\xi((-km+m+1)x)-\mu((-km+m+1)x)\|_*, \|\xi((2km-1)x)-\mu((2km-1)x)\|_*\}. \end{aligned} \tag{45}$$

So, (H2) is valid. By using mathematical induction, we will show that, for each $x \in X_0$, we have

$$\Lambda_m^n \varepsilon_m(x) \leq (\lambda(-km+1) + \lambda(m))\alpha_m^n \varphi(x). \tag{46}$$

From (42), we obtain that inequality (46) holds for $n = 0$. Next, we will assume that (46) holds for $n = r$, where $r \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\Lambda_m^{r+1}\varepsilon_m(x) &= \Lambda_m\left(\Lambda_m^r \varepsilon_m(x)\right) = \max\{\Lambda_m^r\varepsilon_m(mx), \Lambda_m^r\varepsilon_m((km+m-1)x), \Lambda_m^r\varepsilon_m((-km+m+1)x), \Lambda_m^r\varepsilon_m((2km-1)x)\} \\
&\leq (\lambda(m) + \lambda(-km+1))\alpha_m^r \max\{\varphi(mx), \varphi((km+m-1)x), \varphi((-km+m+1)x), \varphi((2km-1)x)\} \\
&\leq (\lambda(m) + \lambda(-km+1))\alpha_m^{r+1}\varphi(x) \\
&x \in X_0.
\end{aligned} \tag{47}$$

This shows that (46) holds for $n = r + 1$. Now, we can conclude that inequality (46) holds for all $n \in \mathbb{N}$. From (46), we obtain $\lim_{n\longrightarrow\infty}\Lambda^n\varepsilon_m(x) = 0$, for all $x \in X_0$ and all $m \in U$. Hence, according to Theorem 1, there exists, for each $m \in U$, a unique solution C_m: $X_0 \longrightarrow Y$ of the equation:

$$\begin{aligned}
C_m(x) &= 2\left(k^3 - k\right)C_m(mx) + kC_m((km+m-1)x) \\
&\quad + kC_m((-km+m+1)x) - C_m((2km-1)x),
\end{aligned} \tag{48}$$

$x \in X_0$, such that

$$\left\|f(x) - C_m(x)\right\|_* \leq \sup_{n\in\mathbb{N}_0}\{(\lambda(m) + \lambda(-km+1))\alpha_m^n\varphi(x)\}, \quad x \in X_0. \tag{49}$$

Moreover, $C_m(x) \coloneqq \lim_{n\longrightarrow\infty}(\mathcal{T}_m^n f)(x)$, for all $x \in X_0$. Now, we show that

$$\begin{aligned}
&\left\|\mathcal{T}_m^n f(kx+y) + \mathcal{T}_m^n f(kx-y) - k\mathcal{T}_m^n f(x+y) - k\mathcal{T}_m^n f(x+y) - 2\left(k^3-k\right)\mathcal{T}_m^n f(x)\right\|_* \\
&\quad \leq \alpha_m^n(\varphi(x) + \varphi(y)),
\end{aligned} \tag{50}$$

for every $x, y \in X_0$ such that $x + y \neq 0$, $x - y \neq 0$, and $n \in \mathbb{N}$. Since the case $n = 0$ is just (39), take $k \in \mathbb{N}$ and assume that (50) holds for $n = r$, where $r \in \mathbb{N}$ and every $x, y \in X_0$ such that $x + y \neq 0$. Then,

$$\begin{aligned}
&\left\|\mathcal{T}_m^{r+1} f(kx+y) + \mathcal{T}_m^{k+1} f(kx-y) - k\mathcal{T}_m^{r+1} f(x+y) - k\mathcal{T}_m^{r+1} f(x-y) - 2\left(k^3-k\right)\mathcal{T}_m^{r+1} f(x)\right\|_* \\
&= \Big\|2\left(k^3-k\right)\mathcal{T}_m^r f(m(kx+y)) + k\mathcal{T}_m^r f((km+m-1)(kx+y)) \\
&\quad + k\mathcal{T}_m^r f((-km+m+1)(kx+y)) - \mathcal{T}_m^r f((2km-1)(kx+y)) + 2\left(k^3-k\right)\mathcal{T}_m^r f((m)(kx-y)) \\
&\quad + k\mathcal{T}_m^r f((km+m-1)(kx-y)) + k\mathcal{T}_m^r f((-km+m+1)(kx-y)) - \mathcal{T}_m^r f((kkm-1)(kx-y)) \\
&\quad - k\left(k^3-k\right)\mathcal{T}_m^r f((m)(x+y)) - k^2\mathcal{T}_m^r f((km+m-1)(x+y)) - k^2\mathcal{T}_m^r f((-km+m+1)(x+y)) \\
&\quad + k\mathcal{T}_m^r f((2km-1)(x+y)) - k\left(k^3-k\right)\mathcal{T}_m^r f((m)(x-y)) - k^2\mathcal{T}_m^r f((km+m-1)(x-y)) \\
&\quad - k^2\mathcal{T}_m^r f((-km+m+1)(x-y))k\mathcal{T}_m^r f((2km-1)(x-y)) - \left(2\left(k^3-k\right)\right)^2\mathcal{T}_m^r f((m)(x)) \\
&\quad - 2k\left(k^3-k\right)\mathcal{T}_m^r f((km+m-1)(x)) - 2k\left(k^3-k\right)\mathcal{T}_m^r f((-km+m+1)(x)) \\
&\quad + 2\left(k^3-k\right)\mathcal{T}_m^r f((2km-1)(x))\Big|_* \\
&\leq \max\Big\{\left\|\mathcal{T}_m^n f(m(kx+y)) + \mathcal{T}_m^n f(m(kx-y)) - k\mathcal{T}_m^n f(m(x+y)) - k\mathcal{T}_m^n f(m(x-y))\right. \\
&\quad \left. -2\left(k^3-k\right)\mathcal{T}_m^n f(mx)\right\|_*,
\end{aligned}$$

$$\begin{aligned}
&\left\|\mathcal{T}_m^n f((km+m-1)(kx+y))+\mathcal{T}_m^n f((km+m-1)(kx-y))-k\mathcal{T}_m^n f((km+m-1)(x+y))\right.\\
&\quad\left.-\mathcal{T}_m^n f((km+m-1)(x-y))-2\left(k^3-k\right)\mathcal{T}_m^n f((km+m-1)x)\right\|_*,\\
&\left\|\mathcal{T}_m^n f((m-km+1)(kx+y))+\mathcal{T}_m^n f((m-km+1)(kx-y))-k\mathcal{T}_m^n f((m-km+1)(x+y))\right.\\
&\quad\left.-\mathcal{T}_m^n f((m-km+1)(x-y))-2\left(k^3-k\right)\mathcal{T}_m^n f((m-km+1)x)\right\|_*,\\
&\left\|\mathcal{T}_m^n f((2km-1)(kx+y))+\mathcal{T}_m^n f((2km-1)(kx-y))-k\mathcal{T}_m^n f((2km-1)(x+y))-\right.\\
&\quad\left.\mathcal{T}_m^n f((2km-1)(x-y))-2\left(k^3-k\right)\mathcal{T}_m^n f((2km-1)x)\right\|_*\Big\},\\
&\le\max\Big\{\alpha_m^k\Big(\varphi\big((mx)+\varphi((my)\big),\alpha_m^k(\varphi(km+m-1)x)+\varphi((Km+m-1)y)\Big)\Big)\\
&\alpha_m^k((\varphi(-km+m+1)x)+\varphi((-km+m+1)y)),\alpha_m^k((\varphi(2km-1)x)+\varphi((2km-1)y))\Big\}\\
&\quad\le\alpha_m^k\max\{\lambda(m),\lambda(km+m-1),\lambda(-km+m+1),\lambda(2km-1)\}(\varphi(x)+\varphi(y))\\
&\quad=\alpha_m^{k+1}(\varphi(x)+\varphi(y)).
\end{aligned}\tag{51}$$

Thus, by induction, we have shown that (50) holds for every $n\in\mathbb{N}$. Letting $n\longrightarrow\infty$ in (50), we obtain that

$$\begin{aligned}
&C_m(kx+y)+C_m(kx-y)\\
&\quad=kC_m(x+y)+kC_m(x-y)+2\left(k^3-k\right)C_m(x),
\end{aligned}\tag{52}$$

for all $x,y\in X_0$ such that $x+y\neq 0$ and $x-y\neq 0$. In this way, we obtain a sequence $\{C_m\}_{m\in U}$ of $k-$cubic functions on X_0 such that

$$\left\|f(x)-C_m(x)\right\|_*\le\sup_{n\in\mathbb{N}_0}\{(\lambda(m)+\lambda(-km+1))\alpha_m^n\varphi(x)\},\quad x\in X_0.\tag{53}$$

This implies that

$$\left\|f(x)-C_m(x)\right\|_*\le(\lambda(m)+\lambda(-km+1))\varphi(x),\quad x\in X_0,\tag{54}$$

because the precedent inequality holds for over $n=0$ and $\alpha_m<1$.

It follows, with $m\longrightarrow\infty$, that f is cubic on X_0. The following corollary is a particular case of Theorem 4 where $\varphi(x):=c\|x\|^p$ with $c\ge 0$ and $p<0$. □

Corollary 2. *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|_*)$ be normed space and ultrametric Banach space, respectively, $c\ge 0$, and $p<0$, and let $f\colon X\longrightarrow Y$ satisfies*

$$\left\|f(kx+y)+f(kx+y)-kf(x+y)-kf(x-y)-2\left(k^3-k\right)f(x)\right\|_*\le c\left(\|x\|^p+\|y\|^p\right),\tag{55}$$

for all $x,y\in X_0$. Then, f is $k-$cubic on X_0.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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