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Research Article

Generalized Hesitant Fuzzy Ideals in Semigroups

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In this paper, as a generalization of the concepts of hesitant fuzzy bi-ideals and hesitant fuzzy right (resp. left) ideals of semigroups, the concepts of hesitant fuzzy (m, n)-ideals and hesitant fuzzy $(m, 0)$-ideals (resp. $(0, n)$-ideals) are introduced. Furthermore, conditions for a hesitant fuzzy (m, n)-ideal ($(m, 0)$-ideal, $(0, n)$-ideal) to be a hesitant fuzzy bi-ideal (right ideal, left ideal) are provided. Moreover, several correspondences between bi-ideals (right ideals, left ideals) and hesitant fuzzy (m, n)-ideals ($(m, 0)$-ideals, $(0, n)$-ideals) are obtained. Also, the characterizations of different classes of semigroups in terms of their hesitant fuzzy (m, n)-ideals and hesitant fuzzy $(m, 0)$-ideals ($(0, n)$-ideals) are investigated.

1. Introduction

The fuzzy set theory introduced by Zadeh has been applied to different fields. Furthermore, in the literature, a number of generalizations and extensions of fuzzy sets have been introduced, for instance, intuitionistic fuzzy sets, interval-valued fuzzy sets, type 2 fuzzy sets, and fuzzy multisets. As a new generalization of fuzzy sets, Torra [1] introduced the notion of hesitant fuzzy sets which permit the membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [1, 2]). Torra [1] defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain. The hesitant fuzzy set offers a more accurate representation of hesitancy among people in expressing their preferences over objects than the fuzzy set or its classical extensions. This is really helpful to express the hesitancy of people in everyday life. The hesitant fuzzy set is a valuable tool to deal with uncertainty, which can be accurately and ideally described in terms of decision makers' opinions.

Torra [1] defined hesitant fuzzy sets as a function returning a collection of membership values for each domain element. The hesitant fuzzy set offers a more accurate representation of hesitancy among people in expressing their preferences over objects than the fuzzy set or its classical extensions. Fuzzy set theory has been applied to different classes in semigroups (see, for e.g., [3–9]). Also, fuzzy ideal theory of algebraic structures has been studied on various aspects in [10–13].

Hesitant fuzzy set theory was applied to many practical problems, particularly in the field of decision-making (see, for e.g., [1, 2, 14–19]). Later on, Jun and Song applied the notion of hesitant fuzzy sets to MTL-algebras and EQ-algebras (see [20, 21]). Recently, hesitant fuzzy set theory has been applied to various algebraic structures on different aspects, namely, Jun et al. applied the hesitant fuzzy set theory to BCK/BCI-algebras and semigroups (see [22–25]), and Muhiuddin et al. applied the hesitant fuzzy set theory to residuated lattices, lattice implication algebras, and BCK/BCI-algebras (see [26–35]). Motivated by a lot of work on hesitant fuzzy sets, we introduce the notions of hesitant fuzzy (m, n)-ideals, hesitant fuzzy $(m, 0)$-ideals, and hesitant fuzzy $(0, n)$-ideals of a semigroup by generalizing the concept of hesitant fuzzy bi-ideals, hesitant fuzzy right ideals, and hesitant fuzzy left ideals. Furthermore, associated properties of these generalized notions are discussed. Moreover, characterizations of different semigroup classes such as (m, n)-regular, $(m, 0)$-regular, and $(0, n)$-regular semigroups in terms of their hesitant fuzzy (m, n)-ideals, hesitant fuzzy $(m, 0)$-ideals, and hesitant fuzzy $(0, n)$-ideals are given.

2. Preliminaries

A nonempty set S endowed with an associative binary operation is called a semigroup. Throughout our discussion, S will denote a semigroup unless otherwise mentioned.

A subset $\varnothing \neq \Omega$ of S is called a sub-semigroup of S if $\Omega\Omega \subseteq \Omega$, and Ω is called the left (resp. right) ideal of S if $S\Omega \subseteq \Omega$ (resp. $\Omega S \subseteq \Omega$). If Ω is both left and right ideals of S, then it is called an ideal of S. A sub-semigroup Ω of S is called a bi-ideal of S if $\Omega S \Omega \subseteq \Omega$.

Let R be a reference set. Then, we define the hesitant fuzzy set (HFS) on R in terms of a function $\widehat{\mathscr{F}^{\mathscr{H}}}$ such that when applied to R, it returns a subset of $[0,1]$.

For a HFS $\widehat{\mathscr{F}^{\mathscr{H}}}$ on S and $\hbar, \kappa \in S$, we use the notations $\widehat{\mathscr{F}^{\mathscr{H}}_{\hbar}} :=\widehat{\mathscr{F}^{\mathscr{H}}}(\hbar)$ and $\mathscr{F}^{\mathscr{H}}_{\hbar}\mathscr{H} :=\widehat{\mathscr{F}^{\mathscr{H}}}(\hbar) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(\kappa)$.

Two HFSs $\widehat{\mathscr{F}^{\mathscr{H}}}$ and $\widehat{\mathscr{F}^{\mathscr{H}}} \cap \widehat{\mathscr{G}^{\mathscr{H}}}$ are defined as follows:

$$\begin{aligned} \widehat{\mathscr{F}^{\mathscr{H}}} \cup \widehat{\mathscr{G}^{\mathscr{H}}} &: S \longrightarrow \mathscr{P}([0,1]), \\ &\hbar \mapsto \widehat{\mathscr{F}^{\mathscr{H}}_{\hbar}} \cup \widehat{\mathscr{G}^{\mathscr{H}}_{\hbar}}, \end{aligned} \quad (1)$$

$$\begin{aligned} \widehat{\mathscr{F}^{\mathscr{H}}} \cap \widehat{\mathscr{G}^{\mathscr{H}}} &: S \longrightarrow \mathscr{P}([0,1]), \\ &\hbar \mapsto \widehat{\mathscr{F}^{\mathscr{H}}_{\hbar}} \cap \widehat{\mathscr{G}^{\mathscr{H}}_{\hbar}}, \end{aligned} \quad (2)$$

respectively.

For any HFSs $\widehat{\mathscr{F}^{\mathscr{H}}}$ and $\widehat{\mathscr{G}^{\mathscr{H}}}$ on S, we define $\widehat{\mathscr{F}^{\mathscr{H}}} \subseteq \widehat{\mathscr{G}^{\mathscr{H}}}$ if $\widehat{\mathscr{F}^{\mathscr{H}}_{\hbar}} \subseteq \widehat{\mathscr{G}^{\mathscr{H}}_{\hbar}}\ \forall \hbar \in S$.

For any two HFSs $\widehat{\mathscr{F}^{\mathscr{H}}}$ and $\widehat{\mathscr{G}^{\mathscr{H}}}$ of S, the HFS $\widehat{\mathscr{F}^{\mathscr{H}}} \circ \widehat{\mathscr{G}^{\mathscr{H}}}$ is defined as

$$\left(\widehat{\mathscr{F}^{\mathscr{H}}} \circ \widehat{\mathscr{G}^{\mathscr{H}}}\right)(\hbar) = \begin{cases} \bigcup_{\hbar=\kappa\ell}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}(\kappa) \cap \widehat{\mathscr{G}^{\mathscr{H}}}(\ell)\right\}, & \text{if there exist } \kappa, \ell \in S \text{ such that } \hbar = \kappa\ell, \\ \varnothing, & \text{otherwise.} \end{cases} \quad (3)$$

For $\Omega \subseteq S$, we denote by $\widehat{\chi^{\mathscr{H}}_{\Omega}}$ the hesitant characteristic fuzzy set of Ω, which is defined as

$$\widehat{\chi^{\mathscr{H}}_{\Omega}}(\hbar) = \begin{cases} [0,1] & \text{if } \hbar \in \Omega, \\ \varnothing & \text{if } \hbar \notin \Omega. \end{cases} \quad (4)$$

We denote the identity HFS by $\widehat{\mathscr{I}^{\mathscr{H}}_{S}}$, and it is defined as follows:

$$\widehat{\mathscr{I}^{\mathscr{H}}_{S}}(\hbar) = [0,1], \quad \forall \hbar \in S. \quad (5)$$

Let $A, B \subseteq S$. Then, we have

(1) $\widehat{\chi^{\mathscr{H}}_{A}} \circ \widehat{\chi^{\mathscr{H}}_{B}} = \widehat{\chi^{\mathscr{H}}_{AB}}$.
(2) $\widehat{\chi^{\mathscr{H}}_{A}} \cap \widehat{\chi^{\mathscr{H}}_{B}} = \widehat{\chi^{\mathscr{H}}_{A\cap B}}$.

A HFS $\widehat{\mathscr{F}^{\mathscr{H}}}$ is called a hesitant fuzzy sub-semigroup (briefly HFSS) of S if $\forall \hbar, \kappa \in S$, $\widehat{\mathscr{F}^{\mathscr{H}}}(\hbar\kappa) \supseteq \widehat{\mathscr{F}^{\mathscr{H}}}(\hbar) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(\kappa)$, and $\widehat{\mathscr{F}^{\mathscr{H}}}$ is called a hesitant fuzzy left (resp. right) ideal (briefly HFLI and HFRI) of S if $\forall \hbar, \kappa \in S$, $\widehat{\mathscr{F}^{\mathscr{H}}}(\hbar\kappa) \supseteq \widehat{\mathscr{F}^{\mathscr{H}}}(\kappa)$ (resp. $\widehat{\mathscr{F}^{\mathscr{H}}}(\hbar\kappa) \supseteq \widehat{\mathscr{F}^{\mathscr{H}}}(\hbar)$). If $\widehat{\mathscr{F}^{\mathscr{H}}}$ is both HFLI and HFRI of S, then it is called a hesitant fuzzy ideal of S. A HFSS $\widehat{\mathscr{F}^{\mathscr{H}}}$ is called a hesitant fuzzy bi-ideal (briefly HFBI) of S if $\widehat{\mathscr{F}^{\mathscr{H}}}(\hbar\ell\kappa) \supseteq \widehat{\mathscr{F}^{\mathscr{H}}}(\hbar) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(\kappa)$ for each $\hbar, \kappa, \ell \in S$.

Throughout the paper, $\widehat{\wp_{\mathscr{H}_R}}, \widehat{\wp_{\mathscr{H}_L}}$, and $\widehat{\wp_{\mathscr{H}_B}}$ will stand for the set of all hesitant fuzzy right ideals, hesitant fuzzy left ideals, and hesitant fuzzy right bi-ideals of S.

The concept of (m,n)-ideals of semigroups was given by Lajos [36]. Also, the study of (m,n)-ideals in different algebraic structures has been conducted by several authors [37–43]. A sub-semigroup A of S is called an (m,n)-ideal of S [36] if $A^m S A^n \subseteq A$, where m and n are nonnegative integers. Here, $A^0 S = SA^0 = S$.

The set of all (m,n)-ideals, $(m,0)$-ideals, and $(0,n)$-ideals will be denoted by $\mathscr{I}_{(m,n)}, \mathscr{I}_{(m,0)}$, and. $\mathscr{I}_{(0,n)}$

3. Main Results

Definition 1. A HFSS $\widehat{\mathscr{F}^{\mathscr{H}}}$ of S is called a hesitant fuzzy (m,n)-ideal of S if $\widehat{\mathscr{F}^{\mathscr{H}}_{r_1 r_2 \ldots r_m z s_1 s_2 \ldots s_n}} \supseteq \widehat{\mathscr{F}^{\mathscr{H}}_{r_1}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{r_2}} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}_{r_m}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_1}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_2}} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_n}}$ for all $r_1, r_2, \ldots, r_m, z, s_1, s_2, \ldots s_n \in S$.

Throughout the paper, $\widehat{\wp_{\mathscr{H}(m,n)}}$ will stand for the set of all hesitant fuzzy (m,n)-ideals of S.

Lemma 1. *Let* $\left\{\widehat{\mathscr{F}^{\mathscr{H}}_{i}}\ t \in n\widehat{\wp_{\mathscr{H}(m,n)}}\ |qih \in {}_I\right\}$. *Then,* $\cap_{i\in I}\widehat{\mathscr{F}^{\mathscr{H}}_{i}} \in \widehat{\wp_{\mathscr{H}(m,n)}}$.

Proof. Straightforward. □

Remark 1. Let $\left\{\widehat{\mathscr{F}^{\mathscr{H}}_{i}}\ t \in n\widehat{\wp_{\mathscr{H}(m,n)}} q\, h|\ xi7 \in CI\right\}$. Then, $\cup_{i\in I}\widehat{\mathscr{F}^{\mathscr{H}}_{i}} \notin \widehat{\wp_{\mathscr{H}(m,n)}}$ in general. We illustrate it by the following example.

Example 1. Let $S = \{\vartheta, \iota, \kappa, \hbar\}$ be a semigroup with the following multiplication table:

Let $\widehat{\mathcal{F}_1^{\mathcal{H}}}$ and $\widehat{\mathcal{F}_2^{\mathcal{H}}}$ be two HFS of S such that

$$\begin{aligned} \widehat{\mathcal{F}_1^{\mathcal{H}}}(\vartheta) &= [0, 0.2], \\ \widehat{\mathcal{F}_1^{\mathcal{H}}}(\iota) &= [0, 0.2], \\ \widehat{\mathcal{F}_1^{\mathcal{H}}}(\kappa) &= \varnothing, \\ \widehat{\mathcal{F}_1^{\mathcal{H}}}(\hbar) &= \varnothing, \\ \widehat{\mathcal{F}_2^{\mathcal{H}}}(\vartheta) &= [0, 0.2], \\ \widehat{\mathcal{F}_2^{\mathcal{H}}}(\iota) &= \varnothing, \\ \widehat{\mathcal{F}_2^{\mathcal{H}}}(\kappa) &= [0, 0.2], \\ \widehat{\mathcal{F}_2^{\mathcal{H}}}(\hbar) &= \varnothing. \end{aligned} \tag{6}$$

Then, $\widehat{\mathcal{F}_1^{\mathcal{H}}}, \widehat{\mathcal{F}_2^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$ but $\widehat{\mathcal{F}_1^{\mathcal{H}}} \cup \widehat{\mathcal{F}_2^{\mathcal{H}}} \notin \widehat{\wp_{\mathcal{H}(m,n)}}$ because $\varnothing = \widehat{\mathcal{F}_1^{\mathcal{H}}}(u) \cup \widehat{\mathcal{F}_1^{\mathcal{H}}}(u) = (\widehat{\mathcal{F}_1^{\mathcal{H}}} \cup t \widehat{\mathcal{F}_2^{\mathcal{H}}})(u) = (\widehat{\mathcal{F}_1^{\mathcal{H}}} \cup t \widehat{\mathcal{F}_2^{\mathcal{H}}})(\mathrm{sc}) \subset (\widehat{\mathcal{F}_1^{\mathcal{H}}} \cup t \widehat{\mathcal{F}_2^{\mathcal{H}}})(s) \cap (\widehat{\mathcal{F}_1^{\mathcal{H}}} \cup t \widehat{\mathcal{F}_2^{\mathcal{H}}})(w) = [0, 0.2]$.

Lemma 2. *Let $\varnothing \neq A \subseteq S$. Then, $A \in \mathcal{I}_{(m,n)} \Leftrightarrow \widehat{\chi_A^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$.*

Proof. ($\Rightarrow$) Let $r_1, r_2, \ldots, r_m, z, s_1, s_2, \ldots, s_n \in S$. Then, the following are observed.

Case 1: if $r_\imath \notin A$ for any $\imath \in \{1, 2, \ldots, \mathrm{m}\}$, then

$$\widehat{\chi_A^{\mathcal{H}}}(r_1 r_2 \ldots r_m z s_1 s_2 \ldots s_n) \supseteq \widehat{\chi_A^{\mathcal{H}}}(r_1) \cap \widehat{\chi_A^{\mathcal{H}}}(r_2) \cap \cdots \cap \widehat{\chi_A^{\mathcal{H}}}(r_m) \cap \widehat{\chi_A^{\mathcal{H}}}(s_1) \cap \widehat{\chi_A^{\mathcal{H}}}(s_2) \cap \cdots \cap \widehat{\chi_A^{\mathcal{H}}}(s_n). \tag{7}$$

Case 2: if $s_\jmath \notin A$ for any $\jmath \in \{1, 2, \ldots, n\}$, then

$$\widehat{\chi_A^{\mathcal{H}}}(r_1 r_2 \ldots r_m z s_1 s_2 \ldots s_n) \supseteq \widehat{\chi_A^{\mathcal{H}}}(r_1) \cap \widehat{\chi_A^{\mathcal{H}}}(r_2) \cap \cdots \cap \widehat{\chi_A^{\mathcal{H}}}(r_m) \cap \widehat{\chi_A^{\mathcal{H}}}(s_1) \cap \widehat{\chi_A^{\mathcal{H}}}(s_2) \cap \cdots \cap \widehat{\chi_A^{\mathcal{H}}}(s_n). \tag{8}$$

Case 3: if $r_\imath \notin A$ and $s_\jmath \notin A \, \forall \imath \in \{1, 2, \ldots, \mathrm{m}\}$, $\jmath \in \{1, 2, \ldots, n\}$, then

$$\widehat{\chi_A^{\mathcal{H}}}(r_1 r_2 \ldots r_m z s_1 s_2 \ldots s_n) \supseteq \widehat{\chi_A^{\mathcal{H}}}(r_1) \cap \widehat{\chi_A^{\mathcal{H}}}(r_2) \cap \cdots \cap \widehat{\chi_A^{\mathcal{H}}}(r_m) \cap \widehat{\chi_A^{\mathcal{H}}}(s_1) \cap \widehat{\chi_A^{\mathcal{H}}}(s_2) \cap \cdots \cap \widehat{\chi_A^{\mathcal{H}}}(s_n). \tag{9}$$

Case 4: if $r_\imath, s_\jmath \in A \, \forall \imath \in \{1, 2, \ldots, \mathrm{m}\}, \jmath \in \{1, 2, \ldots, \mathrm{n}\}$, then $r_1 r_2 \ldots r_m z s_1 s_2 \ldots s_n \in A^m S A^n \subseteq A$. Therefore,

$$\widehat{\chi_A^{\mathcal{H}}}(r_1 r_2 \ldots r_m z s_1 s_2 \ldots s_n) = [0, 1] \supseteq \widehat{\chi_A^{\mathcal{H}}}(r_1) \cap \widehat{\chi_A^{\mathcal{H}}}(r_2) \cap \cdots \cap \widehat{\chi_A^{\mathcal{H}}}(r_m) \cap \widehat{\chi_A^{\mathcal{H}}}(s_1) \cap \widehat{\chi_A^{\mathcal{H}}}(s_2) \cap \cdots \cap \widehat{\chi_A^{\mathcal{H}}}(s_n). \tag{10}$$

Hence, $\widehat{\chi_A^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$.

($\Leftarrow$) Let $x, z, y \in S$. If $x, y \in A$, then $\widehat{\chi_A^{\mathcal{H}}}(x^m z y^n) \supseteq \widehat{\chi_A^{\mathcal{H}}}(x) \cap \widehat{\chi_A^{\mathcal{H}}}(y) = [0, 1]$ implies

$\cdot$	ϑ	ι	κ	$\hbar$
ϑ	ϑ	ϑ	ϑ	ϑ
ι	ϑ	ϑ	$\hbar$	ϑ
κ	ϑ	ϑ	ϑ	ϑ
$\hbar$	ϑ	ϑ	ϑ	ϑ

$\widehat{\chi_A^{\mathscr{H}}}(x^m z y^n) = [0,1]$. Therefore, $x^m z y^n \in A$. Thus, $A^m S A^n \subseteq A$. □

Definition 2. For any HFS $\widehat{\mathscr{F}^{\mathscr{H}}}$ of S, the set

$$\widehat{\mathscr{F}_T^{\mathscr{H}}} = \left\{ x \in S \mid t\widehat{\mathscr{F}_x^{\mathscr{H}}} n \supseteq q T \right\}, \tag{11}$$

where $T \in \mathscr{P}([0,1])$, is said to be a hesitant T-level subset of $\widehat{\mathscr{F}^{\mathscr{H}}}$.

Theorem 1. *Let $\widehat{\mathscr{F}^{\mathscr{H}}}$ be the HFS of S. Then, the hesitant T-level subset $\widehat{\mathscr{F}_T^{\mathscr{H}}} \in \mathscr{I}_{(m,n)} \forall T \in \mathscr{P}([0,1])$, provided $\widehat{\mathscr{F}_T^{\mathscr{H}}} \neq \varnothing \Leftrightarrow \widehat{\mathscr{F}^{\mathscr{H}}} \in \wp_{\widehat{\mathscr{H}(m,n)}}$.*

Proof. ($\Rightarrow$) For any $r_1, r_2, \ldots, r_m, s_1, s_2, \ldots, s_n \in S$ and suppose, to the contrary, that $\widehat{\mathscr{F}^{\mathscr{H}}}_{r_1} \cap \widehat{\mathscr{F}^{\mathscr{H}}}_{r_2} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}}_{r_m} \cap \widehat{\mathscr{F}^{\mathscr{H}}}_{s_1} \cap \widehat{\mathscr{F}^{\mathscr{H}}}_{s_2} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}}_{s_n}^{\widehat{\mathscr{F}^{\mathscr{H}}}_{r_1 r_2 \cdots r_m z s_1 s_2 \cdots s_n}}$. Then, there exists $C \in \mathscr{P}([0,1])$ such that $\widehat{\mathscr{F}^{\mathscr{H}}_{r_1}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{r_2}} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}_{r_m}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_1}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_2}} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_n}} \supseteq C^{\widehat{\mathscr{F}^{\mathscr{H}}}_{r_1 r_2 \cdots r_m z s_1 s_2 \cdots s_n}}$. So, $r_1, r_2, \ldots, r_m, z, s_1, s_2, \ldots, s_n \in \widehat{\mathscr{F}^{\mathscr{H}}_C}$, but $r_1 r_2 \cdots r_m z s_1 s_2 \cdots s_n \notin \widehat{\mathscr{F}^{\mathscr{H}}_C}$, a contradiction. Thus, $\widehat{\mathscr{F}^{\mathscr{H}}}_{r_1 r_2 \cdots r_m z s_1 s_2 \cdots s_n} \supseteq \widehat{\mathscr{F}^{\mathscr{H}}_{r_1}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{r_2}} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}_{r_m}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_1}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_2}} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_n}}$ for all $r_1, r_2, \ldots, r_m, z, s_1, s_2, \ldots, s_n \in S$. Hence, $\widehat{\mathscr{F}^{\mathscr{H}}} \in \wp_{\widehat{\mathscr{H}(m,n)}}$.

($\Leftarrow$) Let $z \in S$ and $r_1, r_2, \ldots, r_m, s_1, s_2, \ldots, s_n \in \widehat{\mathscr{F}_T^{\mathscr{H}}}$, where $T \in \mathscr{P}([0,1])$. Then, $\widehat{\mathscr{F}^{\mathscr{H}}_{r_1}} \supseteq T, \widehat{\mathscr{F}^{\mathscr{H}}_{r_2}} \supseteq T, \ldots, \widehat{\mathscr{F}^{\mathscr{H}}_{x_n}} \supseteq T$ and $\widehat{\mathscr{F}^{\mathscr{H}}_{s_1}} \supseteq T, \widehat{\mathscr{F}^{\mathscr{H}}_{s_2}} \supseteq T, \cdots \widehat{\mathscr{F}^{\mathscr{H}}_{y_m}} \supseteq T$. By definition, $\widehat{\mathscr{F}^{\mathscr{H}}}_{r_1 r_2 \cdots r_m z s_1 s_2 \cdots s_n} \supseteq \widehat{\mathscr{F}^{\mathscr{H}}_{r_1}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{r_2}} \cap \cdots \cap \widehat{\mathscr{F}_{r_m}}^{\mathscr{H}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_1}} \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_2}} \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}_{s_n}} = T$. Therefore, $r_1 r_2 \cdots r_m z s_1 s_2 \cdots s_n \in \widehat{\mathscr{F}_T^{\mathscr{H}}}$. Hence, $\widehat{\mathscr{F}_T^{\mathscr{H}}} \in \mathscr{I}_{(m,n)}$. □

Theorem 2. *Let $\widehat{\mathscr{F}^{\mathscr{H}}}$ be the HFSS of S. Then, $\widehat{\mathscr{F}^{\mathscr{H}}} \in \wp_{\widehat{\mathscr{H}(m,n)}} \Leftrightarrow \widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n \subseteq \widehat{\mathscr{F}^{\mathscr{H}}}$.*

Proof. ($\Rightarrow$) Let $\widehat{\mathscr{F}^{\mathscr{H}}} \in \wp_{\widehat{\mathscr{H}(m,n)}}$ and $\hbar \in S$. If $(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n)(\hbar) = \varnothing$, then $\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n \subseteq \widehat{\mathscr{F}^{\mathscr{H}}}$. If $(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n)(\hbar) \neq \varnothing$, then there exist ℓ, κ in S such that $\hbar = \ell\kappa$, $(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}})(\ell) \neq \varnothing$, and $\widehat{\mathscr{F}^{\mathscr{H}}}^n(\kappa) \neq \varnothing$. We have the following.

Case 1: when $(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}})(\ell) \neq \varnothing$, then

$\exists u_1, v_1 \in S$ such that $\ell = u_1 v_1$ implies $\widehat{\mathscr{F}^{\mathscr{H}}}^m(u_1) \neq \varnothing$ and $\widehat{\mathscr{I}_S^{\mathscr{H}}}(v_1) = [0,1]$

$\exists u_2, v_2 \in S$ such that $u_1 = u_2 v_2 \widehat{\mathscr{F}^{\mathscr{H}}}(u_2) \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}^{m-1}(v_2) \neq \varnothing$

$\exists u_3, v_3 \in S$ such that $v_2 = u_3 v_3 \widehat{\mathscr{F}^{\mathscr{H}}}(u_3) \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}^{m-2}(v_3) \neq \varnothing$

$\vdots$

$\exists u_{m-1}, v_{m-1} \in S$ such that $v_{m-2} = u_{m-1} v_{m-1}$ $\widehat{\mathscr{F}^{\mathscr{H}}}(u_{m-1}) \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}^2(v_{m-1}) = \widehat{\mathscr{F}^{\mathscr{H}}}^{m-m-2}(v_{m-1}) \neq \varnothing$

$\exists u_m, v_m \in S$ such that $v_{m-1} = u_m v_m \widehat{\mathscr{F}^{\mathscr{H}}}(u_m) \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}(v_m) \neq \varnothing$

Case 2: when $\widehat{\mathscr{F}^{\mathscr{H}}}^n(\kappa) \neq \varnothing$, then

$\exists u_1', v_1' \in S$ such that $\kappa = u_1' v_1' \widehat{\mathscr{F}^{\mathscr{H}}}(u_1') \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}^{n-1}(v_1') \neq \varnothing$

$\exists u_2', v_2' \in S$ such that $v_1' = u_2' v_2' \widehat{\mathscr{F}^{\mathscr{H}}}(u_2') \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}^{n-2}(v_2') \neq \varnothing$

$\exists u_3', v_3' \in S$ such that $v_2' = u_3' v_3' \widehat{\mathscr{F}^{\mathscr{H}}}(u_3') \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}^{n-3}(v_3') \neq \varnothing$

$\vdots$

$\exists u_{n-2}', v_{n-2}' \in S$ such that $v_{n-3}' = u_{n-2}' v_{n-2}' \widehat{\mathscr{F}^{\mathscr{H}}}(u_{n-2}') \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}^{n-(n-2)}(v_{n-2}') \neq \varnothing$

$\exists u_{n-1}', v_{n-1}' \in S$ such that $v_{n-2}' = u_{n-1}' v_{n-1}' \widehat{\mathscr{F}^{\mathscr{H}}}(u_{n-1}') \neq \varnothing$ and $\widehat{\mathscr{F}^{\mathscr{H}}}(v_{n-1}') \neq \varnothing$

Now, we have

$$\begin{aligned}
&\left(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n\right)(a) = \bigcup_{a=\ell\kappa}\left\{\left(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}}\right)(\ell) \cap \widehat{\mathscr{F}^{\mathscr{H}}}^n(\kappa)\right. \\
&= \bigcup_{a=\ell\kappa}\left\{\bigcup_{\ell=u_1v_1}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}^m(u_1) \cap \widehat{\mathscr{I}_S^{\mathscr{H}}}(v_1) \cap \bigcup_{\kappa=u_1'v_1'}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}(u_1') \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{n-1}(v_1')\right\}\right\} = \\
&\bigcup_{a=\ell\kappa}\bigcup_{\ell=u_1v_1}\bigcup_{\kappa=u_1'v_1'}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}^m(u_1) \cap \widehat{\mathscr{I}_S^{\mathscr{H}}}(v_1) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(u_1') \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{n-1}(v_1')\right\} \\
&= \bigcup_{a=\ell\kappa}\bigcup_{\ell=u_1v_1}\bigcup_{\kappa=u_1'v_1'}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}^m(u_1) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(u_1') \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{n-1} \cap (v_1')\right\} \\
&= \bigcup_{a=\ell\kappa}\bigcup_{\ell=u_1v_1}\bigcup_{\kappa=u_1'v_1'}\left\{\bigcup_{u_1=u_2v_2}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}(u_2) \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{m-1}(v_2) \cap \bigcup_{u_1'=u_2'v_2'}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}(u_2') \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{n-2}(v_2') \cap \widehat{\mathscr{F}^{\mathscr{H}}}(v_1')\right\}\right.\right. \\
&= \bigcup_{a=\ell\kappa}\bigcup_{\ell=u_1v_1}\bigcup_{\kappa=u_1'v_1'}\bigcup_{u_1=u_2v_2}\bigcup_{u_1'=u_2'v_2'}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}(u_2) \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{m-1}(v_2) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(u_2') \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{n-2}(v_2') \cap \widehat{\mathscr{F}^{\mathscr{H}}}(v_1')\right\} \\
&= \bigcup_{a=\ell\kappa}\bigcup_{\ell=u_1v_1}\bigcup_{\kappa=u_1'v_1'}\bigcup_{u_1=u_2v_2}\bigcup_{u_1'=u_2'v_2'}\cdots\bigcup_{u_{m-1}=u_mv_m}\bigcup_{u_{n-2}'=u_{n-1}'v_{n-1}'}\left\{\begin{array}{c}\widehat{\mathscr{F}^{\mathscr{H}}}(u_2) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(u_3) \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}}(u_m) \cap \\ \widehat{\mathscr{F}^{\mathscr{H}}}(v_m) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(u_{n-1}') \cap \widehat{\mathscr{F}^{\mathscr{H}}}(v_{n-1}') \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}}(v_2') \cap \widehat{\mathscr{F}^{\mathscr{H}}}(v_1')\end{array}\right\} \\
&\subseteq \bigcup_{a=\ell\kappa}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}_{u_2u_3\ldots v_mv_mv_1u_1'u_2'\ldots u_{n-1}'v_{n-1}'}\right\} = \bigcup_{a=\ell\kappa}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}(\ell\kappa)\right\} \\
&(\text{since } \ell = u_2u_3\ldots v_mv_mv_1 \text{ and } \kappa = u_1'u_2'\ldots u_{n-1}'v_{n-1}') = \widehat{\mathscr{F}^{\mathscr{H}}}(a).
\end{aligned} \tag{12}$$

($\Leftarrow$) Assume that $\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n \subseteq \widehat{\mathscr{F}^{\mathscr{H}}}$. For any $r_1, r_2, \ldots, r_m, z, s_1, s_2, \ldots, s_n \in S$, let $a = r_1r_2 \cdots r_m z s_1 s_2 \cdots s_n$. Since $\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n \subseteq \widehat{\mathscr{F}^{\mathscr{H}}}$, we have

$$\begin{aligned}
&\widehat{\mathscr{F}^{\mathscr{H}}}(r_1r_2\ldots r_mzs_1s_2\ldots s_n) = \widehat{\mathscr{F}^{\mathscr{H}}}(a) \supseteq \left(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n\right)(a) \\
&= \bigcup_{a=pq}\left\{\left(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}}\right)(p) \cap \widehat{\mathscr{F}^{\mathscr{H}}}^n(q)\right\} \supseteq \left\{\left(\widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{I}_S^{\mathscr{H}}}\right)(r_1r_2\ldots r_mz) \cap \widehat{\mathscr{F}^{\mathscr{H}}}^n(s_1s_2\ldots s_n)\right. \\
&\supseteq \left\{\bigcup_{r_1r_2\ldots r_mz=uv}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}^m(u) \cap \widehat{\mathscr{I}_S^{\mathscr{H}}}(u)\right\} \cap \bigcup_{s_1s_2\ldots s_n=u',v'}\left\{\widehat{\mathscr{F}^{\mathscr{H}}}(u') \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{n-1}(v')\right\}\right\} \\
&\supseteq \left\{\left\{\widehat{\mathscr{F}^{\mathscr{H}}}^m(r_1r_2\ldots r_m) \cap \widehat{\mathscr{I}_S^{\mathscr{H}}}(z)\right\} \cap \left\{\widehat{\mathscr{F}^{\mathscr{H}}}(s_1) \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{n-1}(s_2\ldots y_{n-1}s_n)\right\}\right\} \\
&\supseteq \left\{\widehat{\mathscr{F}^{\mathscr{H}}}^m(r_1r_2\ldots r_m) \cap \widehat{\mathscr{F}^{\mathscr{H}}}^{n-1}(s_1s_2\ldots y_{n-1}) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(s_n)\right\} \\
&\vdots \\
&\supseteq \widehat{\mathscr{F}^{\mathscr{H}}}(r_1) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(r_2) \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}}(r_m) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(s_1) \cap \widehat{\mathscr{F}^{\mathscr{H}}}(s_2) \cap \cdots \cap \widehat{\mathscr{F}^{\mathscr{H}}}(s_n).
\end{aligned} \tag{13}$$

Hence, $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,n)}}$. □

Lemma 3. *Let $\widehat{\mathscr{F}^{\mathscr{H}}}$ be the HFS of S. Then, $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}_B}} \Rightarrow \widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,n)}}\ \forall m, n \in \mathbb{N}$.*

·	ϑ	ι	κ	$\hbar$
ϑ	ϑ	ϑ	ϑ	ϑ
ι	ι	ι	ι	ι
κ	κ	κ	κ	κ
$\hbar$	ϑ	ϑ	ι	ϑ

Proof. Straightforward. □

Remark 2. In general, $\widehat{\mathcal{F}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(m,n)}} \nRightarrow \widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}_B}}$.

Example 2. Let $S = \{\vartheta, \iota, \kappa, \hbar\}$ be a semigroup with the following multiplication table:

Define the HFS $\widehat{\mathcal{F}^{\mathcal{H}}}$ of S as follows: $\widehat{\mathcal{F}^{\mathcal{H}}}(\vartheta) = \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) = [0,1]$ and $\widehat{\mathcal{F}^{\mathcal{H}}}(\iota) = \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) = 0$. Then, $\widehat{\mathcal{F}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(m,n)}}$, but $\widehat{\mathcal{F}^{\mathcal{H}}} \notin \widehat{\wp_{\mathcal{H}_B}}$.

Definition 3. A semigroup S is called (m,n)-regular if $\forall\, \hbar \in S\, \exists\, \ell \in S$ such that $\hbar = \hbar^m \ell \hbar^n$.

Lemma 4. *Let $\widehat{\mathcal{F}^{\mathcal{H}}}$ be the HFS of (m,n)-regular semigroup S. Then, $\widehat{\mathcal{F}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(m,n)}} \Rightarrow \widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}_B}}$.*

Proof. Suppose that $\widehat{\mathcal{F}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(m,n)}}$ and $r, \hbar, s \in S$. Since S is (m,n)-regular, $r\hbar s = r^m p r^n \hbar s^m q s^n$ for some $p, q \in S$. We have

$$\begin{aligned}\widehat{\mathcal{F}^{\mathcal{H}}}(r\hbar s) &= \widehat{\mathcal{F}^{\mathcal{H}}}(r^m p r^n \hbar s^m q s^n) = \widehat{\mathcal{F}^{\mathcal{H}}}(r^m (p r^n \hbar s^m q) s^n)\\ &\supseteq \widehat{\mathcal{F}^{\mathcal{H}}}(r) \cap \widehat{\mathcal{F}^{\mathcal{H}}}(s),\end{aligned} \tag{14}$$

as required. □

Lemma 5. *Let $\widehat{\mathcal{F}^{\mathcal{H}}}$ be the HFS. Then, $\widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}^k(\hbar^k)$ for any $k \in \mathbb{N}$ and $\hbar \in S$.*

Proof. Let $\hbar \in S$. As $\hbar^k = \hbar\hbar^{k-1}$, we have

$$\begin{aligned}\widehat{\mathcal{F}^{\mathcal{H}}}^k(\hbar^k) &= \bigcup_{\hbar^k = uv}\left\{f(u) \cap \widehat{\mathcal{F}^{\mathcal{H}}}^{k-1}(v)\right\} \supseteq \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \cap \widehat{\mathcal{F}^{\mathcal{H}}}^{k-1}(\hbar^{k-1})\\ &= \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \cap \bigcup_{\hbar^{k-1} = u'v'}\left\{\widehat{\mathcal{F}^{\mathcal{H}}}(u')t \cap n\widehat{\mathcal{F}^{\mathcal{H}}}^{k-2} q(v')\right\} \supseteq \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \cap \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \cap \widehat{\mathcal{F}^{\mathcal{H}}}^{k-2}(\hbar^{k-2})\\ &\;\;\vdots\\ &\supseteq \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \cap \cdots \cap \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \cap \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) = \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar).\end{aligned} \tag{15}$$

□

Theorem 3. *S is (m,n)-regular $\Leftrightarrow \widehat{\mathcal{F}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{J}^{\mathcal{H}}_S} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n$ for each HFS $\widehat{\mathcal{F}^{\mathcal{H}}}$ of S.*

Proof. (⇒) Take any $\hbar \in S$. Then, $\hbar = \hbar^m \ell \hbar^n$ for some $\ell \in S$. We have

$$\begin{aligned}&\left(\widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{J}^{\mathcal{H}}_S} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n\right)(\hbar) = \bigcup_{\hbar = rs}\left\{\left(\widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{J}^{\mathcal{H}}_S}\right)(r) \cap \left(\widehat{\mathcal{F}^{\mathcal{H}}}^n\right)(s)\right\}\\ &\supseteq \left(\widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{J}^{\mathcal{H}}_S}\right)(\hbar^m x) \cap \left(\widehat{\mathcal{F}^{\mathcal{H}}}^n\right)(\hbar^n) = \bigcup_{\hbar^m x = pq}\left\{\left(\widehat{\mathcal{F}^{\mathcal{H}}}^m\right)(p) \cap \widehat{\mathcal{J}^{\mathcal{H}}_S}(q)\right\} \cap \left(\widehat{\mathcal{F}^{\mathcal{H}}}^n\right)(\hbar^n)\\ &\supseteq \widehat{\mathcal{F}^{\mathcal{H}}}^m(\hbar^m) \cap \widehat{\mathcal{J}^{\mathcal{H}}_S}(x) \cap \widehat{\mathcal{F}^{\mathcal{H}}}^n(\hbar^n) = \widehat{\mathcal{F}^{\mathcal{H}}}^m(\hbar^m) \cap \widehat{\mathcal{F}^{\mathcal{H}}}^n(\hbar^n)\\ &\supseteq \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \cap \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar) \text{ by Lemma 2.14} = \widehat{\mathcal{F}^{\mathcal{H}}}(\hbar).\end{aligned} \tag{16}$$

Therefore, $\widehat{\mathcal{F}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n$.

($\Leftarrow$) Suppose that $\hbar \in S$. Since $\widehat{\chi_\hbar^{\mathcal{H}}}$ is the HFS of S, by hypothesis, $\widehat{\chi_\hbar^{\mathcal{H}}} \subseteq \widehat{\chi_\hbar^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\chi_\hbar^{\mathcal{H}}}^n = \widehat{\chi_{\hbar^m S \hbar^n}^{\mathcal{H}}}$. Therefore, $\hbar \in \hbar^m S \hbar^n$. Hence, S is (m, n)-regular.□ □

Theorem 4. *S is (m, n)-regular $\Leftrightarrow \widehat{\mathcal{F}^{\mathcal{H}}} = \widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n$ $\forall \widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$*

Proof. ($\Rightarrow$) Let $\widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$. Then, by hypothesis and Theorems 2 and 4, $\widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}$ and $\widehat{\mathcal{F}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n$. Hence, $\widehat{\mathcal{F}^{\mathcal{H}}} = \widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n$.

($\Leftarrow$) Let $B \in \mathcal{I}_{(m,n)}$ and $b \in B$, so by hypothesis, we have $(\widehat{\chi_B^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\chi_B^{\mathcal{H}}}^n)(b) = \chi_A(b) = 1$ implies $(\widehat{\chi_B^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\chi_B^{\mathcal{H}}}^n)(b) = 1$. This implies that there exist elements $\hbar, \kappa$ in S with $b = \hbar\kappa$ such that

$$\left(\widehat{\chi_B^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}}\right)(\hbar) = [0,1] \text{ and } \widehat{\chi_B^{\mathcal{H}}}^n(\kappa) = [0,1]. \tag{17}$$

This implies that there exist elements u, v in S with $\hbar = uv$ such that

$$\widehat{\chi_B^{\mathcal{H}}}^m(u) = [0,1] \text{ and } \widehat{\mathcal{I}_S^{\mathcal{H}}}(v) = [0,1]. \tag{18}$$

So, $[0,1] = \widehat{\chi_B^{\mathcal{H}}}^n(\kappa) = \widehat{\chi_{B^n}^{\mathcal{H}}}(\kappa)$ and $[0,1] = \widehat{\chi_B^{\mathcal{H}}}^m(u) = \widehat{\chi_{B^m}^{\mathcal{H}}}(u)$, and it follows that $\kappa \in B^n$ and $u \in B^n$. Since $a = \hbar\kappa$ and $\hbar = uv$, therefore, $a = \hbar\kappa = uv\kappa \in B^m S B^n$. Thus, $B \subseteq B^m S B^n$. Therefore, $B = B^m S B^n$. Hence, by Theorem 2of [44], S is (m, n)-regular. □

Lemma 6. *If $\widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}}$ is a HFSS of S such that*

$$\widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n \subseteq \widehat{\mathcal{G}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}, \tag{19}$$

then $\widehat{\mathcal{G}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$.

Proof. Since $\widehat{\mathcal{G}^{\mathcal{H}}}$ is a HFSS of S, by Theorem 2, it is sufficient to show that $\widehat{\mathcal{G}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}^n \subseteq \widehat{\mathcal{G}^{\mathcal{H}}}$. Now,

$$\left(\widehat{\mathcal{G}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}^n\right)(a) \subseteq \left(\widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{I}_S^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n\right)(a) \subseteq \widehat{\mathcal{G}^{\mathcal{H}}}(a). \tag{20}$$

Hence, $\widehat{\mathcal{G}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$. □

Lemma 7. *Let $\widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}}$ be a HFS of S. If $\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}$ or $\widehat{\mathcal{G}^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}$, then*

(1) $\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$.

(2) $\widehat{\mathcal{G}^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$.

Proof. When $\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}$, we have

$$\left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}\right) \circ \left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}\right) = \widehat{\mathcal{F}^{\mathcal{H}}} \circ \left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ t \widehat{\mathcal{G}^{\mathcal{H}}}\right) = \widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}. \tag{21}$$

Therefore, $\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}$ is a HFSS of S. Also, we have

$$\begin{aligned}\left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}\right)^m \circ S \circ \left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}\right)^n &= \left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}\right)^m \circ S \circ \left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}\right)^{n-1} \circ \left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ t \widehat{\mathcal{G}^{\mathcal{H}}}\right) \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}^m \circ S \circ \widehat{\mathcal{F}^{\mathcal{H}}}^{n-1} \\ &\circ \left(\widehat{\mathcal{F}^{\mathcal{H}}} \circ t \widehat{\mathcal{G}^{\mathcal{H}}}\right) \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}^m \circ \widehat{\mathcal{G}^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}}^n \circ \widehat{\mathcal{G}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}}.\end{aligned} \tag{22}$$

Thus, $\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$. Similarly, when $\widehat{\mathcal{G}^{\mathcal{H}}} \circ \widehat{\mathcal{F}^{\mathcal{H}}} \subseteq \widehat{\mathcal{F}^{\mathcal{H}}}$, then $\widehat{\mathcal{F}^{\mathcal{H}}} \circ \widehat{\mathcal{G}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,n)}}$.

(2)Similar to (1). □

4. Hesitant Fuzzy $(m, 0)$-Ideals and Hesitant Fuzzy $(0, n)$-Ideals

Definition 4. A HFSS $\widehat{\mathcal{F}^{\mathcal{H}}}$ of S is called a hesitant fuzzy $(m, 0)$-ideal of S if

$$\widehat{\mathcal{F}^{\mathcal{H}}}_{r_1 r_2 \ldots r_m \hbar} \supseteq \widehat{\mathcal{F}^{\mathcal{H}}}_{r_1} \cap \widehat{\mathcal{F}^{\mathcal{H}}}_{r_2} \cap \cdots \cap \widehat{\mathcal{F}^{\mathcal{H}}}_{r_m}, \tag{23}$$

for all $r_1, r_2, \ldots, r_m, \hbar \in S$.

Dually, a hesitant fuzzy $(0, n)$-ideal of S can be defined.

Lemma 8. *Let $\widehat{\mathcal{F}^{\mathcal{H}}}$ be the HFS of S. Then, $\widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}_R}}$ (resp. $\widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}_L}}$) $\Rightarrow \widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(m,0)}}$ (resp. $\widehat{\mathcal{F}^{\mathcal{H}}} \in \widehat{\wp_{\mathcal{H}(0,n)}}$) $\forall m, n \in \mathbb{N}$.*

Proof. Straightforward. □

Remark 3. In general, converse of Lemma 8 does not hold.

Example 3. In Example 2, the HFS $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,0)}}, \widehat{\wp_{\mathscr{H}(0,n)}}\ \forall m,n \in \mathbb{N},\ m,n \geq 2$, but $\widehat{\mathscr{F}^{\mathscr{H}}} \notin \widehat{\wp_{\mathscr{H}_R}}, \widehat{\wp_{\mathscr{H}_L}}$

Definition 5. A semigroup S is called $(m,0)$-regular (resp. $(0,n)$-regular) if $\forall \hbar \in S\, \exists \kappa \in S$ such that $\hbar = \hbar^m\kappa\,(\text{resp.}\ \hbar = \kappa\hbar^n)$.

Lemma 9. *In S, the following assertions hold:*

(1) In $(m,0)$-regular semigroup S, $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,0)}} \Rightarrow \widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}_R}}$

(2) In $(0,n)$-regular semigroup S, $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(0,n)}} \Rightarrow \widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}_L}}$

Proof. Let $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,0)}}$ and $\hbar,\kappa \in S$. Since S is $(m,0)$-regular, $\exists \ell \in S$ such that $\hbar\kappa = \hbar^m\ell\kappa$. Therefore, we have

$$\widehat{\mathscr{F}^{\mathscr{H}}}(\hbar\kappa) = \widehat{\mathscr{F}^{\mathscr{H}}}(\hbar^m\ell\kappa) = \widehat{\mathscr{F}^{\mathscr{H}}}(\hbar^m(\ell\kappa)) \supseteq \widehat{\mathscr{F}^{\mathscr{H}}}(\hbar). \tag{24}$$

Hence, $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}_R}}$.

(2) Similar to the proof of (1). □

Lemma 10. *Let $\varnothing \neq \Omega \subseteq S$. Then, $\Omega \in \mathscr{I}_{(m,0)}$ (resp. $\Omega \in \mathscr{I}_{(0,n)}$) $\Leftrightarrow$ HFS $\widehat{\chi_\Omega^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,0)}}$ (resp. $\widehat{\chi_\Omega^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(0,n)}}$).*

Proof. ($\Rightarrow$) Let $r_1, r_2, \ldots, r_m, z \in S$. If $x_i \notin \Omega$ for any $i \in \{1,2,\ldots,m\}$, then

$$\widehat{\chi_\Omega^{\mathscr{H}}}(r_1r_2\ldots r_m z) \supseteq \widehat{\chi_\Omega^{\mathscr{H}}}(r_1) \cap \widehat{\chi_\Omega^{\mathscr{H}}}(r_2) \cap \cdots \cap \widehat{\chi_\Omega^{\mathscr{H}}}(r_m). \tag{25}$$

If $r_i \in \Omega$ for each $i \in \{1,2,\ldots,m\}$, then $r_1r_2\cdots r_m z \in \Omega^m S \subseteq \Omega$. Therefore,

$$\widehat{\chi_\Omega^{\mathscr{H}}}(r_1r_2\ldots r_m z) = 1 \supseteq \widehat{\chi_\Omega^{\mathscr{H}}}(r_1) \cap \widehat{\chi_\Omega^{\mathscr{H}}}(r_2) \cap \cdots \cap \widehat{\chi_\Omega^{\mathscr{H}}}(r_m). \tag{26}$$

Hence, $\Omega \in \mathscr{I}_{(m,0)}$.

($\Leftarrow$) Suppose that $x,z \in S$. If $x \in \Omega$, then $\widehat{\chi_\Omega^{\mathscr{H}}}(x^m z) \supseteq \widehat{\chi_\Omega^{\mathscr{H}}}(x) = 1$ implies $\widehat{\chi_\Omega^{\mathscr{H}}}(x^m z) = 1$. Therefore, $x^m z \in \Omega$. Thus, $\Omega^m S \subseteq \Omega$. □

Theorem 5. *Let $\widehat{\mathscr{G}^{\mathscr{H}}}$ be the HFS of S. Then, $[\widehat{\mathscr{G}^{\mathscr{H}}}]_T \in \mathscr{I}_{(m,0)}$ (resp. $[\widehat{\mathscr{G}^{\mathscr{H}}}]_T \in \mathscr{I}_{(m,0)}$) $\forall T \in \mathscr{P}([0,1])$, provided $[\widehat{\mathscr{G}^{\mathscr{H}}}]_T \neq \varnothing \Leftrightarrow \widehat{\mathscr{G}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,0)}}$ (resp. $\widehat{\mathscr{G}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(0,n)}}$).*

Proof. ($\Rightarrow$) Suppose that $z \in S$ and $r_1, r_2, \ldots, r_m \in \widehat{\mathscr{G}_T^{\mathscr{H}}}$, where $T \in \mathscr{P}([0,1])$. Then, $\widehat{\mathscr{G}_{r_1}^{\mathscr{H}}} \supseteq T, \widehat{\mathscr{G}_{r_2}^{\mathscr{H}}} \supseteq T, \ldots, \widehat{\mathscr{G}_{r_m}^{\mathscr{H}}} \supseteq T$. By Definition 4, $\widehat{\mathscr{G}_{r_1r_2\cdots r_m z}^{\mathscr{H}}} \supseteq \widehat{\mathscr{G}_{r_1}^{\mathscr{H}}} \cap \widehat{\mathscr{G}_{r_2}^{\mathscr{H}}} \cap \cdots \cap \widehat{\mathscr{G}_{r_m}^{\mathscr{H}}} = T$. Therefore, $r_1r_2\cdots r_m z \in [\widehat{\mathscr{G}^{\mathscr{H}}}]_T$. Hence, $[\widehat{\mathscr{G}^{\mathscr{H}}}]_T \in \mathscr{I}_{(m,n)}$.

($\Leftarrow$) Let $[\widehat{\mathscr{G}^{\mathscr{H}}}]_T \in \mathscr{I}_{(m,0)}\ \forall T \in \mathscr{P}([0,1])$ and $x,y \in S$. Suppose, to the contrary, that $\widehat{\mathscr{G}_{r_1}^{\mathscr{H}}} \cap \widehat{\mathscr{G}_{r_2}^{\mathscr{H}}} \cap \cdots \cap \widehat{\mathscr{G}_{r_m}^{\mathscr{H}}}^{\widehat{\mathscr{G}_{r_1r_2\cdots r_m z}^{\mathscr{H}}}}$. Then, there exists $C \in \mathscr{P}([0,1])$ such that $\widehat{\mathscr{G}_{r_1}^{\mathscr{H}}} \cap \widehat{\mathscr{G}_{r_2}^{\mathscr{H}}} \cap \cdots \cap \widehat{\mathscr{G}_{r_m}^{\mathscr{H}}} \supseteq C^{\widehat{\mathscr{G}_{r_1r_2\cdots r_m z}^{\mathscr{H}}}}$. This implies that $r_1, r_2, \ldots, r_m, z, \in [\widehat{\mathscr{G}^{\mathscr{H}}}]_C$, but $r_1r_2\cdots r_m z \notin [\widehat{\mathscr{G}^{\mathscr{H}}}]_C$, a contradiction. Thus, $\widehat{\mathscr{G}_{r_1r_2\cdots r_m z}^{\mathscr{H}}} \supseteq \widehat{\mathscr{G}_{r_1}^{\mathscr{H}}} \cap \widehat{\mathscr{G}_{r_2}^{\mathscr{H}}} \cap \cdots \cap \widehat{\mathscr{G}_{r_m}^{\mathscr{H}}}$ for all $r_1, r_2, \ldots, r_m, z \in S$. Hence, by Definition 4, $\widehat{\mathscr{G}^{\mathscr{H}}}$ is a hesitant fuzzy $(m,0)$-ideal of S. □

Theorem 6. *Let $\widehat{\mathscr{F}^{\mathscr{H}}}$ be any HFSS of S. Then, $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,0)}}$ (resp. $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(0,n)}}$) $\Leftrightarrow \widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{J}_S^{\mathscr{H}}} \subseteq \widehat{\mathscr{F}^{\mathscr{H}}}$ (resp. $\widehat{\mathscr{J}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n \subseteq \widehat{\mathscr{F}^{\mathscr{H}}}$).*

Proof. On the similar lines to the proof of Theorem 2. □

Lemma 11. *If S is an (m,n)-regular semigroup, then $\widehat{\mathscr{F}^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}} = \widehat{\mathscr{F}^{\mathscr{H}}}\ \forall \widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,0)}}$ and $\widehat{\mathscr{G}^{\mathscr{H}}} \circ \widehat{\mathscr{G}^{\mathscr{H}}} = \widehat{\mathscr{G}^{\mathscr{H}}}\ \forall \widehat{\mathscr{G}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(0,n)}}$.*

Proof. Let S be an (m,n)-regular semigroup and $\widehat{\mathscr{F}^{\mathscr{H}}} \in \widehat{\wp_{\mathscr{H}(m,0)}}$. Then, $\widehat{\mathscr{F}^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}} \subseteq \widehat{\mathscr{F}^{\mathscr{H}}}$. As S is (m,n)-regular, we have

$$\begin{aligned}\widehat{\mathscr{F}^{\mathscr{H}}} &\subseteq \widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{J}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n = \widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{J}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^{n-1} \circ \widehat{\mathscr{F}^{\mathscr{H}}} \\ &\subseteq \widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{J}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^{n-1} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{J}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^n \subseteq \widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{J}_S^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}}^m \circ \widehat{\mathscr{J}_S^{\mathscr{H}}} \subseteq \widehat{\mathscr{F}^{\mathscr{H}}} \circ \widehat{\mathscr{F}^{\mathscr{H}}},\end{aligned} \tag{27}$$

and so, we obtain $\widehat{\mathcal{F}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}$. Hence, $\widehat{\mathcal{F}^{\mathcal{H}}}=\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}$. Similarly, we may prove that $\widehat{\mathcal{G}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}=\widehat{\mathcal{G}^{\mathcal{H}}}\,\forall\,\widehat{\mathcal{G}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(0,n)}}$. □

Theorem 7. *The following statements hold in S:*

(1) *S is $(m,0)$-regular $\Leftrightarrow \widehat{\mathcal{F}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}$ for each HFS $\widehat{\mathcal{F}^{\mathcal{H}}}$ of S*

(2) *S is $(0,n)$-regular $\Leftrightarrow \widehat{\mathcal{F}^{\mathcal{H}}}\subseteq\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}^{n}$ for each HFS $\widehat{\mathcal{F}^{\mathcal{H}}}$ of S*

Proof

(1) $(\Rightarrow)$ Take any $\hbar\in S$. Then, $\exists\,\kappa\in S$ such that $\hbar=\hbar^m\kappa$. Now, we have

$$\begin{aligned}\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\right)(\hbar)&=\bigcup_{\hbar=rs}\left\{\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\right)(r)\cap\widehat{\mathcal{I}_S^{\mathcal{H}}}(s)\right\}\supseteq\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\right)(\hbar^m)\cap\widehat{\mathcal{I}_S^{\mathcal{H}}}(\kappa)\\&=\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\right)(\hbar^m)=\widehat{\mathcal{F}^{\mathcal{H}}}(\hbar).\end{aligned}\tag{28}$$

Therefore, $\widehat{\mathcal{F}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}$.

$(\Leftarrow)$ Let $\hbar\in S$. Since $\widehat{\chi_\hbar^{\mathcal{H}}}$ is the HFS of S, by hypothesis, $\widehat{\chi_\hbar^{\mathcal{H}}}\subseteq\widehat{\chi_\hbar^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\chi_\hbar^{\mathcal{H}}}^{n}=\mathcal{H}_{\hbar^m S\hbar^n}$. So, $\hbar\in\hbar^m S\hbar^n$, and hence, S is (m,n)-regular.

(2) Similar to the proof of (1). □

Theorem 8. *The following assertions are true in S:*

(1) *S is $(m,0)$-regular $\Leftrightarrow\widehat{\mathcal{F}^{\mathcal{H}}}=\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\,\forall\,\widehat{\mathcal{F}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(m,0)}}$*

(2) *S is $(0,n)$-regular $\Leftrightarrow\widehat{\mathcal{G}^{\mathcal{H}}}=\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}^{n}\,\forall\,\widehat{\mathcal{G}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(0,n)}}$*

Proof

(1) $(\Rightarrow)$ Let $\widehat{\mathcal{F}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(m,0)}}$. Then, by hypothesis and Theorems 7 and 6, we have $\widehat{\mathcal{F}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}$ and $\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}$. Hence, $\widehat{\mathcal{F}^{\mathcal{H}}}=\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}$.

$(\Leftarrow)$ Let R be any $(m,0)$-ideal of S, and take $a\in R$. Then, by hypothesis, we have $(\widehat{\mathcal{H}_R^m}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}})(a)=\mathcal{H}_R(a)=[0,1]$ implies $(\mathcal{H}_R^m\circ\mathcal{I}_S^{\mathcal{H}})(a)=[0,1]$. Therefore, there exist elements x,y in S with $a=xy$ such that $\mathcal{H}_R^m(x)=[0,1]$ and $\mathcal{I}_S^{\mathcal{H}}(y)=[0,1]$. As we have $\mathcal{H}_R^m(x)=[0,1]$, $\mathcal{H}_{R^m}(x)=[0,1]$, and it follows that $x\in R^m$. Since $a=xy$, therefore, $a=xy\in R^mS$. Thus, $R\subseteq R^mS$. Since R is $(m,0)$-ideal of S, $R^mS\subseteq R$. Therefore, $R=R^mS$. Hence, by Theorem 1 of [44], S is $(m,0)$-regular.

(2) Similar to the proof of (1). □

Theorem 9. *A semigroup S is (m,n)-regular $\Leftrightarrow\widehat{\mathcal{F}^{\mathcal{H}}}\cap\widehat{\mathcal{G}^{\mathcal{H}}}=\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\cap\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}^{n}\,\forall\,\widehat{\mathcal{F}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(m,0)}}$ and $\forall\,\widehat{\mathcal{G}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(0,n)}}$.*

Proof.

$(\Rightarrow)$ Let $\widehat{\mathcal{F}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(m,0)}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(0,n)}}$. As S is (m,n)-regular, we have $\widehat{\mathcal{F}^{\mathcal{H}}}\cap\widehat{\mathcal{G}^{\mathcal{H}}}\subseteq(\widehat{\mathcal{F}^{\mathcal{H}}}\cap t\widehat{\mathcal{G}^{\mathcal{H}}})^{m}\circ\widehat{\chi^{\mathcal{H}}}\circ(\widehat{\mathcal{F}^{\mathcal{H}}}\cap t\widehat{\mathcal{G}^{\mathcal{H}}})^{n}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}^{n}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{G}^{\mathcal{H}}}$,and so, $\widehat{\mathcal{F}^{\mathcal{H}}}\cap\widehat{\mathcal{G}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{G}^{\mathcal{H}}}$. Similarly, $\widehat{\mathcal{F}^{\mathcal{H}}}\cap\widehat{\mathcal{G}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}^{n}$. Thus, $\widehat{\mathcal{F}^{\mathcal{H}}}\cap\widehat{\mathcal{G}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\cap\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}^{n}$. As $\widehat{\mathcal{F}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(m,0)}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(0,n)}}$, $\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\cap\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}^{n}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\cap\widehat{\mathcal{G}^{\mathcal{H}}}$. Therefore, $\widehat{\mathcal{F}^{\mathcal{H}}}\cap\widehat{\mathcal{G}^{\mathcal{H}}}=\widehat{\mathcal{F}^{\mathcal{H}}}^{m}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\cap\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}^{n}$.

$(\Leftarrow)$ Let $a\in R\cap L$ for $R\in\mathcal{I}_{(m,0)}$ and $L\in\mathcal{I}_{(0,n)}$. By Lemma 10, $\widehat{\chi_R^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(m,0)}}$ and $\widehat{\chi_L^{\mathcal{H}}}\in\widehat{\wp_{\mathcal{H}(0,n)}}$. Therefore, by hypothesis,

$$\left(\widehat{\chi_R^{\mathcal{H}}}\cap t\widehat{\chi_L^{\mathcal{H}}}\right)(a)=\left(\widehat{\chi_R^{\mathcal{H}}}^{m}\circ\widehat{\chi_L^{\mathcal{H}}}\right)(a)\cap\left(\widehat{\chi_R^{\mathcal{H}}}\circ t\widehat{\chi_L^{\mathcal{H}}}^{n}\right)(a)=\left(\widehat{\chi_{R^m}^{\mathcal{H}}}\circ t\widehat{\chi_L^{\mathcal{H}}}\right)(a)\cap\left(\widehat{\chi_R^{\mathcal{H}}}\circ t\widehat{\chi_{L^n}^{\mathcal{H}}}\right)(a).\tag{29}$$

Since $\mathcal{H}_R(a)=[0,1]$ and $\mathcal{H}_L(a)$, $(\widehat{\chi_R^{\mathcal{H}}}\cap t\widehat{\chi_L^{\mathcal{H}}})(a)=[0,1]$, $(\widehat{\chi_{R^m}^{\mathcal{H}}}\circ t\widehat{\chi_L^{\mathcal{H}}})(a)=[0,1]$ and $(\widehat{\chi_R^{\mathcal{H}}}\circ t\widehat{\chi_{L^n}^{\mathcal{H}}})(a)=[0,1]$. This implies that there exist x, y, u, and v in S with $a=xy$ and $a=uv$ such that $\widehat{\chi_{R^m}^{\mathcal{H}}}(x)=[0,1]$, $\widehat{\chi_{L^n}^{\mathcal{H}}}(y)=[0,1]$, and $\widehat{\chi_R^{\mathcal{H}}}(u)=[0,1]$, $\widehat{\chi_{L^n}^{\mathcal{H}}}(v)=[0,1]$, and it follows that $x\in R^m$, $y\in L$ and $u\in R$, $v\in L^n$. As $a=xy$ and $a=uv$, $a=xy\in R^mL$ and $a=uv\in RL^n$ imply $a\in R^mL\cap RL^n$. Thus, we obtain

$R\cap L\subseteq R^mL\cap RL^n$. Also, $R^mL\cap RL^n\subseteq R\cap L$. Therefore, $R\cap L = R^mL\cap RL^n$. Hence, by Theorem 3 of [44], S is (m,n)-regular. □

Proposition 1. *Let* $\widehat{\mathcal{F}^{\mathcal{H}}}\in\wp_{\widehat{\mathcal{H}(m,0)}}$ *and* $\widehat{\mathcal{G}^{\mathcal{H}}}\in\wp_{\widehat{\mathcal{H}(0,n)}}$. *If* $\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}} = \widehat{\mathcal{G}^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}$, *then the product* $\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\in\wp_{\widehat{\mathcal{H}(m,n)}}$.

Proof. Let $\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}} = \widehat{\mathcal{G}^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}$. Then, we have

$$\left(\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\left(\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\right) = \widehat{\mathcal{F}^{\mathcal{H}}}\circ\left(\widehat{\mathcal{G}^{\mathcal{H}}}\circ t\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\widehat{\mathcal{F}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}. \tag{30}$$

Therefore, $\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}$ is a HFSS of S. Also, we have

$$\begin{aligned}&\left(\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\right)^m\circ\widehat{\chi^{\mathcal{H}}}\circ\left(\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\right)^n = \widehat{\mathcal{F}^{\mathcal{H}}}^m\circ\widehat{\mathcal{G}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}^n\circ\widehat{\mathcal{G}^{\mathcal{H}}}^n\\&\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}^n\circ\widehat{\mathcal{F}^{\mathcal{H}}}^n\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^m\circ\widehat{\mathcal{G}^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}^n\\&\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}},\end{aligned}\tag{31}$$

as required. □

Lemma 12. *Let* $\widehat{\mathcal{F}^{\mathcal{H}}}$ *be a HFS of* S. *Then,* $\cup_{i=1}^{m}\widehat{\mathcal{F}^{\mathcal{H}}}\cup\widehat{\mathcal{F}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\in\wp_{\widehat{\mathcal{H}(m,0)}}$ (*resp.* $\cup_{i=1}^{n}\widehat{\mathcal{F}^{\mathcal{H}}}\cup\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{F}^{\mathcal{H}}}^n\in\wp_{\widehat{\mathcal{H}(0,n)}}$).

Proof. Straightforward. □

Lemma 13. *If* S *is* (m,n)*-regular, then* $\forall\,\widehat{\mathcal{J}^{\mathcal{H}}}\in\widehat{\mathcal{I}^{\mathcal{H}}}_{(m,n)}$ *there exist* $\widehat{\mathcal{F}^{\mathcal{H}}}\in\widehat{\mathcal{I}^{\mathcal{H}}}_{(m,0)}$ *and* $\widehat{\mathcal{G}^{\mathcal{H}}}\in\widehat{\mathcal{I}^{\mathcal{H}}}_{(0,n)}$ *such that* $\widehat{\mathcal{J}^{\mathcal{H}}} = \widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}$.

Proof. Let $\widehat{\mathcal{J}^{\mathcal{H}}}\in\widehat{\mathcal{I}^{\mathcal{H}}}_{(m,n)}$. Then, $\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\subseteq\widehat{\mathcal{J}^{\mathcal{H}}}$. As S is (m,n)-regular, $\widehat{\mathcal{J}^{\mathcal{H}}}\subseteq\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n$. Therefore, $\widehat{\mathcal{J}^{\mathcal{H}}} = \widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n$. Let $\widehat{\mathcal{F}^{\mathcal{H}}} = \cup_{i=1}^{m}\widehat{\mathcal{J}^{\mathcal{H}}}^i\cup\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}} = \cup_{i=1}^{n}\widehat{\mathcal{J}^{\mathcal{H}}}^i\cup\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n$. By Lemma 12, $\widehat{\mathcal{F}^{\mathcal{H}}}\in\wp_{\widehat{\mathcal{H}(m,0)}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}}\in\wp_{\widehat{\mathcal{H}(0,n)}}$. As S is (m,n)-regular, $\widehat{\mathcal{F}^{\mathcal{H}}} = \cup_{i=1}^{m}\widehat{\mathcal{J}^{\mathcal{H}}}^i\cup\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}} = \widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}} = \cup_{i=1}^{n}\widehat{\mathcal{J}^{\mathcal{H}}}^i\cup\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n = \widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n$. Thus,

$$\begin{aligned}\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}} &= \widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\subseteq\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n = \widehat{\mathcal{J}^{\mathcal{H}}}\\ \widehat{\mathcal{J}^{\mathcal{H}}} &= \widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n = \left(\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\mathcal{I}_S^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\right)^m\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\\ &= \underbrace{\left(\left(\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\right)\circ\left(\widehat{\mathcal{J}^{\mathcal{H}}}^m\right)\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\ldots\left(\widehat{\mathcal{J}^{\mathcal{H}}}^m\right)\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\right)}_{m-\text{times}}\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\\ &\subseteq\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\circ\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\circ\ldots\circ\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\\ &= \widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\left(\widehat{\mathcal{J}^{\mathcal{H}}}^n\circ\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\circ\ldots\circ\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\circ\widehat{\chi^{\mathcal{H}}}\right)\circ\widehat{\mathcal{J}^{\mathcal{H}}}^n\\ &\subseteq\widehat{\mathcal{J}^{\mathcal{H}}}^m\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\chi^{\mathcal{H}}}\circ\widehat{\mathcal{H}}^n = \widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}},\end{aligned}\tag{32}$$

as required. □

Lemma 14. *If S is (m,n)-regular, then* $\forall\,\widehat{\mathcal{F}^{\mathcal{H}}}\in\wp_{\widehat{\mathcal{H}(m,0)}}$, *and for each HFS* $\widehat{\mathcal{G}^{\mathcal{H}}}$ *of* S, $\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\in\wp_{\widehat{\mathcal{H}(m,n)}}$.

Proof. Let $\widehat{\mathcal{F}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(m,0)}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}}$ be the HFS of S. We have

$$\begin{aligned}
\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)^{m}\circ\widehat{\mathcal{J}_{S}^{\mathcal{H}}}\circ\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)^{n} &= \underbrace{\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\ldots^{\circ}\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)}_{m-\text{times}}\circ\widehat{\mathcal{J}_{S}^{\mathcal{H}}}\circ \\
&\quad \underbrace{\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)^{\circ}\ldots^{\circ}\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)}_{n-\text{times}} \\
&= \Bigg(\widehat{\mathcal{F}^{\mathcal{H}}}\circ t\widehat{\mathcal{G}^{\mathcal{H}}})\circ\underbrace{\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\ldots^{\circ}\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)}_{\text{m}-1-\text{times}}\circ\widehat{\mathcal{J}_{S}^{\mathcal{H}}} \\
&\quad \circ\underbrace{\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)^{\circ}\ldots^{\circ}\left(\widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}\right)}_{\text{n}-1-\text{times}}\overset{\circ}{}\left(\widehat{\mathcal{F}^{\mathcal{H}}}\circ t\widehat{\mathcal{G}^{\mathcal{H}}}\right)\subseteq\left(\widehat{\mathcal{F}^{\mathcal{H}}}\circ t\widehat{\mathcal{G}^{\mathcal{H}}}\right)\circ\widehat{\mathcal{J}_{S}^{\mathcal{H}}}\circ\widehat{\mathcal{J}_{S}^{\mathcal{H}}}\circ\widehat{\mathcal{J}_{S}^{\mathcal{H}}} \\
&\quad \circ\left(\widehat{\mathcal{F}^{\mathcal{H}}}\circ t\widehat{\mathcal{G}^{\mathcal{H}}}\right)\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{J}_{S}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}^{\text{m}}\circ\widehat{\mathcal{J}_{S}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}\,(\text{by Lemma 5})\subseteq\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}}.
\end{aligned} \tag{33}$$

Therefore, $\widehat{\mathcal{F}^{\mathcal{H}}}\circ\widehat{\mathcal{G}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(\text{m,n})}}$.

By Lemmas 13 and 14, we have the following. □

Corollary 1. *If S is (m,n)-regular, then $\widehat{\mathcal{J}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(m,n)}} \Leftrightarrow$ there exist $\widehat{\mathcal{F}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(m,0)}}$ and $\widehat{\mathcal{G}^{\mathcal{H}}} \in \wp_{\widehat{\mathcal{H}(0,n)}}$ such that $\widehat{\mathcal{J}^{\mathcal{H}}} = \widehat{\mathcal{F}^{\mathcal{H}}}^{\circ}\widehat{\mathcal{G}^{\mathcal{H}}}$.*

5. Conclusion

The principal objective of this paper is to establish the notions of the hesitant fuzzy - ideal, hesitant fuzzy $(m,0)$-ideal, and hesitant fuzzy $(0,n)$-ideal and to improve the understanding of various semigroup classes through the use of these notions. In particular, if we take $m = 1 = n$ in the hesitant fuzzy (m,n)-ideal, hesitant fuzzy $(m,0)$-ideal, and hesitant fuzzy $(0,n)$-ideal, then we get the hesitant fuzzy bi-ideal, hesitant fuzzy right ideal, and hesitant fuzzy left ideal. The concepts presented in this paper are therefore more general. Furthermore, if we put $m = 1 = n$ in the results of this paper, then most of the results of the paper [?] are deduced as corollaries which are the key application of the findings of this paper and a proof of the genuineness of the notions presented in this paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript.

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