Hindawi
Journal of Mathematics
Volume 2020, Article ID 8857459, 10 pages
https://doi.org/10.1155/2020/8857459

Research Article

Picture Fuzzy Ideals of Near-Rings

Awais Asif,[1] Hassen Aydi,[2,3,4,5] Muhammad Arshad,[1] Abdul Rehman,[6] and Usman Tariq[7]

[1]*Department of Math & Stats, International Islamic University Islamabad, Pakistan*
[2]*Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[3]*Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[4]*Universit'e de Sousse, Institut Sup'erieur d'Informatique et des Techniques de Communication, Hammam Sousse 4000, Tunisia*
[5]*China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[6]*Department of Computer Science and Engineering, Kyungpook National University, Daegu 41566, Republic of Korea*
[7]*Department of Computer Science, Nankai University Tianjin, Tianjin 300071, China*

Correspondence should be addressed to Hassen Aydi; hassen.aydi@tdtu.edu.vn

Received 24 September 2020; Revised 24 October 2020; Accepted 27 October 2020; Published 24 December 2020

Academic Editor: Heng Liu

A picture fuzzy set (PFS) is an augmentation of Atanassov's intuitionistic fuzzy set (IFS). The PFS-based models are useful in the circumstances when we face uncertain and vague information, especially in the case when we need more answers of the form "indeed," "avoid," "no," and "refusal." It has been considered as an essential tool to deal with unsure data during an investigation. In this manuscript, we explore the idea of a picture fuzzy near-ring (PFNR) and a picture fuzzy ideal (PFI) of a near-ring (NR). We illustrate some basic properties such as union, intersection, homomorphic image, and preimage of PFIs of a NR. Furthermore, there is discussion about the direct product of PFIs of a NR.

1. Introduction

Fuzzy set (FS) was first introduced by Zadeh [1]. It discusses the grade of truth values belonging to a unit interval. FS is a valuable technique to cope with vague and difficult information in a real-life decision. It has received extensive attention from researchers and has been utilized in decision-making in many ways (see [2, 3]). However, in some cases, FS cannot work reliably. For coping such types of issues, Atanassov [4] explored the intuitionistic fuzzy set (IFS) to deal with more complex problems. IFS contains truth and falsity grades, whose sum is belonging to a unit interval. IFS is a more modified version of a FS, and many researchers have used it in decision-making [5], such as medical diagnosis [6] and pattern recognition [7]. Although FS and IFS are applicable in various fields, however, there exist problems when a decision maker face various kinds of opinions of human beings such as "yes," "abstinence," "no," and "refusal." To coping with such kinds of problems, Cuong [8] explored the concept of a picture fuzzy set (PFS), containing the grades of truth, abstinence, falsity, and refusal, whose sum is belonging to a unit interval. To deal with uncertain and difficult information in real-life decisions, PFS is more effective than the IFS and FS. Joshi [9] presented a new setting as criteria of fuzzy entropy for the PFS and proposed a picture fuzzy information model based on Tsallis–Havrda–Charvat entropy. Qyias et al. [10] established aggregation operators based on linguistic PFSs and their application in decision-making problems, while Ganie et al. [11] examined the new correlation coefficient for the PFS.

The concept of an ideal in the frame of a near-ring is discussed by numerous scholars by using intuitionistic fuzzy sets. To investigate complicated and awkward information, a picture fuzzy set is a more influential tool than the fuzzy set and intuitionistic fuzzy sets. Thus, the concept of a picture fuzzy ideal of near-rings is presented in this article. Our results clearly show that the idea of a picture fuzzy set is more proficient and reliable than the aforementioned concepts. The notion of fuzzy subnear-rings (FSNR) and fuzzy ideals (FI) of a NR was introduced by Abou-Zaid [12]. This concept was further discussed by Kim [13] and Dutta and Biswas [14]. Chinnadurai and Kadalarasi [15] discussed the direct product of n ($n = 1, 2, \ldots, k$) FSNR, FI, and fuzzy R-subgroups. The advantages of a PFS became the motivation of our main theorems. The results presented are entirely

new and more beneficial than the existing results in the literature. The aims of this manuscript are given as follows:

(1) To explore the idea of a picture fuzzy near-ring (PFNR) and a picture fuzzy ideal (PFI) of a near-ring (NR)

(2) To describe some basic properties such as union, intersection, homomorphic image, and preimage of PFIs of a NR

(3) To elaborate the direct product of PFIs of a NR

The summary of this manuscript is as follows: in Section 2, we review some notions such as groups, subgroups, ideals, and fuzzy sets. In Section 3, we explore the idea of a picture fuzzy near-ring (PFNR) and a picture fuzzy ideal (PFI) of a near-ring (NR). In Section 4, we investigate some basic properties such as union, intersection, homomorphic image, and preimage of a PFI of a NR. In Section 5, we discuss about the direct product of PFIs of a NR. The conclusion of this manuscript is discussed in Section 6.

2. Preliminaries

The aim of this study is to recall some basic notions such as near-rings, ideals of a near-ring, fuzzy sub-near-rings, fuzzy ideals, and picture fuzzy sets.

Definition 1 (see [13]). For any nonempty set E with two binary operations "+" and "·," we say that E is called a near-ring if the following conditions hold:

(i) $(E,+)$ is a group

(ii) $(E,\cdot)$ is a semigroup

(iii) $((h+k)\cdot t = h\cdot t + k\cdot t)$ for $h,k,t \in E$

Definition 2 (see [14]). An ideal of a NR E is a subgroup A of E such that

(i) $(A,+)$ is a normal subgroup of $(E,+)$,

(ii) $AE \subseteq A$,

(iii) $(k(h+i) - kh \in A)$, for $h,k \in E$ and $i \in A$,

where A is a left ideal of E whenever A satisfies (i) and (ii) and a right ideal of E if A satisfies (i) and (iii).

Definition 3 (see [16]). A FS in a NR E is said to be a fuzzy sub-near-ring of E if

(i) $B(h-k) \geq \min(B(h), B(k))$, for all $h,k \in E$

(ii) $B(hk) \geq \min(B(h), B(k))$, for all $h,k \in E$

Definition 4 (see [16]). A FS B in a NR E is said to be a fuzzy ideal of E if

(i) $B(h-k) \geq \min(B(h), B(k))$, for all $h,k \in E$

(ii) $B(k+h-k) \geq B(h)$, for all $h,k \in E$

(iii) $B(hk) \geq B(k)$, for all $h,k \in E$

(iv) $B((h+t)k - hk) \geq B(t)$, for all $h,k \in E$

A fuzzy set satisfying (i)–(iii) is called a fuzzy left ideal of E, whereas a fuzzy set with (i) and (ii) is called a fuzzy right ideal of E.

Definition 5 (see [8]). A PFS P on the universe of discourse U characterized by a truth membership function $B_P(h)$, an indeterminacy function $A_P(h)$, and a falsity membership function $F_P(h)$ is defined as

$$P = \{\langle h, B_P(h), A_P(h), F_P(h)\rangle : h \in U\}, \tag{1}$$

where $B_P, A_P, F_P : U \longrightarrow [0,1]$ and $0 \leq B_P(h) + A_P(h) + F_P(h) \leq 1$.

Definition 6 (see [8]). Let P and Q be PFSs of U. Then,

$$P \cup Q = \{\langle h, B_{P\cup Q}(h), A_{P\cup Q}(h), F_{P\cup Q}(h)\rangle\}, \tag{2}$$

where

$$\begin{aligned} B_{P\cup Q}(h) &= \max(B_P(h), B_Q(h)), \\ A_{P\cup Q}(h) &= \min(A_P(h), A_Q(h)), \\ F_{P\cup Q}(h) &= \min(F_P(h), F_Q(h)), \end{aligned} \tag{3}$$

for all $h \in U$.

$$P \cap Q = \{\langle h, B_{P\cap Q}(h), A_{P\cap Q}(h), F_{P\cap Q}(h)\rangle\}, \tag{4}$$

where

$$\begin{aligned} B_{P\cap Q}(h) &= \min(B_P(h), B_Q(h)), \\ A_{P\cap Q}(h) &= \max(A_P(h), A_Q(h)), \\ F_{P\cap Q}(h) &= \max(F_P(h), F_Q(h)), \end{aligned} \tag{5}$$

for all $h \in U$.

3. Picture Fuzzy Ideals of Near-Rings

The aim of this study is to explore the idea of a PFNR and a PFI of a NR.

Definition 7. A PFS P in a NR E is called a PFSNR of E if

(i) $B_P(h-k) \geq \min(B_P(h), B_P(k))$, $A_P(h-k) \leq \max(A_P(h), A_P(k))$, $F_P(h-k) \leq \max(F_P(h), F_P(k))$.

(ii) $B_P(hk) \geq \min(B_P(h), B_P(k))$, $A_P(hk) \leq \max(A_P(h), A_P(k))$ and $F_P(hk) \leq \max(F_P(h), F_P(k))$.

Definition 8. Let E be a NR. A PFS P in a NR E is called a PFS of E if

(i) $B_P(h-k) \geq \min(B_P(h), B_P(k))$, $A_P(h-k) \leq \max(A_P(h), A_P(k))$, and $F_P(h-k) \leq \max(F_P(h), F_P(h))$.

(ii) $B_P(k+h-k) \geq B_P(h)$, $A_P(k+h-k) \leq A_P(h)$ and $F_P(k+h-k) \leq F_P(h)$.

(iii) $B_P(hk) \geq B_P(k)$, $A_P(hk) \leq A_P(k)$, and $F_P(hk) \leq F_P(k)$.

(iv) $B_P((h+t)k-hk) \geq B_P(t)$, $A_P((h+t)k-hk) \leq A_P(t)$, and $F_P((h+t)k-hk) \leq F_P(t)$.

A picture fuzzy subset with the above conditions (i)–(iii) is called a picture fuzzy left ideal of E, whereas a picture fuzzy subset with (i), (ii), and (iv) is called a picture fuzzy right ideal of E.

Theorem 1. *Let P and Q be PFIs of E. If $P \subset Q$, then $P \cup Q$ is a PFI of E.*

Proof. Let P and Q be PFIs of E. Let $h, k, t \in E$; then,

$$\begin{aligned} B_{P\cup Q}(h-k) &= \max\left(B_P(h-k), B_Q(h-k)\right) \\ &\geq \max\left(\min\left(B_P(h), B_P(k)\right), \min\left(B_Q(h), B_Q(k)\right)\right) \\ &= \min\left\{\max\left(B_P(h), B_Q(h)\right), \max\left(B_P(k), B_Q(k)\right)\right\} \\ &= \min\left(B_{P\cup Q}(h), B_{P\cup Q}(k)\right). \end{aligned} \tag{6}$$

Similarly, for the abstinence degree, we write

$$\begin{aligned} A_{P\cup Q}(h-k) &= \min\left(A_P(h-k), A_Q(h-k)\right) \\ &\leq \min\left(\max\left(A_P(h), A_P(k)\right)\right), \max\left(A_Q(h), A_Q(k)\right) \\ &= \max\left\{\min\left(A_P(h), A_Q(h)\right), \max\left(A_P(k), A_Q(k)\right)\right\} \\ &= \max\left(A_{P\cup Q}(h), A_{P\cup Q}(k)\right), \end{aligned} \tag{7}$$

and for falsity grade, we have

$$\begin{aligned} F_{P\cup Q}(h-k) &= \min\left(F_P(h-k), F_Q(h-k)\right), \\ &\leq \min\left(\max\left(F_P(h), F_P(k)\right)\right), \max\left(F_Q(h), F_Q(k)\right) \\ &= \max\left\{\min\left(F_P(h), F_Q(h)\right), \max\left(F_P(k), F_Q(k)\right)\right\} \\ &= \max\left(F_{P\cup Q}(h), F_{P\cup Q}(k)\right). \end{aligned} \tag{8}$$

Next, we write

$$\begin{aligned} B_{P\cup Q}(k+h-k) &= \max\left(B_P(k+h-k), B_Q(k+h-k)\right), \\ &\geq \max\left(B_P(h), B_Q(h)\right), \\ &= B_{P\cup Q}(h). \end{aligned} \tag{9}$$

For the abstinence degree, we obtain

$$\begin{aligned} A_{P\cup Q}(k+h-k) &= \min\left(A_P(k+h-k), A_Q(k+h-k)\right), \\ &\leq \min\left(A_P(h), A_Q(h)\right) \\ &= A_{P\cup Q}(h), \end{aligned} \tag{10}$$

and for falsity grade, we get

$$\begin{aligned} F_{P\cup Q}(k+h-k) &= \min\left(F_P(k+h-k), F_Q(k+h-k)\right), \\ &\leq \min\left(F_P(h), F_Q(h)\right), \\ &= F_{P\cup Q}(h). \end{aligned} \tag{11}$$

Furthermore, we deduce that

$$\begin{aligned} B_{P\cup Q}(hk) &= \max\left(B_P(hk), B_Q(hk)\right) \geq \max\left(B_P(k), B_Q(k)\right) \\ &= B_{P\cup Q}(k), \\ A_{P\cup Q}(hk) &= \min\left(A_P(hk), A_Q(hk)\right) \leq \min\left(A_P(k), A_Q(k)\right) \\ &= A_{P\cup Q}(k), \\ F_{P\cup Q}(hk) &= \min\left(F_P(hk), F_Q(hk)\right) \leq \min\left(F_P(k), F_Q(k)\right) \\ &= F_{P\cup Q}(k). \end{aligned} \tag{12}$$

At last, we obtain

$$\begin{aligned} B_{P\cup Q}((h+t)k-hk) &= \max\left\{B_P((h+t)k-hk), B_Q((h+t)k-hk)\right\} \\ &\geq \max\left(B_P(t), B_Q(t)\right) = B_{P\cup Q}(t), \\ A_{P\cup Q}((h+t)k-hk) &= \min\left\{A_P((h+t)k-hk), A_Q((h+t)k-hk)\right\} \\ &\leq \min\left(A_P(t), A_Q(t)\right) = A_{P\cup Q}(t), \\ B_{P\cup Q}((h+t)k-hk) &= \min\left\{F_P((h+t)k-hk), F_Q((h+t)k-hk)\right\} \\ &\leq \min\left(F_P(t), F_Q(t)\right) = F_{P\cup Q}(t). \end{aligned} \tag{13}$$

Therefore, $P \cup Q$ is a PFI of E. □

Theorem 2. *Let P and Q be PFIs of E. If $P \subset Q$, then $P \cap Q$ is a PFI of E.*

Proof. Let P and Q be PFIs of E. Let $h, k, t \in E$. Then, the following are obtained.

For truth grade, we get

$$\begin{aligned} B_{P\cap Q}(h-k) &= \min\left(B_P(h-k), B_Q(h-k)\right) \\ &\geq \min\left(\min\left(B_P(h), B_P(k)\right), \min\left(B_Q(h), B_Q(k)\right)\right) \\ &= \min\left\{\min\left(B_P(h), B_Q(h)\right), \min\left(B_P(k), B_Q(k)\right)\right\} \\ &= \min\left(B_{P\cap Q}(h), B_{P\cap Q}(k)\right). \end{aligned} \tag{14}$$

Similarly, for abstinence grade, we write

$$\begin{aligned} A_{P\cap Q}(h-k) &= \max\left(A_P(h-k), A_Q(h-k)\right) \\ &\leq \max\left(\max\left(A_P(h), A_P(k)\right), \max\left(A_Q(h), A_Q(k)\right)\right) \\ &= \max\left\{\max\left(A_P(h), A_Q(h)\right), \max\left(A_P(k), A_Q(k)\right)\right\} \\ &= \max\left(A_{P\cap Q}(h), A_{P\cap Q}(k)\right), \end{aligned} \tag{15}$$

and for falsity grade, we obtain

$$\begin{aligned} F_{P\cap Q}(h-k) &= \max\left(F_P(h-k), F_Q(h-k)\right) \\ &\le \max\left(\max\left(F_P(h), F_P(k)\right)\right), \max\left(F_Q(h), F_Q(k)\right) \\ &= \max\left\{\max\left(F_P(h), F_Q(h)\right), \max\left(F_P(k), F_Q(k)\right)\right\}, \\ &= \max\left(F_{P\cap Q}(h), F_{P\cap Q}(k)\right). \end{aligned} \tag{16}$$

Next, we obtain

$$\begin{aligned} B_{P\cap Q}(k+h-k) &= \min\left(B_P(k+h-k), B_Q(k+h-k)\right) \\ &\ge \min\left(B_P(h), B_Q(h)\right) \\ &= B_{P\cap Q}(h). \end{aligned} \tag{17}$$

Similarly,

$$\begin{aligned} A_{P\cap Q}(k+h-k) &= \max\left(A_P(k+h-k), A_Q(k+h-k)\right) \\ &\le \max\left(A_P(h), A_Q(h)\right) \\ &= A_{P\cap Q}(h), \\ F_{P\cap Q}(k+h-k) &= \max\left(F_P(k+h-k), F_Q(k+h-k)\right) \\ &\le \max\left(F_P(h), F_Q(h)\right) \\ &= F_{P\cap Q}(h). \end{aligned} \tag{18}$$

Furthermore, we deduce that

$$\begin{aligned} B_{P\cap Q}(hk) &= \min\left(B_P(hk), B_Q(hk)\right) \ge \min\left(B_P(k), B_Q(k)\right) \\ &= B_{P\cap Q}(k), \\ A_{P\cap Q}(hk) &= \max\left(A_P(hk), A_Q(hk)\right) \le \max\left(A_P(k), A_Q(k)\right) \\ &= A_{P\cap Q}(k), \\ F_{P\cap Q}(hk) &= \max\left(F_P(hk), F_Q(hk)\right) \le \max\left(F_P(k), F_Q(k)\right) \\ &= F_{P\cap Q}(k). \end{aligned} \tag{19}$$

Finally, we conclude that

$$\begin{aligned} B_{P\cap Q}((h+t)k-hk) &= \min\left\{B_P((h+t)k-hk), B_Q((h+t)k-hk)\right\} \\ &\ge \min\left(B_P(t), B_Q(t)\right) = B_{P\cap Q}(t), \\ A_{P\cap Q}((h+t)k-hk) &= \max\left\{A_P((h+t)k-hk), A_Q((h+t)k-hk)\right\} \\ &\le \max\left(A_P(t), A_Q(t)\right) = A_{P\cap Q}(t), \\ B_{P\cap Q}((h+t)k-hk) &= \max\left\{F_P((h+t)k-hk), F_Q((h+t)k-hk)\right\} \\ &\le \max\left(F_P(t), F_Q(t)\right) = F_{P\cap Q}(t). \end{aligned} \tag{20}$$

Therefore, $P\cap Q$ is a PFI of E. □

Corollary 1. *If $(P_1, P_2, \ldots, P_n)$ are PFIs of E, then $P = \cap_{i=1}^{n} P_i$ is a PFI of E.*

Lemma 1. *For all $h, b \in A$ and i is any positive integer, if $h = b$, then*

(1) $h^i \le b^i$.

(2) $[\min(h, b)]^i = \min(h^i, b^i)$.

(3) $[\max(h, b)]^i = \max(h^i, b^i)$.

Theorem 3. *Let P be a PFI of E. Then,*

$$P^m = \left\{\langle h, B_{P^m}(h), A_{P^m}(h), F_{P^m}(h)\rangle : h \in E\right\} \tag{21}$$

is a PFI of E, where m is a positive integer and $B_{P^m}(h) = (B_P(h))^m$, $A_{P^m}(h) = (A_P(h))^m$, and $F_{P^m}(h) = (F_P(h))^m$.

Proof. Let P be a PFI of E. Let $h, k, t \in E$. Then, the following are observed.

For truth grade, we can write

$$\begin{aligned} B_{P^m}(h-k) &= \left(B_P(h-k)\right)^m \ge \left(\min\left(B_P(h), B_P(k)\right)\right)^m \\ &= \min\left(\left(B_P(h)\right)^m, \left(B_P(k)\right)^m\right) \\ &= \min\left(B_{P^m}(h), B_{P^m}(k)\right). \end{aligned} \tag{22}$$

Similarly, for abstinence grade, we get

$$\begin{aligned} A_{P^m}(h-k) &= \left(A_P(h-k)\right)^m \le \left(\max\left(A_P(h), A_P(k)\right)\right)^m \\ &= \max\left(\left(A_P(h)\right)^m, \left(A_P(k)\right)^m\right) \\ &= \max\left(A_{P^m}(h), A_{P^m}(k)\right), \end{aligned} \tag{23}$$

and for falsity grade, we obtain the following:

$$\begin{aligned} F_{P^m}(h-k) &= \left(F_P(h-k)\right)^m \le \left(\max\left(F_P(h), F_P(k)\right)\right)^m \\ &= \max\left(\left(F_P(h)\right)^m, \left(F_P(k)\right)^m\right) \\ &= \max\left(F_{P^m}(h), F_{P^m}(k)\right). \end{aligned} \tag{24}$$

Next, it is obtained that

$$\begin{aligned} B_{P^m}(k+h-k) &= \left(B_P(k+h-k)\right)^m \ge \left(B_P(h)\right)^m = B_{P^m}(h), \\ A_{P^m}(k+h-k) &= \left(A_P(k+h-k)\right)^m \le \left(A_P(h)\right)^m = A_{P^m}(h), \\ F_{P^m}(k+h-k) &= \left(F_P(k+h-k)\right)^m \le \left(F_P(h)\right)^m = F_{P^m}(h). \end{aligned} \tag{25}$$

Also, we examine that

$$\begin{aligned} B_{P^m}(hk) &= \left(B_P(hk)\right)^m \ge \left(B_P(k)\right)^m = B_{P^m}(k), \\ A_{P^m}(hk) &= \left(A_P(hk)\right)^m \le \left(A_P(k)\right)^m = A_{P^m}(k), \\ F_{P^m}(hk) &= \left(F_P(hk)\right)^m \le \left(F_P(k)\right)^m = F_{P^m}(k). \end{aligned} \tag{26}$$

At last, we write that

$$\begin{aligned} B_{P^m}((h+t)k-hk) &= \left(B_P((h+t)k-hk)\right)^m \geq \left(B_P(t)\right)^m \\ &= B_{P^m}(t), \\ A_{P^m}((h+t)k-hk) &= \left(A_P((h+t)k-hk)\right)^m \leq \left(A_P(t)\right)^m \\ &= A_{P^m}(t), \\ F_{P^m}((h+t)k-hk) &= \left(F_P((h+t)k-hk)\right)^m \leq \left(F_P(t)\right)^m \\ &= F_{P^m}(t). \end{aligned} \tag{27}$$

Therefore, P^m is a PFI of E^m. □

4. Direct Product of Picture Fuzzy Ideals of Near-Rings

This section throws light on some basic properties such as union, intersection, homomorphic image, and preimage of PFIs of a NR.

Definition 9. Let P and Q be picture fuzzy subsets of near-rings (PFSSNR) E_1 and E_2, respectively. Then, the direct product of PFSSNRs is defined by $P \times Q$: $E_1 \times E_2 \longrightarrow [0,1]$ such that

$$\begin{aligned} P \times Q = \{ &\langle (h,k), B_{P\times Q}(h,k), A_{P\times Q}(h,k), F_{P\times Q}(h,k)\rangle : \\ &h \in E_1, k \in E_2\}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} B_{P\times Q}(h,k) &= \min\left(B_P(h), B_Q(k)\right), \\ A_{P\times Q}(h,k) &= \max\left(A_P(h), A_Q(k)\right), \\ F_{P\times Q}(h,k) &= \max\left(F_P(h), F_Q(k)\right). \end{aligned} \tag{29}$$

Example 1. Suppose $R = \{a,b,c,d\}$ is a near-ring with binary operations "+" and "·," defined by

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	b	a

and

–	a	b	c	D
a	a	a	a	A
b	a	a	a	A
c	a	a	a	a
d	a	a	b	b

respectively. Suppose PFSs $P = (\alpha_P, \beta_P, \gamma_P)$ and $Q = (\alpha_Q, \beta_Q, \gamma_Q)$ defined by

$$\begin{aligned} &\alpha_P(a) = \alpha_Q(a) = 0.7, \alpha_P(b) = \alpha_Q(b) = 0.5, \\ &\alpha_P(c) = \alpha_P(d) = \alpha_Q(c) = \alpha_Q(d) = 0.3, \\ &\beta_P(a) = \beta_Q(a) = 0.2, \beta_P(b) = \beta_Q(b) = 0.3, \\ &\beta_P(c) = \beta_P(d) = \beta_Q(c) = \beta_Q(d) = 0.6, \\ &\gamma_P(a) = \gamma_Q(a) = 0.02, \gamma_P(b) = \gamma_Q(b) = 0.03, \\ &\gamma_P(c) = \gamma_P(d) = \gamma_Q(c) = \gamma_Q(d) = 0.07. \end{aligned} \tag{30}$$

Then, we say that $P = (\alpha_P, \beta_P, \gamma_P)$ and $Q = (\alpha_Q, \beta_Q, \gamma_Q)$ are PFI of R. Now,

$$P \times Q = \{(0.7, 0.2, 0.02), (0.5, 0.3, 0.03), (0.3, 0.6, 0.06)\}. \tag{31}$$

Note that this example can be extended to construct an example for Definition 10.

Definition 10. Let P and Q be PFSSNRs E_1 and E_2, respectively. Then, $P \times Q$ is a PFI of $E_1 \times E_2$ if it satisfies the following conditions:

$$\begin{aligned} B_{P\times Q}((h_1,h_2)-(k_1,k_2)) &\geq \min\left(B_{P\times Q}(h_1,h_2), B_{P\times Q}(k_1,k_2)\right), \\ A_{P\times Q}((h_1,h_2)-(k_1,k_2)) &\leq \max\left(A_{P\times Q}(h_1,h_2), A_{P\times Q}(k_1,k_2)\right), \\ F_{P\times Q}((h_1,h_2)-(k_1,k_2)) &\leq \max\left(F_{P\times Q}(h_1,h_2), F_{P\times Q}(k_1,k_2)\right), \\ B_{P\times Q}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) &\geq \min\left(B_{P\times Q}(h_1,h_2)\right), \\ A_{P\times Q}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) &\leq \max\left(A_{P\times Q}(h_1,h_2)\right), \\ F_{P\times Q}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) &\leq \max\left(F_{P\times Q}(h_1,h_2)\right), \\ B_{P\times Q}((h_1,h_2)(k_1,k_2)) &\geq \left(B_{P\times Q}(k_1,k_2)\right), \\ A_{P\times Q}((h_1,h_2)(k_1,k_2)) &\leq \left(A_{P\times Q}(k_1,k_2)\right), \\ F_{P\times Q}((h_1,h_2)(k_1,k_2)) &\leq \left(F_{P\times Q}(k_1,k_2)\right), \\ B_{P\times Q}([(h_1,h_2)+(t_1,t_2)](k_1,k_2)-(h_1,h_2)(k_1,k_2)) &\geq B_{P\times Q}(t_1,t_2), \\ A_{P\times Q}([(h_1,h_2)+(t_1,t_2)](k_1,k_2)-(h_1,h_2)(k_1,k_2)) &\leq A_{P\times Q}(t_1,t_2), \\ F_{P\times Q}([(h_1,h_2)+(t_1,t_2)](k_1,k_2)-(h_1,h_2)(k_1,k_2)) &\leq F_{P\times Q}(t_1,t_2). \end{aligned} \tag{32}$$

Theorem 4. *Let P and Q be PFIs of E_1 and E_2, respectively. Then, $P \times Q$ is a PFI of $E_1 \times E_2$.*

Proof. Let P and Q be PFIs of E_1 andE_2, respectively. Let $(h_1, h_2), (k_1, k_2), (t_1, t_2) \in E_1 \times E_2$. Then, the following are obtained.

For truth grade, we obtain the following:

$$\begin{aligned} B_{P\times Q}((h_1,h_2)-(k_1,k_2)) &= B_{P\times Q}(h_1-k_1,h_2-k_2),\\ &= \min\left(B_P(h_1-k_1),B_Q(h_2-k_2)\right),\\ &\geq \min\left\{\min\left[B_P(h_1),B_P(k_1)\right],\ \min\left[B_Q(h_2),B_Q(k_2)\right]\right\},\\ &= \min\left\{\min\left[B_P(h_1),B_Q(h_1)\right],\ \min\left[B_P(k_1),B_Q(k_2)\right]\right\},\\ &= \min\left(B_{P\times Q}(h_1,h_2),B_{P\times Q}(k_1,k_2)\right). \end{aligned} \tag{33}$$

Similarly, for abstinence grade, we write

$$\begin{aligned} A_{P\times Q}((h_1,h_2)-(k_1,k_2)) &= A_{P\times Q}(h_1-k_1,h_2-k_2),\\ &= \max\left(A_P(h_1-k_1),A_Q(h_2-k_2)\right),\\ &\leq \max\left\{\max\left[A_P(h_1),A_P(k_1)\right],\ \max\left[A_Q(h_2),A_Q(k_2)\right]\right\},\\ &= \max\left\{\max\left[A_P(h_1),A_Q(h_2)\right],\ \max\left[A_P(k_1),A_Q(k_2)\right]\right\},\\ &= \max\left(A_{P\times Q}(h_1,h_2),A_{P\times Q}(k_1,k_2)\right), \end{aligned} \tag{34}$$

and for falsity grade, we get

$$\begin{aligned} F_{P\times Q}((h_1,h_2)-(k_1,k_2)) &= F_{P\times Q}(h_1-k_1,h_2-k_2),\\ &= \max\left(F_P(h_1-k_1),F_Q(h_2-k_2)\right),\\ &\leq \max\left\{\max\left[F_P(h_1),F_P(k_1)\right],\ \max\left[F_Q(h_2),F_Q(k_2)\right]\right\},\\ &= \max\left\{\max\left[F_P(h_1),F_Q(h_2)\right],\ \max\left[F_P(k_1),F_Q(k_2)\right]\right\},\\ &= \max\left(F_{P\times Q}(h_1,h_2),F_{P\times Q}(k_1,k_2)\right). \end{aligned} \tag{35}$$

Next, it is clear that

$$\begin{aligned} B_{P\times Q}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) &= B_{P\times Q}(k_1+h_1-k_1,k_2+h_2-k_2),\\ &= \min\left(B_P(k_1+h_1-k_1),B_Q(k_2+h_2-k_2)\right) \geq \min\left(B_P(h_1),B_Q(h_2)\right),\\ &= B_{P\times Q}((h_1,h_2)). \end{aligned} \tag{36}$$

Similarly,

$$\begin{aligned} A_{P\times Q}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) &= A_{P\times Q}(k_1+h_1-k_1,k_2+h_2-k_2),\\ &= \max\big(A_P(k_1+h_1-k_1),A_Q(k_2+h_2-k_2)\big)\le \max\big(A_P(h_1),A_Q(h_2)\big),\\ &= A_{P\times Q}((h_1,h_2)),\\ F_{P\times Q}((k_1,k_2)+(h_1,h_2)-(k_1,k_2)) &= F_{P\times Q}(k_1+h_1-k_1,k_2+h_2-k_2),\\ &= \max\big(F_P(k_1+h_1-k_1),F_Q(k_2+h_2-k_2)\big)\le \max\big(F_P(h_1),F_Q(h_2)\big),\\ &= F_{P\times Q}((h_1,h_2)). \end{aligned} \tag{37}$$

Moreover, we deduce the following inequalities:

$$\begin{aligned} B_{P\times Q}((h_1,h_2)(k_1,k_2)) &= B_{P\times Q}(h_1k_1,h_2k_2)\\ &= \min\big(B_P(h_1k_1),B_Q(h_2k_2)\big)\\ &\ge \min\big(B_P(k_1),B_Q(k_2)\big)\\ &= B_{P\times Q}(k_1,k_2). \end{aligned} \tag{38}$$

Similarly,

$$\begin{aligned} A_{P\times Q}((h_1,h_2)(k_1,k_2)) &= A_{P\times Q}(h_1k_1,h_2k_2)\\ &= \max\big(A_P(h_1k_1),A_Q(h_2k_2)\big)\\ &\le \max\big(A_P(k_1),A_Q(k_2)\big)\\ &= A_{P\times Q}(k_1,k_2),\\ F_{P\times Q}((h_1,h_2)(k_1,k_2)) &= F_{P\times Q}(h_1k_1,h_2k_2)\\ &= \max\big(F_P(h_1k_1),F_Q(h_2k_2)\big)\\ &\le \max\big(F_P(k_1),F_Q(k_2)\big)\\ &= F_{P\times Q}(k_1,k_2). \end{aligned} \tag{39}$$

Finally, we prove that

$$\begin{aligned} &B_{P\times Q}((h_1,h_2)+(t_1,t_2)-(k_1,k_2)-(h_1,h_2)(k_1,k_2))\\ &\quad= B_{P\times Q}([h_1+t_1]k_1-h_1k_1,[h_2+t_2]k_2-h_2k_2)\\ &\quad= \min\big(B_P([h_1+t_1]k_1-h_1k_1),B_Q([h_2+t_2]k_2-h_2k_2)\big)\\ &\quad\ge \min\big(B_P(t_1),B_Q(t_2)\big)=B_{P\times Q}(t_1,t_2). \end{aligned} \tag{40}$$

Also,

$$\begin{aligned} &A_{P\times Q}((h_1,h_2)+(t_1,t_2)-(k_1,k_2)-(h_1,h_2)(k_1,k_2))\\ &\quad= A_{P\times Q}([h_1+t_1]k_1-h_1k_1,[h_2+t_2]k_2-h_2k_2)\\ &\quad= \max\big((A_P[h_1+t_1]k_1-h_1k_1),(A_Q[h_2+t_2]k_2-h_2k_2)\big)\\ &\quad\le \max\big(A_P(t_1),A_Q(t_2)\big)=A_{P\times Q}(t_1,t_2),\\ &F_{P\times Q}((h_1,h_2)+(t_1,t_2)-(k_1,k_2)-(h_1,h_2)(k_1,k_2))\\ &\quad= F_{P\times Q}([h_1+t_1]k_1-h_1k_1,[h_2+t_2]k_2-h_2k_2)\\ &\quad= \max\big((F_P[h_1+t_1]k_1-h_1k_1),(F_Q[h_2+t_2]k_2-h_2k_2)\big)\\ &\quad\le \max\big(F_P(t_1),F_Q(t_2)\big)=F_{P\times Q}(t_1,t_2). \end{aligned} \tag{41}$$

Therefore, $P\times Q$ is a PFI of $E_1\times E_2$. □

5. Homomorphism of Picture Fuzzy Ideals of Near-Rings

This section is concerned with the direct product of PFIs of a NR.

Definition 11. Let R and S be two NRs. Then, the mapping $f: R\longrightarrow S$ is called a near-ring homomorphism (NRH) if for all $h,k\in R$, the following hold:

(1) $f(h+k)=f(h)+f(k)$.

(2) $f(hk)=f(h)f(k)$.

Definition 12. Let U and Y be two nonempty sets and $f: U\longrightarrow Y$ be a function.

(1) If Q is a PFS in Y, then the preimage of Q under f, denoted by $f^{-1}(Q)$, is the PFS in U defined by $f^{-1}(Q)=\{\langle(h),f^{-1}(B_Q(h)),f^{-1}(A_Q(h)),f^{-1}(F_Q(h))\rangle: h\in U\}$, where $f^{-1}(B_Q(h))=B_Q(f(h))$, and so on.

(2) If $P$is a PFS in U, then the image of P under f, denoted by $f(P)$, is the PFS in Y defined by $f(P)=\{\langle(k),f(B_P\ (k)),f(A_P(k)),f(F_P(k))\rangle: k\in Y\}$, where

$$\begin{aligned} f(B_P(k)) &= \begin{cases} \mathrm{Sup}_{h\in f^{-1}(k)}B_P(h), & \text{if } f^{-1}(k)\ne 0_E,\\ 0, & \text{otherwise,}\end{cases}\\ f(A_P(k)) &= \begin{cases} \mathrm{Sup}_{h\in f^{-1}(k)}A_P(h), & \text{if } f^{-1}(k)\ne 0_E,\\ 0, & \text{otherwise,}\end{cases}\\ f(F_P(k)) &= \begin{cases} \mathrm{Sup}_{h\in f^{-1}(k)}F_P(h), & \text{if } f^{-1}(k)\ne 0_E,\\ 0, & \text{otherwise,}\end{cases} \end{aligned} \tag{42}$$

where $f(F_P(k))=(1-f(1-F_P))(k)$.

Theorem 5. *Let E and $E\prime$ be two NRs and f be a homomorphism of E onto $E\prime$. If $P\prime$ is a PFI of $E\prime$, then $f^{-1}(P)$ is a PFI of E.*

Proof. Suppose $h, k, t \in E$. Then, we can deduce the following inequalities:

For membership grade,

$$\begin{aligned} f^{-1}(B_P)(h-k) &= B_P(f(h-k)) \\ &= B_P(f(h)-f(k)) \\ &\geq \min(B_P(f(h), B_P(f(k)))) \\ &= \min\left(f^{-1}(B_P)(h), f^{-1}(B_P)(k)\right). \end{aligned} \tag{43}$$

Similarly, for abstinence grade, we obtain

$$\begin{aligned} f^{-1}(A_P)(h-k) &= A_P(f(h-k)) \\ &= A_P(f(h)-f(k)) \\ &\leq \max(A_P(f(h), A_P(f(k)))) \\ &= \max\left(f^{-1}(A_P)(h), f^{-1}(A_P)(k)\right), \end{aligned} \tag{44}$$

and for falsity grade, we write

$$\begin{aligned} f^{-1}(F_P)(h-k) &= F_P(f(h-k)) \\ &= F_P(f(h)-f(k)) \\ &\leq \max(F_P(f(h), F_P(f(k)))) \\ &= \max\left(f^{-1}(F_P)(h), f^{-1}(F_P)(k)\right). \end{aligned} \tag{45}$$

Also, we acquire the following:

$$\begin{aligned} f^{-1}(B_P)(k+h-k) &= B_P(f(k+h-k)) \\ &= B_P(f(k)+f(h)-f(k)) \\ &\geq \min(B_P(f(h))) \\ &= \min\left(f^{-1}(B_P)(h)\right). \end{aligned} \tag{46}$$

Similarly,

$$\begin{aligned} f^{-1}(A_P)(k+h-k) &= A_P(f(k+h-k)) \\ &= A_P(f(k)+f(h)-f(k)) \\ &\leq \max(A_P(f(h))) \\ &= \max\left(f^{-1}(A_P)(h)\right), \\ f^{-1}(F_P)(k+h-k) &= F_P(f(k+h-k)) \\ &= F_P(f(k)+f(h)-f(k)) \\ &\leq \max(F_P(f(h))) \\ &= \max\left(f^{-1}(F_P)(h)\right). \end{aligned} \tag{47}$$

Furthermore, for truth grade, we obtain

$$\begin{aligned} f^{-1}(B_P)(hk) &= B_P(f(hk)) \\ &= B_P(f(h)f(k)) \\ &\geq \min(B_P(f(k))) \\ &= \min\left(f^{-1}(B_P)(k)\right). \end{aligned} \tag{48}$$

Similarly, for abstinence grade, we get

$$\begin{aligned} f^{-1}(A_P)(hk) &= A_P(f(hk)) \\ &= A_P(f(h)f(k)) \\ &\leq \max(A_P(f(k))) \\ &= \max\left(f^{-1}(A_P)(k)\right), \end{aligned} \tag{49}$$

and for falsity, we note

$$\begin{aligned} f^{-1}(F_P)(hk) &= F_P(f(hk)) \\ &= F_P(f(h)f(k)) \\ &\leq \max(F_P(f(k))) \\ &= \max\left(f^{-1}(F_P)(k)\right). \end{aligned} \tag{50}$$

Finally, for truth grade, we obtain

$$\begin{aligned} f^{-1}(B_P)((h+t)k-hk) &= B_P(f[(h+t)k-hk]) \\ &= B_P([f(h)+f(t)]f(k) \\ &\quad - f(h)f(k)) \\ &\geq \min(B_P(f(t))) \\ &= \min\left(f^{-1}(B_P)(t)\right). \end{aligned} \tag{51}$$

Similarly, for abstinence grade,

$$\begin{aligned} f^{-1}(A_P)((h+t)k-hk) &= A_P(f[(h+t)k-hk]) \\ &= A_P([f(h)+f(t)]f(k) \\ &\quad - f(h)f(k)) \\ &\leq \max(A_P(f(t))) \\ &= \max\left(f^{-1}(A_P)(t)\right), \end{aligned} \tag{52}$$

and for falsity grade,

$$\begin{aligned} f^{-1}(F_P)((h+t)k-hk) &= F_P(f[(h+t)k-hk]) \\ &= F_P([f(h)+f(t)]f(k) \\ &\quad - f(h)f(k)) \\ &\leq \max(F_P(f(t))) \\ &= \max\left(f^{-1}(F_P)(t)\right). \end{aligned} \tag{53}$$

Therefore, $f^{-1}(P)$ is a PFI of E. □

Theorem 6. *Let E_1 and E_2 be two NRs and f be a homomorphism of E_1 and E_2. If A is a PFI of E_1, then $f(P)$ is a PFI of E_2.*

Proof. Let $k_1, k_2, k_3 \in E_2$ and $h_1, h_2, h_1 \in E_1$. Then, the following are observed.

For truth grade, we can write

$$\begin{aligned} f(B_P(k_1-k_2)) &= \sup_{h_1,h_2\in f^{-1}(E_2)} B_P(h_1-h_2) \\ &\geq \sup_{h_1,h_2\in f^{-1}(E_2)} \min(B_P(h_1),B_P(h_2)) \\ &= \min\left(\sup_{h_1\in f^{-1}(E_2)} B_P(h_1), \sup_{h_2\in f^{-1}(E_2)} B_P(h_2)\right) \\ &= \min(f(B_P(k_1)), f(B_P(k_2))). \end{aligned} \tag{54}$$

Also,

$$\begin{aligned} f(B_P(k_1+k_2-k_1)) &= \sup_{h_1,h_2\in f^{-1}(E_2)} B_P(h_1+h_2-h_1) \\ &\geq \sup_{h_1\in f^{-1}(E_2)} B_P(h_1) \\ &= f(B_P(k_1)). \end{aligned} \tag{55}$$

Furthermore, we write

$$\begin{aligned} f(B_P(k_1k_2)) &= \sup_{h_1,h_2\in f^{-1}(E_2)} B_P(h_1h_2) \geq \sup_{h_2\in f^{-1}(E_2)} B_P(h_2) \\ &= f(B_P(k_2)). \end{aligned} \tag{56}$$

Finally, we get

$$\begin{aligned} &f(B_P)((k_1+k_3)k_2-k_1k_2) \\ &\quad= \sup_{h_1,h_2,h_3\in f^{-1}(E_2)} B_P((h_1+h_3)h_2-h_1h_2) \\ &\quad\geq \sup_{h_1,h_2,h_3\in f^{-1}(E_2)} B_P(h_3) = f(B_P(k_3)). \end{aligned} \tag{57}$$

Similarly, we can deduce for abstinence grade that

$$\begin{aligned} f(A_P(k_1-k_2)) &\leq \max(f(A_P(k_1)), f(A_P(k_2))), \\ f(A_P(k_1+k_2-k_1)) &\leq f(A_P(k_2)), \\ f(A_P(k_1k_2)) &\leq f(A_P(k_2)), \\ f(A_P((k_1+k_3)k_2+k_1k_2)) &\leq f(A_P(k_3)), \end{aligned} \tag{58}$$

and for falsity grade, we deduce that

$$\begin{aligned} f(F_P(k_1-k_2)) &\leq \max(f(F_P(k_1)), f(F_P(k_2))), \\ f(F_P(k_1+k_2-k_1)) &\leq f(F_P(k_2)), \\ f(F_P(k_1k_2)) &\leq f(F_P(k_2)), \\ f(F_P((k_1+k_3)k_2+k_1k_2)) &\leq f(F_P(k_3)). \end{aligned} \tag{59}$$

Hence, $f(P)$ is a PFI of E_2.

Note that, by choosing the value of nonmembership as zero, the main results of the article are obtained in the setting of fuzzy sets. However, taking the value of abstinence as zero, we acquire the mentioned results in intuitionistic fuzzy sets. These are the extensions of the presented work. □

6. Conclusion

The theory of a picture fuzzy set is basically the extension of the fuzzy set, containing the grade of membership, the grade of abstinence, and the grade of nonmembership with a rule, provided the sum of all grades cannot exceed the unit interval. Keeping in mind the advantages of the picture fuzzy set, in this manuscript, we presented the PFSNR and PFIs of NRs and explained some of their algebraic properties. We have proved that the union of two PFIs of a NR is a PFI of that NR. Also, we have proved that the positive integral powers of a PFI of a NR are a PFI. We have defined the direct product on PFIs of NRs and proved that the direct product of any two PFIs of NRs is a PFI. Moreover, we showed that we can extend the result for a finite number of PFIs of NRs.

In future, we will investigate the decision-making problems based on Pythagorean fuzzy sets [17, 18]. An investigation will be carried out about the Pythagorean fuzzy near-rings and Pythagorean ideals of a near-ring, and their algebraic properties will be discussed.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally to writing this article. All authors read and approved the final manuscript.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] C. Kahraman, *Fuzzy Multi-Criteria Decision Making: Theory and Applications with Recent Developments*, Springer Science & Business Media (16), 2008.

[3] Y. Deng, Z. Zhenfu, and L. Qi, "Ranking fuzzy numbers with an area method using radius of gyration," *Computers & Mathematics with Applications*, vol. 51, no. 6-7, pp. 1127–1136, 2006.

[4] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Intuitionistic Fuzzy Sets, Physica, Heidelberg, Germany, 1999.

[5] D.-F. Li, "Multiattribute decision making models and methods using intuitionistic fuzzy sets," *Journal of Computer and System Sciences*, vol. 70, no. 1, pp. 73–85, 2005.

[6] S. K. De, R. Biswas, and A. R. Roy, "An application of intuitionistic fuzzy sets in medical diagnosis," *Fuzzy Sets and Systems*, vol. 117, no. 2, pp. 209–213, 2001.

[7] L. Dengfeng and C. Chuntian, "New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions," *Pattern Recognition Letters*, vol. 23, no. 1-3, pp. 221–225, 2002.

[8] B. C. Cuong, "Picture fuzzy sets," *Journal of Computer Science and Cybernetics*, vol. 30, no. 4, p. 409, 2014.

[9] R. Joshi, "A new picture fuzzy information measure based on Tsallis–Havrda–Charvat concept with applications in

presaging poll outcome," *Computational and Applied Mathematics*, vol. 39, no. 2, pp. 1–24, 2020.

[10] M. Qiyas, S. Abdullah, S. Ashraf, and M. Aslam, "Utilizing linguistic picture fuzzy aggregation operators for multiple-attribute decision-making problems," *International Journal of Fuzzy Systems*, vol. 22, no. 1, pp. 310–320, 2020.

[11] A. H. Ganie, S. Singh, and P. K. Bhatia, "Some new correlation coefficients of picture fuzzy sets with applications," *Neural Computing and Applications*, vol. 32, pp. 1–17, 2020.

[12] S. Abou-Zaid, "On fuzzy subnear-rings and ideals," *Fuzzy Sets and Systems*, vol. 44, no. 1, pp. 139–146, 1991.

[13] K. H. Kim, "On fuzzy *R*-subgroup of near-rings," *Journal of Fuzzy Mathematics*, vol. 8, pp. 549–558, 2000.

[14] T. K. Dutta and B. K. Biswas, "Fuzzy ideal of a near-ring," *Bull. Cal. Math. Soc*, vol. 89, no. 6, pp. 447–456, 1997.

[15] V. Chinnadurai and S. Kadalarasi, "Direct product of fuzzy ideals of near-rings," *Annals of Fuzzy Mathematics and Informatics*, vol. 539, p. 540, 2016.

[16] A. Solairaju and S. Thiruveni, "Neutrosophic Fuzzy Ideals of Near-Rings," *International Journal of Pure and Applied Mathematics*, vol. 118, no. 6, 2018.

[17] L. Wang and N. Li, "Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 35, no. 1, pp. 150–183, 2020.

[18] L. Wang, H. Garg, and N. Li, "Pythagorean fuzzy interactive Hamacher power aggregation operators for assessment of express service quality with entropy weight," *Soft Computing*, pp. 1–21, 2020.