

Research Article

Paradox of Enrichment in a Stochastic Predator-Prey Model

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We propose a stochastic predator-prey model to study a novel idea that involves investigating random noises effects on the enrichment paradox phenomenon. Existence and stochastic boundedness of a unique positive solution with positive initial conditions are proved. The global asymptotic stability is studied to determine the occurrence of the enrichment paradox phenomenon. We show theoretically that intensive noises play an important role in the occurrence of the phenomenon, where increasing intensive noises lead to occurrence of the paradox of enrichment. We perform numerical simulations to verify and demonstrate the theoretical results. The new results in this study may contribute to increasing attention to study the random noise effects on some ecological and biological phenomena as the paradox of enrichment.

1. Introduction

Theoretical ecology has grown to the extent that we regularly see papers in mathematical ecology discussing models in purely mathematical terms that from time to time bear little resemblance to real ecological processes. There is a desire to reexamine the situation and study the diverse constructing blocks once more. Mathematical modeling is a beneficial tool to reveal how a process works and predict how it will progress [1]. However, the difficulty in determining ecological principles makes formulating ecological problems a complicated process [2, 3].

Differential equations are a central tool that are used to describe many ecological problems mathematically. Predator-prey interaction is one of the most significant topics in applied mathematics and mathematical biology [4–8]. Different styles of mathematical models have been used to study predator-prey interaction. Logistic predator-prey models have been used by some researchers due to their realistic descriptions of the growth rates of species [1]. Logistic models display a carrying capacity term that represents limiting species sustainable by the environment. In this paper, we use logistic models of predator and prey equations, in which the carrying capacity of predator is a proportional to the available amount of prey.

Paradoxical phenomena have attracted much attention than normal observations. One of the most important paradoxes in ecology is the paradox of enrichment, which was firstly mentioned by Rosenzweig [9]. The paradox of enrichment states that coexistence equilibrium point will destabilize when the carrying capacity is increased, and the destabilization may lead to stochastic extinction for one of the species or all species. Mathematically, destabilization can be explained by the oscillations closer to one of the axes or both axes of phase space. Although the enrichment paradox phenomenon is interpreted as the variance between the real world and the mathematical construction in many experimental studies, the paradox of enrichment has occurred as explained in some recent experimental studies [10–12].

Stochastic process has many applications in different sciences [13–17]. Random noises are a ubiquitous characteristic of ecological systems, since nearly all environments are subject to some unexpected factors, which have an important role in an ecosystem component [18–23]. Random noises are important in ecology with regard to focusing on the variability of ecological systems and moving away from thinking in terms of equilibrium dynamics. Some recent studies investigated stochastic predator-prey models to study the random noise effects on the dynamic behaviors such as [24–27]. We highlight the main points that have been

focused in the literature on stochastic predator-prey models topic as follows: firstly, using different stochastic predator-prey models; secondly, establishing existence and uniqueness of global solution of these systems; thirdly, establishing sufficient conditions for the existence of a unique ergodic stationary distribution; finally, obtaining the persistence and extinction conditions.

Alebraheem [28] studied the occurrence of the paradox of enrichment with deterministic models. Motivated by the results in the previous study, we study a novel idea in this paper that involves investigating stochastic concept to study the random noise effects on the occurrence of the paradox of enrichment in a predator-prey model. In order to study our idea, we propose a stochastic predator-prey model and prove the global stability of the stochastic model by constructing suitable Lyapunov function and using Itô formula, which determine the occurrence of the phenomenon. This study may contribute to increasing attention to study the random noise effects on some ecological and biological phenomena as the paradox of enrichment.

The rest of this paper is organized as follows. In the next section, we introduce a stochastic predator-prey model. In Section 3, we prove existence and stochastic boundedness of a unique positive solution with positive initial conditions. Section 4 presents the global asymptotic stability and the occurrence of the enrichment paradox phenomenon in stochastic and deterministic models, in addition to numerical simulations that verify and demonstrate the theoretical results. Finally, in Section 5, discussion and conclusions are presented.

2. Stochastic Predator-Prey Model

We investigate the random noises in a predator-prey model to study its effects on the dynamic behaviors of this model, which have effects on prey and predator species.

The standard Itô stochastic differential equation has the following form [21]:

$$dX(t) = F(X(t), t)dt + G(X(t), t)dW(t), X(t_0) = X_0, \text{ for } t \geq t_0, \tag{1}$$

where the function $F(t, X(t))$ is called the drift and $G(t, X(t))$ is the diffusion matrix. W is a standard Wiener or Brownian motion processes.

We use stochastic differential equations to describe the random noises of continuous-time model that have been applied by some studies [23–27].

We define

$$\mathbb{R}_+^n = \{X = (X_1, X_2, \dots, X_n) \in \mathbb{R}^n: X_i > 0, 1 \leq i \leq n\}. \tag{2}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and suppose that the constant initial value $X_0 \in \mathbb{R}_+^n$.

The differential operator L of equation (1) is defined by the following formula [21]:

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^n f_i(X(t), t) \frac{\partial}{\partial X_i} + \frac{1}{2} \sum_{i,j=1}^n [g(X(t), t), g(X(t), t)^T]_{ij} \frac{\partial^2}{\partial X_i \partial X_j}. \tag{3}$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}_+^n, \mathbb{R}_+)$, then

$$LV(X(t), t) = V_t(X(t), t) + V_X(X(t), t)f(X(t), t) + \frac{1}{2} \text{trace}[g^T(X(t), t)V_{XX}(X(t), t)g(X(t), t)], \tag{4}$$

where $V_t = (\partial V / \partial t)$, $V_X = (\partial V / \partial X_1, \partial V / \partial X_2, \dots, \partial V / \partial X_n)$, and $V_{XX} = (\partial^2 V / \partial X_i \partial X_j)_{n \times n}$. By Itô's formula, we can attain

$$dV(X(t), t) = LV(X(t), t)dt + V_X(X(t), t)g(X(t), t)dW(t). \tag{5}$$

In this paper, we use the continuous-time predator-prey model with stochastic perturbations as follows:

$$dX = X \left(\rho \left(1 - \frac{X}{k} \right) - \alpha Y \right) dt + \sigma_1 X dW_1, \tag{6a}$$

$$dY = Y(-u + e\alpha X - e\alpha Y)dt + \sigma_2 Y dW_2. \tag{6b}$$

Subjecting to initial conditions,

$$X(0) = X_0 > 0, Y(0) = Y_0 > 0, \tag{7}$$

where σ_i represent the strength of noise and dW_i is a standard Wiener or Brownian motion process for $i = 1, 2$.

The meaning of variables and parameters of model ((6a) and (6b)) are summarized as follows:

- X : density of prey species
- Y : density of predator species
- ρ : inherent growth rate of prey
- k : carrying capacity of the environment
- u : death rates of predator
- α : attack rate of predator
- e : efficiency of converting consumed prey into predator birth

3. Existence, Uniqueness, and Boundedness of the Positive Solution

Since the model ((6a) and (6b)) is a biological model, the following theorem shows the existence and uniqueness of the global positive solution to the model ((6a) and (6b)). For a model of population dynamics, we have a unique global (i.e., no explosion in a finite time) solution for any given initial value which means that there is a finite population size at a finite time [29] while the solution is positive because of biological feasibility.

Theorem 1. *If any initial condition $(X_0, Y_0) \in \mathbb{R}_+^2$, then there is a unique positive solution $(X(t), Y(t))$ of the model ((6a) and (6b)) almost surely for all $t \geq 0$ with probability one.*

Proof. We assume the solution of the model ((6a) and (6b)) is $(X(t), Y(t))$ for $t \in (0, \tau_e)$, where τ_e indicates the explosion time.

Let us consider $q_1(t) = \ln X(t)$ and $q_2 = \ln Y(t)$. We apply Itô formula on the system ((6a) and (6b)) and then the equation (6a) becomes as follows:

$$\begin{aligned} d(\ln X(t)) &= \left[\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial X} X \left(\rho \left(1 - \frac{X}{k} \right) - \alpha Y \right) + \frac{1}{2} \frac{\partial^2 q_1}{\partial X^2} (\sigma_1 X)^2 \right] dt \\ &\quad + \frac{\partial q_1}{\partial X} \sigma_1 X dW_1 \\ &= \left[0 + \frac{\partial q_1}{\partial X} X \left(\rho \left(1 - \frac{X}{k} \right) - \alpha Y \right) - \frac{1}{2} \frac{1}{X^2} \sigma_1^2 X^2 \right] dt \\ &\quad + \frac{1}{X} \sigma_1 X dW_1 \\ &= \left[\rho - \frac{\rho X}{k} - \alpha Y - \frac{1}{2} \sigma_1^2 \right] dt + \sigma_1 dW_1, \\ d(q_1) &= \left[\rho - \frac{\rho}{k} e^{q_1} - \alpha e^{q_2} - \frac{1}{2} \sigma_1^2 \right] dt + \sigma_1 dW_1. \end{aligned} \tag{8}$$

Similarly, for equation (6b), we obtain

$$d(q_2) = \left[-u + e \alpha e^{q_1} - e \alpha e^{q_2} - \frac{1}{2} \sigma_2^2 \right] dt + \sigma_2 dW_2, \tag{9}$$

with initial values $q_1(0) = \ln X_0$ and $q_2(0) = \ln Y_0$.

The transformed system becomes

$$d(q_1) = \left[\rho - \frac{\rho}{k} e^{q_1} - \alpha e^{q_2} - \frac{1}{2} \sigma_1^2 \right] dt + \sigma_1 dW_1, \tag{10a}$$

$$d(q_2) = \left[-u + e \alpha e^{q_1} - e \alpha e^{q_2} - \frac{1}{2} \sigma_2^2 \right] dt + \sigma_2 dW_2, \tag{10b}$$

with $q_1(0) = \ln X_0$ and $q_2(0) = \ln Y_0$.

The coefficients of the system ((10a) and (10b)) satisfy local Lipschitz condition; this means that the system ((10a) and (10b)) has unique local solution $(q_1(t), q_2(t))$ for $t \in [0, \tau_e)$. By using the Ito formula, the unique positive solution of the system ((6a) and (6b)) with initial value (X_0, Y_0) is $(X(t), Y(t)) = (e^{q_1(t)}, e^{q_2(t)})$ for $t \in [0, \tau_e)$.

We show that the solution is global if we verify $\tau_e = \infty$ a.s.

Since the solution is positive on $[0, \tau_e)$, we find

$$dX \leq \rho X dt + \sigma_1 X dW_1. \tag{11}$$

Let $\xi_1(t)$ be the unique solution of the equation

$$d(\xi_1) = \rho \xi_1 dt + \sigma_1 \xi_1 dW_1, \tag{12}$$

with $\xi_1(0) = X_0$.

Let $U = (1/\xi_1(t))$. Now, we apply Itô formula, and we obtain

$$\begin{aligned} dU &= \left[-\frac{\rho \xi_1}{\xi_1^2} + \sigma_1^2 \frac{\xi_1^2}{\xi_1^3} \right] dt - \frac{\sigma_1 \xi_1}{\xi_1^2} dW_1 \\ &= \left[-\frac{\rho}{\xi_1} + \frac{\sigma_1^2}{\xi_1} \right] dt - \frac{\sigma_1}{\xi_1} dW_1 \\ &= (-\rho + \sigma_1^2) U dt - \sigma_1 U dW_1, \\ dU &= (-\rho + \sigma_1^2) U dt - \sigma_1 U dW_1, \end{aligned} \tag{13}$$

with $U(0) = (1/X_0)$.

The unique solution of stochastic differential equation (13) is

$$U = \frac{1}{X_0} e^{(-\rho + \sigma_1^2)t - \sigma_1 W_1}, \tag{14}$$

$$\text{i.e. } \xi_1 = X_0 e^{(\rho - \sigma_1^2)t + \sigma_1 W_1}.$$

$\therefore X(t) \leq \xi_1(t)$.

We have from the second equation of the system ((6a) and (6b)) that

$$dY = Y(-u + e \alpha X - e \alpha Y) dt + \sigma_2 Y dW_2, \tag{15}$$

$$dY \leq e \alpha X Y dt + \sigma_2 Y dW_2.$$

Let $\xi_2(t)$ be the unique solution of the equation

$$d\xi_2 = e \alpha \xi_1 \xi_2 dt + \sigma_2 \xi_2 dW_2, \tag{16}$$

with $\xi_2(0) = Y_0$.

Applying the Itô formula, as doing the same procedure for first equation of the system ((6a) and (6b)), we obtain the unique solution of stochastic differential equation (16) as

$$\xi_2(t) = Y_0 e^{e \alpha \int_0^t \xi_1(s) ds + \sigma_2 W_2}. \tag{17}$$

$\therefore Y(t) \leq \xi_2(t)$.

Furthermore, for the first equation of the system ((6a) and (6b)), we have

$$dX \geq X \left(\rho \left(1 - \frac{X}{k} \right) - \alpha Y \right) dt + \sigma_1 X dW_1. \tag{18}$$

We can say that

$$dX \geq X \left(\rho \left(1 - \frac{X}{k} \right) - \alpha \xi_2 \right) dt + \sigma_1 X dW_1. \tag{19}$$

Assume $H_1(t)$ to be the unique solution of the equation

$$dH_1(t) = H_1(t) \left(\rho \left(1 - \frac{H_1(t)}{k} \right) - \alpha \xi_2 \right) dt + \sigma_1 H_1(t) dW_1, \tag{20}$$

with $H_1(0) = X_0$.

By applying the Ito formula, we have the unique solution of stochastic differential equation (20) as

$$H_1(t) = \frac{e^{(\rho-\sigma_1^2)t-\alpha} \int_0^t \xi_2(s)ds + \sigma_1 W_1}{(1/X_0) + (\rho/k) \int_0^t e^{(\rho-\sigma_1^2)s-\alpha} \int_0^s \xi_2(v)dv + \sigma_1 W_1}. \tag{21}$$

∴ $X(t) \geq H_1(t)$.

Through comparison theorems for stochastic differential equations [30], we get from (14) and (21) that

$$H_1(t) \leq X(t) \leq \xi_1(t) \text{ a.s. for } t \in [0, \tau_e]. \tag{22}$$

In the same manner, for the second equation of the system ((6a) and (6b)), we have

$$dY \geq -uYdt + \sigma_2 YdW_2. \tag{23}$$

Suppose that $H_2(t)$ be the unique solution of the equation

$$dH_2(t) = -uH_2dt + \sigma_2 H_2dW_2, \tag{24}$$

with $H_2(0) = Y_0$.

By applying Itô formula, we have the unique solution of stochastic differential equation (24) as

$$H_2(t) = Y_0 e^{-(u+\sigma_2^2)t+\sigma_2 W_2}. \tag{25}$$

∴ $Y(t) \geq H_2(t)$.

Also, through comparison theorems for stochastic differential equations [30], we obtain from (17) and (25) that

$$H_2(t) \leq Y(t) \leq \xi_2(t) \text{ a.s. for } t \in [0, \tau_e]. \tag{26}$$

It follows that $\xi_1(t), \xi_2(t), H_1(t)$, and $H_2(t)$ exist for all $t \geq 0$ and $\tau_e = \infty$ and we can conclude that $(X(t), Y(t))$ globally exists. This proves the theorem.

Through the previous proof, we have the following theorem. □

Theorem 2. *If the unique positive solution $(X(t), Y(t))$ of the system ((6a) and (6b)) has any initial condition $(X_0, Y_0) \in R_+^2$, then there exist the functions $\xi_1(t), \xi_2(t), H_1(t)$, and $H_2(t)$ defined to satisfy*

$$\begin{aligned} H_1(t) &\leq X(t) \leq \xi_1(t), \\ H_2(t) &\leq Y(t) \leq \xi_2(t), \text{ a.s., for all } t \geq 0. \end{aligned} \tag{27}$$

4. Global Asymptotic Stability and Paradox of Enrichment

4.1. Theoretical Analysis. In this section, we study the global stability of the stochastic model ((6a) and (6b)) that is an equivalent of the stationary distribution and ergodicity to stochastic models by constructing suitable Lyapunov function and using Ito formula [31]. There is not positive time-independent equilibrium point as deterministic systems. In this paper, the global asymptotic stability of the stochastic system ((6a) and (6b)) (i.e., ergodic property) is studied to determine the occurrence of the enrichment paradox phenomenon.

Definition 1 (see [32]). The trivial solution of stochastic differential equation (1) is defined to be the following:

- (i) Stochastically stable $\forall \varepsilon \in (0, 1)$ and $r > 0$, \exists a $\delta = \delta(\varepsilon, r) > 0$ such that

$$\mathbb{P}\{|X(t, X_0)| < r \forall t \geq 0\} \geq 1 - \varepsilon. \tag{28}$$

- (ii) Stochastically asymptotically stable if it is stochastically stable and, moreover, $\forall \varepsilon \in (0, 1)$, $\exists \delta_0 = \delta_0(\varepsilon) > 0$ such that

$$\mathbb{P}\left\{\lim_{t \rightarrow \infty} X(t, X_0) = 0\right\} \geq 1 - \varepsilon, \tag{29}$$

whenever $|X_0| < \delta_0$.

- (iii) Globally stochastically asymptotically stable if it is stochastically stable and, moreover, $\forall X_0 \in \mathbb{R}$ with

$$\mathbb{P}\left\{\lim_{t \rightarrow \infty} X(t, X_0) = 0\right\} = 1. \tag{30}$$

If we assume $\sigma_1 = 0$ and $\sigma_2 = 0$, then the model ((6a) and (6b)) becomes a deterministic model as given by Alebraheem [28], but in this paper, inherent growth rate of prey (ρ) is an unknown value, and it was fixed to be 1 in [28]:

$$\begin{aligned} \frac{dX}{dt} &= \rho X \left(1 - \frac{X}{k}\right) - \alpha XY, \\ \frac{dY}{dt} &= -uY + e\alpha XY - e\alpha Y^2. \end{aligned} \tag{31}$$

The coexistence equilibrium point of the model (31) is

$$E_2 = (X, Y) = \left(\frac{k(u + \rho e)}{e\alpha k + e}, \frac{e\rho\alpha k - u}{e\alpha^2 k + e\alpha}\right). \tag{32}$$

The coexistence equilibrium point exists under the following condition:

$$e\rho\alpha k > u. \tag{33}$$

Theorem 3. *If $k < (\rho(X - \dot{X})^2/1/2e(\dot{X}\sigma_1^2 + \dot{Y}\sigma_2^2) - \alpha(Y - \dot{Y})^2)$ and for any initial condition $(X_0, Y_0) \in R_+^2$, then there is a unique positive solution $(X(t), Y(t))$ of the system ((6a) and (6b)) for all $t \geq 0$, which is globally asymptotically stable almost surely (a.s.).*

Proof. Define Lyapunov functions as

$$\begin{aligned} V_1 &= X - \dot{X} - \dot{X} \ln\left(\frac{X}{\dot{X}}\right), \\ V_2 &= Y - \dot{Y} - \dot{Y} \ln\left(\frac{Y}{\dot{Y}}\right). \end{aligned} \tag{34}$$

We apply Ito formula and so we have

$$\begin{aligned}
L(V_1) &= \left(1 - \frac{X}{\dot{X}}\right)dX + \frac{1}{2} \frac{\dot{X}}{X^2} (dX)^2 \\
&= (X - \dot{X}) \left[\left(\rho \left(1 - \frac{X}{k}\right) - \alpha Y \right) dt + \sigma_1 dW_1 \right] + \frac{1}{2} \dot{X} \sigma_1^2 dt \\
&= (X - \dot{X}) \left[\left(\rho - \frac{\rho X}{k} - \alpha Y \right) dt + \sigma_1 dW_1 \right] + \frac{1}{2} \dot{X} \sigma_1^2 dt \\
&= (X - \dot{X}) \left[\left(\frac{\rho \dot{X}}{k} + \alpha \dot{Y} - \frac{\rho X}{k} - \alpha Y \right) dt + \sigma_1 dW_1 \right] + \frac{1}{2} \dot{X} \sigma_1^2 dt \\
&= \left[-\frac{\rho}{k} (X - \dot{X})^2 - \alpha (X - \dot{X})(Y - \dot{Y}) + \frac{1}{2} \dot{X} \sigma_1^2 \right] dt + \sigma_1 (X - \dot{X}) dW_1.
\end{aligned} \tag{35}$$

Similarly, we obtain

$$\begin{aligned}
L(V_2) &= \left(1 - \frac{Y}{\dot{Y}}\right)dY + \frac{1}{2} \frac{\dot{Y}}{Y^2} (dY)^2 \\
&= (Y - \dot{Y}) [(-u + e\alpha X - e\alpha Y) dt + \sigma_2 dW_2] + \frac{1}{2} \dot{Y} \sigma_2^2 dt \\
&= (Y - \dot{Y}) [e\alpha X - e\alpha \dot{X} - e\alpha Y + e\alpha \dot{Y}] dt + \sigma_2 dW_2 + \frac{1}{2} \dot{Y} \sigma_2^2 dt \\
&= (Y - \dot{Y}) [e\alpha (X - \dot{X}) - e\alpha (Y - \dot{Y})] dt + \sigma_2 dW_2 + \frac{1}{2} \dot{Y} \sigma_2^2 dt \\
&= \left[e\alpha (X - \dot{X})(Y - \dot{Y}) - e\alpha (Y - \dot{Y})^2 + \frac{1}{2} \dot{Y} \sigma_2^2 \right] dt + \sigma_2 (Y - \dot{Y}) dW_2.
\end{aligned} \tag{36}$$

Now, we define

$$\begin{aligned}
V &= \left(X - \dot{X} - \dot{X} \ln \left(\frac{X}{\dot{X}} \right) \right) + B \left(Y - \dot{Y} - \dot{Y} \ln \left(\frac{Y}{\dot{Y}} \right) \right), \\
L(V) &= L(V_1) + BL(V_2) \\
&:= L(V) dt + \sigma_1 (X - \dot{X}) dW_1 + \sigma_2 (Y - \dot{Y}) dW_2,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
L(V) &= -\frac{\rho}{k} (X - \dot{X})^2 - e\alpha (Y - \dot{Y})^2 - \alpha (X - \dot{X})(Y - \dot{Y}) \\
&\quad + e\alpha (X - \dot{X})(Y - \dot{Y}) + \frac{1}{2} \dot{X} \sigma_1^2 + \frac{1}{2} \dot{Y} \sigma_2^2.
\end{aligned} \tag{38}$$

We select $B = (1/e)$ to simplify the mathematical analysis; then,

$$L(V) = -\frac{\rho}{k} (X - \dot{X})^2 - \alpha (Y - \dot{Y})^2 + \frac{1}{2e} \dot{X} \sigma_1^2 + \frac{1}{2e} \dot{Y} \sigma_2^2. \tag{39}$$

Therefore, the number of terms is reduced.

If $k < (\rho (X - \dot{X})^2 / (1/2e) (\dot{X} \sigma_1^2 + \dot{Y} \sigma_2^2) - \alpha (Y - \dot{Y})^2)$, then this implies $L(V) < 0$ along all trajectories.

Through the previous analysis and by removing stochastic terms, the dynamics of deterministic model (31) always coexist in a globally stable state and there is no bifurcation under any conditions. Therefore, the paradox of enrichment does not arise with deterministic model (31) as it was proved by Alebraheem [28]. \square

Corollary 1. *If $k < (\rho (X - \dot{X})^2 / (1/2e) (\dot{X} \sigma_1^2 + \dot{Y} \sigma_2^2) - \alpha (Y - \dot{Y})^2)$ is violated, then coexistence equilibrium point $E_2 = (X, Y)$ destabilizes.*

Corollary 2. *The paradox of enrichment arises with the stochastic model ((6a) and (6b)).*

We conclude that the intensive noises affect the stability as shown through Theorem 3; this leads to occurrence of the paradox of enrichment in the stochastic model ((6a) and (6b)) as shown through Corollary 2. The occurrence of the enrichment paradox phenomenon depends on the carrying capacity in one side and stochastic terms in another side as it is shown in Theorem 3 and Corollary 1.

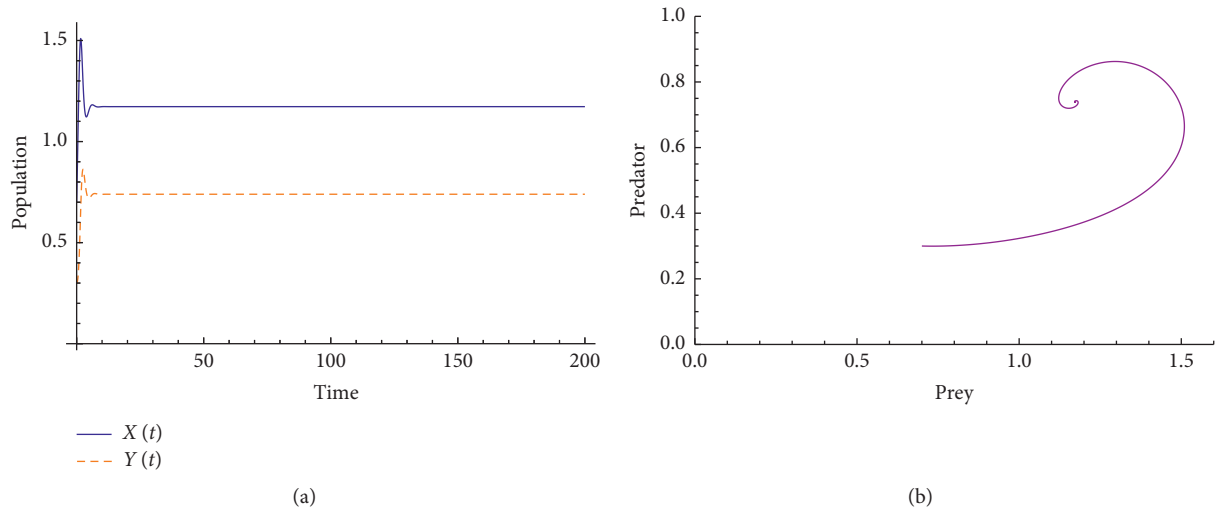


FIGURE 1: Dynamics model (31) when $(\sigma = 0.0)$: (a) time series; (b) phase plane.

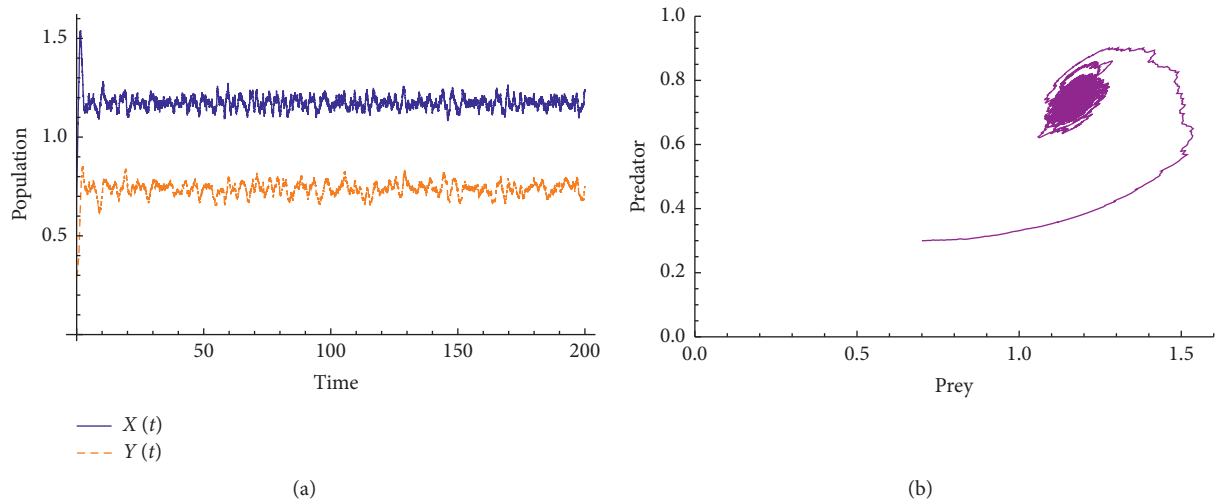


FIGURE 2: Dynamics model (6a) and (6b) when $(\sigma = 0.4)$: (a) time series; (b) phase plane.

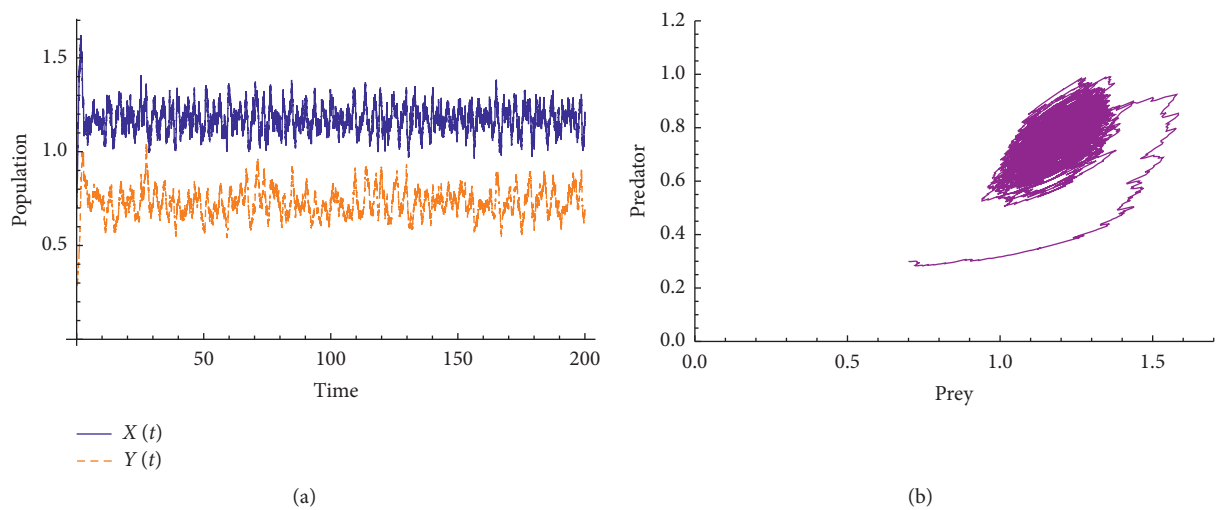


FIGURE 3: Dynamics model (6a) and (6b) when $(\sigma = 1.0)$: (a) time series; (b) phase plane.

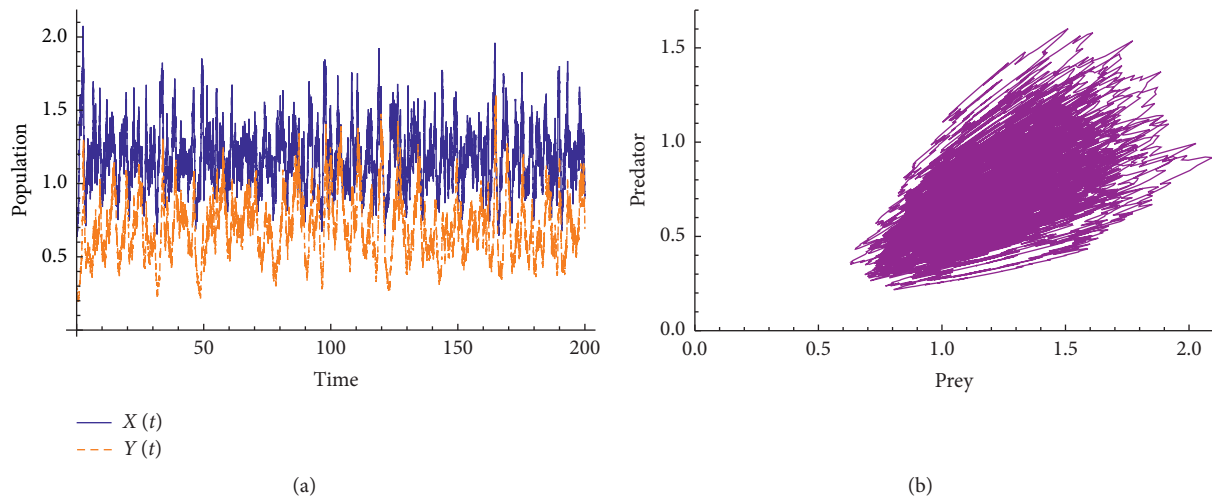


FIGURE 4: Dynamics model (6a) and (6b) when $(\sigma = 3.0)$: (a) time series; (b) phase plane.

4.2. Numerical Simulations. We perform numerical simulations to show the effects of stochastic process on the occurrence of the enrichment paradox phenomenon. The stochastic Runge–Kutta method based on Itô process is used to solve the model ((6a) and (6b)). We use a command “StochasticRungeKuttaScalarNoise” in MATHEMATICA 11.3 program as a method to execute the numerical simulations, according to Wolfram website [33]. The command uses stage Rossler Stochastic Runge–Kutta method for scalar noise that has order $3/2$. The values of the parameters are selected to satisfy condition (33) with regard to the coexistence of prey and predator. The values of the parameters and initial conditions are fixed. However, we use different values of noise strength. The values are as follows:

$$\begin{aligned} r = 1.5, k = 3, \alpha = 1.5, e = 1.0, u = 0.65, \\ X(0) = 0.7, Y(0) = 0.3. \end{aligned} \quad (40)$$

Model (31) presents stable dynamic behavior as shown in Figure 1 when $\sigma = 0$, which refers to deterministic model (31). In our simulations, we observe that the dynamic behavior changes from stable to oscillated when investigating the random noises to be $\sigma = 0.4$, $\sigma = 1.0$, and $\sigma = 3.0$, as shown in Figures 2–4, respectively. However, the difference between the second, third, and fourth cases is the size of the oscillations; with increased values of σ , the size of the oscillations increases. Then, we obtain that the random noises have effects on the occurrence of the paradox of enrichment, moving from stable dynamic behavior to oscillated dynamic behavior.

5. Discussion and Conclusions

In this paper, we investigated a stochastic predator-prey model to study the random noise effects on the enrichment paradox phenomenon. To the best of our knowledge, this is the first study to discuss the random noise effects on the occurrence of the enrichment paradox phenomenon. We proved the existence, uniqueness, and boundedness of the model as shown in Theorems 1 and 2. The global stability was

studied to determine the occurrence of the enrichment paradox phenomenon; therefore, we proved the global stability of the stochastic model by constructing suitable Lyapunov function and using Itô formula as demonstrated in Theorem 3. This is an equivalent of the stationary distribution and ergodicity to stochastic models.

The theoretical results showed that intensive noises play an important role in occurrence of the paradox of enrichment, where increasing intensive noises lead to occurrence of the paradox of enrichment as demonstrated in Theorem 3 and Corollaries 1 and 2. Finally, we performed the numerical simulations to verify and demonstrate the theoretical results. In addition, our simulations showed that the size of the oscillations increases with increased values of intensive noises.

Overall, this study corresponds with the literature, in which the random noises focus on the variability of ecological systems and moving away from thinking in terms of equilibrium dynamics. We proved theoretically and showed by the numerical simulations that the random noises make our model destabilize with increased values of intensive noises, although with no noises, our model is stable without conditions. Hence, the paradox of enrichment arises with the random noises. This study may contribute to increasing attention to study the random noise effects on some ecological and biological phenomena as the paradox of enrichment for future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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