Research Article

Theta Cone Metric Spaces and Some Fixed Point Theorems

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This paper introduces the concept of the theta cone metric, studies its various topological properties, and gives some examples of it. Furthermore, it proves some lemmas and then uses them to give further generalizations of some well-known fixed point theorems. Specifically, Theorem 2 of the paper is a generalization of Reich's fixed point theorem.

1. Introduction and Preliminaries

In 2007, Huang and Zhang [1] introduced cone metric spaces as a generalization of metric spaces. Let $\mathscr{E}$ be a real Banach space, $C \subset \mathscr{E}$. C is called a cone in $\mathscr{E}$ if

(1) C is nonempty, closed, and $C \neq \{\Theta\}$, where Θ is zero (neutral element) of $\mathscr{E}$

(2) $\alpha C + \beta C \subset C$ for all nonnegative real numbers α, β

(3) $C \cap -C = \{\Theta\}$

If $\operatorname{int} C$ is the set of all interior points of C, then a cone C in a normed space $\mathscr{E}$ induces the following ordered relations [1, 2]:

$$
\begin{aligned}
u \prec v &\Longleftrightarrow v - u \in C, \\
u < v &\Longleftrightarrow (v - u \in C \text{ and } u \neq v), \\
u \underset{\neq}{\prec} v &\Longleftrightarrow v - u \in \operatorname{int} C.
\end{aligned}
\tag{1}
$$

A sequence $\{w_n\}_{n \in N}$ in $\mathscr{E}$ is bounded above by $w \in \mathscr{E}$ iff

$$
w_n \underset{\neq}{\prec} w \quad \forall n \in \mathbb{N}.
\tag{2}
$$

The cone C is called normal if there is a number $M > 0$ such that

$$
\forall u, v \in \mathscr{E}, \Theta \prec u \prec v \Longrightarrow \|u\| \le M\|v\|.
\tag{3}
$$

The least positive number satisfying the above is called the normal constant of C.

Huang and Zhang [1] supposed that $\mathscr{E}$ is a Banach space, C is a cone in $\mathscr{E}$ with nonempty interior, and $\prec$ is partial ordering with respect to C. If X is a nonempty set, the distance $d(x, y)$ of two elements x and y in the space X is defined to be a vector in the cone C of the ordered Banach space $(\mathscr{E}, \prec)$.

(1) $\Theta \underset{\neq}{\prec} d(x, y) \, \forall x, y \in X$.

(2) $d(x, y) = \Theta \prec x = y$.

(3) $d(x, y) = d(y, x) \, \forall x, y \in X$.

(4) $d(x, z) \prec [d(x, y) + d(y, z)] \, \forall x, y, z \in X$.

The triple (X, C, d) is known as the cone metric space. They carefully studied convergence and completeness and then proved some fixed point theorems for the contractive type of mappings in this setting.

In 2010, Haghi et al. [3] showed that some generalizations in fixed point theory are really consequences of Huang and Zhang results.

In contrast, in 2012, Cakally et al. [4] obtained that any cone metric space (X, d) is equivalent to the usual metric space $(X, d^\star)$, where the real-valued metric function $d^\star$ is defined by a nonlinear scalarization function [5, 6].

In 2013, Liu and Xu [2] introduced the concept of cone metric spaces with Banach algebras; they replaced Banach spaces $\mathscr{E}$ by Banach algebras $\mathscr{A}$, and they proved some fixed point theorems of generalized Lipschitz mappings with weaker conditions on generalized Lipschitz constants (the

constant k is a vector in a normal cone of the Banach algebra, and the essential conditions on the contraction constant k are neither order relations nor norm relations but spectrum radius). A well-formulated example shows that their main results concerning the fixed point theorems in the setting of cone metric spaces with Banach algebras are more useful than the standard results in cone metric spaces presented in the literature.

In 2013, Khojasteh et al. [7] proposed the notion of the θ-metric as a proper generalization of a metric.

Definition 1. Let $\theta: [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ be a continuous mapping with respect to both variables. Let $\mathrm{Im}(\theta) = \{\theta(s, t): s \geq 0, t \geq 0\}$. The mapping θ is called a B-action if and only if it satisfies the following conditions:

(1) $\theta(0, 0) = 0$ and $\theta(u, v) = \theta(v, u)$ for every $u, v \geq 0$.

(2)

$$\theta(u, v) < \theta(w, t) \text{ if either } \begin{cases} u < w \text{ and } v \leq t \\ \text{or} \\ u \leq w \text{ and } v < t. \end{cases} \quad (4)$$

(3) For every $u \in \mathrm{Im}(\theta)$ and every $v \in [0, u]$, there is $w \in [0, u]$ such that $\theta(v, w) = u$.

(4) $\theta(u, 0) \leq u$ for every $u > 0$.

They functionally formulated the notion of θ-metric spaces, and then, they gave the terminology of open and closed sets. Furthermore, they gave a detailed and comprehensive study of convergence and Cauchyness of sequences in this frame of work.

Now, we replace $[0, \infty)$ by a cone in a normed space and introduce the following analogous generalization of the definition of the θ function.

Definition 2. Let $(\mathscr{E}, \prec)$ be an ordered normed space, where $\prec$ is the ordered relation induced by a cone $C \subset \mathscr{E}$. Let $\theta: C \times C \longrightarrow C$ be a continuous mapping with respect to each variable. Let $\mathrm{Im}(\theta) = \{t: t \in C \text{ such that } \exists u_0, v_0 \in \mathscr{E}, \theta(u_0, v_0) = t\}$. A mapping θ is called an ordered action on $\mathscr{E}$ if and only if it satisfies the following conditions:

(1) $\theta(\Theta, \Theta) = \Theta$ and $\theta(u, v) = \theta(v, u)$ for every $u, v \in C$.

(2)

$$\theta(u, v) < \theta(w, t) \text{ if either } \begin{cases} u \prec w, v \prec t \\ \text{or} \\ u \prec w, v < t. \end{cases} \quad (5)$$

(3) For every $u \in \mathrm{Im}(\theta)$ and every $\Theta \prec v \prec u$, there is $\Theta \prec w \prec u$ such that $\theta(v, w) = u$.

(4) $\theta(u, \Theta) \prec u$ for every $u \in C/\{\Theta\}$.

Because $x - \Theta \in C$ for every $x \in C$, one can write instead $\Theta \prec x$ for every $x \in C$, $[\Theta < x \text{ for every } x \in C/\{\Theta\}]$.

Example 1. Let $\mathbb{R}^m$ be the vector space of all finite m sequences of real numbers with the usual operations of addition and scalar multiplication, for $a = \{a_k\}_{k=1}^m \in \mathbb{R}^m$. Denote $\|a\|_p = \sqrt[p]{\sum_{k=1}^m |a_k|^p}$ and $\|a\|_\infty = \max_{k=1}^m |a_k|$. If $1 \leq p$, then $(\mathbb{R}^m, \|.\|_p)$ is the Banach space. If $0 \leq p < 1$, then $(\mathbb{R}^m, \|.\|_p)$ is the quasi-normed space.

Let $\mathscr{E} = \mathbf{M_n}(\mathbb{R}^m)$ be the vector space of all n-square matrices whose entries are elements in $\mathbb{R}^m$:

$$\mathscr{E} = \left\{ \begin{array}{c} u: u = \{u_{ij}\}_{1 \leq i, j \leq n}, \\ u_{ij} = \{u_k^{ij}\}_{k=1}^m \in \mathbb{R}^m, \quad \forall 1 \leq i, j \leq n \end{array} \right\}. \quad (6)$$

Operations on $\mathscr{E}$ are defined as follows:

$$\alpha u = \alpha \{u_{ij}\}_{1 \leq i, j \leq n} = \left\{ \{\alpha u_k^{ij}\}_{k=1}^m \right\}_{1 \leq i, j \leq n}, \quad (7)$$

$$u \oplus v = \{u_{ij}\}_{1 \leq i, j \leq n} \oplus \{v_{ij}\}_{1 \leq i, j \leq n} = \left\{ \{u_k^{ij} + v_k^{ij}\}_{k=1}^m \right\}_{1 \leq i, j \leq n}.$$

The space $\mathscr{E}$ is a Banach space endowed with the following norms:

$$\|u\|_p = \sum_{1 \leq i, j \leq n} \left\| \{u_k^{ij}\}_{k=1}^m \right\|_p,$$

$$or \quad (8)$$

$$\|u\|_\infty = \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \left\| \{u_k^{ij}\}_{k=1}^m \right\|_p.$$

The zero element of the space $\mathscr{E}$, is the matrix θ, and every entry of the matrix θ is the all of its entries are the zero element of $\mathbb{R}^m$, $0 = \{0, 0, \ldots, 0\}$.

Let $C = \left\{ u: u = \left\{ \{u_k^{ij}\}_{k=1}^m \right\}_{1 \leq i, j \leq n}, 1 \leq i, j \leq n, \text{ and } u_k^{ij} \geq 0 \text{ for all } 1 \leq i, j \leq n, \text{ and } 1 \leq k \leq m \right\}$, Each entry is a vector of the space $\mathbb{R}^m$, and the entries of these vectors are nonnegative real numbers. Then, C is a cone in $\mathscr{E}$. The cone C induces the ordered relation $u = \left\{ \{u_k^{ij}\}_{k=1}^m \right\}_{1 \leq i, j \leq n} \prec v = \left\{ \{v_k^{ij}\}_{k=1}^m \right\}_{1 \leq i, j \leq n}$ if and only if $v_k^{ij} \geq u_k^{ij}$ for every $k \in \{1, 2, \ldots, m\}$ and every $i, j, 1 \leq i, j \leq n$.

Now, let k be nonnegative real number such that $0 \leq k < 1$. Define $\theta_q: C \times C \longrightarrow C, q \in \{1, 2, 3, 4, 5\}$ by

$$\theta_1(u, v) = [u \oplus v] = \left\{ \{u_k^{ij} + v_k^{ij}\}_{k=1}^m \right\}_{1 \leq i, j \leq n},$$

$$\theta_2(u, v) = k[u \oplus v] = \left\{ \{k(u_k^{ij} + v_k^{ij})\}_{k=1}^m \right\}_{1 \leq i, j \leq n},$$

$$\theta_3(u, v) = \left\{ \{k(u_k^{ij} + v_k^{ij} + u_k^{ij} \times v_k^{ij})\}_{k=1}^m \right\}_{1 \leq i, j \leq n}, \quad (9)$$

$$\theta_4(u, v) = \left\{ \{u_k^{ij} + v_k^{ij} + u_k^{ij} \times v_k^{ij}\}_{k=1}^m \right\}_{1 \leq i, j \leq n},$$

$$\theta_5(u, v) = \left\{ \{(u_k^{ij} + v_k^{ij})(1 + u_k^{ij} \times v_k^{ij})\}_{k=1}^m \right\}_{1 \leq i, j \leq n}.$$

Then, the functions $\theta_q, q = 1, 2, \ldots, 5$, are all ordered actions on $\mathscr{E}$.

Furthermore, we replace $[0, \infty)$ by a cone in a normed space and use ordered actions to introduce the concept of action function cone metric spaces.

Definition 3. Let $(\mathscr{E}, \prec)$ be an ordered normed space, where $\prec$ is the ordered relation induced by a cone $C \subset \mathscr{E}$, X be a

nonempty set, and θ be an ordered action on $\mathscr{E}$. Then, the function $d_\theta\colon X \times X \longrightarrow C$ is called θ-cone-metric on X if and only if d_θ satisfies the following conditions:

(1) $d_\theta(x, y) = \Theta \Longleftrightarrow x = y$.

(2) $d_\theta(x, y) = d_\theta(y, x) \, \forall x, y \in X$.

(3) $d_\theta(x, y) \prec \theta\,(d_\theta(x, z), d_\theta(z, y)) \, \forall x, y, z \in X$.

The triple (X, C, d_θ) is defined to be a θ-cone-metric space or equivalently action function cone metric space.

Remark 1. We mention that the class of metric spaces is included in the class of θ-metric spaces if we consider $\theta(u, v) = u + v, u, v \in [0, \infty)$. Also, we mention that the class of θ-metric spaces is included in the class of θ-cone-metric spaces if we take $(\mathscr{E}, \prec) = ([0, \infty), \leq)$.

Example 2. Let $\mathscr{E}$, C, and θ_1 be given as in Example 1 $X = \mathbf{M_n}(\mathbb{R}^m)$ be the space of all $n \times n$ matrices whose entries are elements of the space $\mathbb{R}^m$. Then, X is a nonempty set. The function d_{θ_1}, $d_{\theta_1}\colon X \times X \longrightarrow C$, defined by

$$d_{\theta_1}(x, y) = \left\{\left\{\left\|x_{ij} - y_{ij}\right\|_p \delta_{jk}\right\}_{k=1}^m\right\}_{1 \leq i, j \leq n} = \left\{\left\{\sqrt[p]{\sum_{l=1}^k \left|x_l^{ij} - y_l^{ij}\right|^p} \delta_{jk}\right\}_{k=1}^m\right\}_{1 \leq i, j \leq n} \tag{10}$$

is a θ_1-cone metric on X, and (X, C, d_{θ_1}) is the θ_1-cone-metric space. In fact, conditions (1) and (2) are clear, and for any $x, y, z \in X$, we have

$$\left[d_{\theta_1}(x, z) + d_{\theta_1}(z, y)\right] = \left[\left\{\left\{\left\|x_{ij} - z_{ij}\right\|_p \delta_{jk}\right\}_{k=1}^m\right\}_{1 \leq i, j \leq n} + \left\{\left\{\left\|z_{ij} - y_{ij}\right\|_p \delta_{jk}\right\}_{k=1}^m\right\}_{1 \leq i, j \leq n}\right]$$

$$= \left[\left\{\left\{\left\|x_{ij} - z_{ij}\right\|_p \delta_{jk}\right\}_{k=1}^m + \left\{\left\|z_{ij} - y_{ij}\right\|_p \delta_{jk}\right\}_{k=1}^m\right\}_{1 \leq i, j \leq n}\right] = \left\{\left\{\left[\left\|x_{ij} - z_{ij}\right\|_p + \left\|z_{ij} - y_{ij}\right\|_p\right]\delta_{jk}\right\}_{k=1}^m\right\}_{1 \leq i, j \leq n}$$

$$= \left\{\left\{\left[\sqrt[p]{\sum_{l=1}^k \left|x_l^{ij} - z_l^{ij}\right|^p} + \sqrt[p]{\sum_{l=1}^k \left|z_l^{ij} - y_l^{ij}\right|^p}\right]\delta_{jk}\right\}_{k=1}^m\right\}_{1 \leq i, j \leq n}.$$

$$\tag{11}$$

Hence, $d_{\theta_1}(x, y) \prec \theta_1\,(d_{\theta_1}(x, z), d_{\theta_1}(z, y)) \, \forall x, y, x \in X$ because

$$\sqrt[p]{\sum_{l=1}^k \left|x_l^{ij} - y_l^{ij}\right|^p} \leq \left[\sqrt[p]{\sum_{l=1}^k \left|x_l^{ij} - z_l^{ij}\right|^p} + \sqrt[p]{\sum_{l=1}^k \left|z_l^{ij} - y_l^{ij}\right|^p}\right],$$

$$\forall 1 \leq i, j \leq n, 1 \leq k \leq m. \tag{12}$$

Example 3. Let $\mathscr{E}$, C, and θ_4 be given as in Example 1 and $X = \{x, y, z\}$ be three elements. Then, the function d_{θ_4}, $d_{\theta_4}\colon X \times X \longrightarrow C$, is defined by

$$d_{\theta_4}(x, x) = d_{\theta_4}(y, y) = d_{\theta_4}(z, z) = \Theta,$$

$$d_{\theta_4}(x, y) = d_{\theta_4}(y, x) = \left\{u_{ij}\right\}_{1 \leq i, j \leq n}, \quad \text{where } u_{11} = \{2, 0, 0, \ldots\}, u_{ij} = \{0, 0, 0, \ldots, 0\} \, \forall 1 < i, j \leq n,$$

$$d_{\theta_4}(y, z) = d_{\theta_4}(z, y) = \left\{v_{ij}\right\}_{1 \leq i, j \leq n}, \quad \text{where } v_{11} = \{3, 0, 0, \ldots\}, v_{ij} = \{0, 0, 0, \ldots, 0\} \, \forall 1 < i, j \leq n,$$

$$d_{\theta_4}(x, z) = d_{\theta_4}(z, x) = \left\{w_{ij}\right\}_{1 \leq i, j \leq n}, \quad \text{where } w_{11} = \{6, 0, 0, \ldots\}, w_{ij} = \{0, 0, 0, \ldots, 0\} \, \forall 1 < i, j \leq n.$$

$$\tag{13}$$

Note that d_{θ_4} is not a metric on X because $d_{\theta_4}(x, z) > d_{\theta_4}(x, y) + d_{\theta_4}(y, z)$, but it is a θ_4-cone metric on X, and (X, C, d_{θ_4}) is a θ_4-cone-metric space.

In 2017, J. Fernandez et al. introduced F-cone-metric spaces over Banach algebra and gave some generalization of some previous fixed point theorems.

In 2017, Suzuki [8] introduced different generalized approaches for the strongest sequentially compatible topology on a ν-generalized metric space and studied its characterizations.

In 2019, Abou Bakr [9] gave some common fixed point theorems (and, in particular, fixed points) of the generalized contraction type of cyclic mappings defined on cone metric spaces on Banach algebra.

In 2020, Abou Bakr [10] gave a study on the common fixed point of joint generalized types of contraction mappings in quasi-metric spaces.

2. Main Results

We have the following two sections.

2.1. Convergence and Cauchyness in θ-Cone-Metric Spaces.
Let us start with the following definition.

Definition 4. Let (X, C, d_θ) be a θ-cone-metric space. An open ball $N_{d_\theta}(x, u)$ at center $x \in X$ with a radius $u \in \text{Im}(\theta)$ (neighborhood of x at radius u) is defined as

$$N_{d_\theta}(x, u) =: \{y: y \in X, d_\theta(x, y) < u\}. \tag{14}$$

A subset $Y \subset X$ is bounded if and only if there are $x \in X$ and $u \in \text{Im}(\theta)$ such that $Y \subset N_{d_\theta}(x, u)$. Limit and interior points are defined in the usual way. Additionally, bounded, open, and closed sets in X are also defined.

Remark 2. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, C, d_θ) converges to x whenever for each $u \in \text{Im}(\theta)$ with $\Theta < u$, there is $n_0 \in \mathbb{N}$ such that $d_\theta(x_n, x) < u$ for all $n \geq n_0$. We instead write $x_n \xrightarrow{d_\theta}_{n \longrightarrow \infty} x$.

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, C, d_θ) is Cauchy whenever for each $u \in C$ with $\Theta < u$, there is $n_0 \in \mathbb{N}$ such that $d_\theta(x_n, x_m) < u$ for all $n, m \geq n_0$.

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, C, d_θ) is bounded if and only if there are $x \in X$ and $u \in \text{Im}(\theta)$ such that $\{x_n\}_{n \in \mathbb{N}} \subset N_{d_\theta}(x, u)$. Equivalently, a sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, C, d_θ) is bounded if and only if there are $x \in X$ and $u \in \text{Im}(\theta)$ such that $d_\theta(x_n, x) < u$ for all $n \in \mathbb{N}$.

We have the following lemmas.

Lemma 1. *Every neighborhood in (X, C, d_θ) is an open subset.*

Proof. Let $x \in X$ be an arbitrary element and $N_{d_\theta}(x, u)$ be any neighborhood of x with any radius $u \in C$; we show that, for every $y \in N_{d_\theta}(x, u)$, there is some $w \in \text{Im}(\theta)$ such that $N_{d_\theta}(y, w) \subset N_{d_\theta}(x, u)$. Set $v = d_\theta(x, y)$, and we have $u \in \text{Im}(\theta)$ and $\Theta < v < u$. Using the definition of θ, there is w, $\Theta < w < u$, such that $\theta(v, w) = u$. We claim that $N_{d_\theta}(y, w)$ is the required neighborhood. In fact, if $z \in N_{d_\theta}(y, w)$ is an arbitrary element, then $d_\theta(y, z) < w$. Now, using the definition of d_θ and then the definition of θ, we have the following:

$$d_\theta(x, z) < \theta(d_\theta(x, y), d_\theta(y, z)) < \theta(v, w) = u. \tag{15}$$

Hence, $z \in N_{d_\theta}(x, u)$; this completes the proof. □

Lemma 2. *A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, C, d_θ) converges to x if and only if $d_\theta(x_n, x) \longrightarrow_{n \longrightarrow \infty} \Theta$ converges in $\mathscr{E}$.*

Proof. Suppose that $x_n \xrightarrow{d_\theta}_{n \longrightarrow \infty} x$, and let $\varepsilon > 0$. Then, choose $u \in \text{Im}(\theta)$, $\Theta < u$ with $M\|u\| < \varepsilon$, where M is the normal constant of C; for this u, there is $n_0 \in \mathbb{N}$ such that $d_\theta(x_n, x) < u$ for all $n \geq n_0$; hence, $\|d_\theta(x_n, x)\| < M\|u\|$ for all $n \geq n_0$, and accordingly, we have $\|d_\theta(x_n, x)\| < \varepsilon$ for all $n \geq n_0$. This proves that $d_\theta(x_n, x) \longrightarrow_{n \longrightarrow \infty} \Theta$.

Conversely, let $d_\theta(x_n, x) \longrightarrow_{n \longrightarrow \infty} \Theta [\|d_\theta(x_n, x)\| \longrightarrow_{n \longrightarrow \infty} 0]$; we show that $x_n \xrightarrow{d_\theta}_{n \longrightarrow \infty} x$. Let $u \in \text{Im}(\theta)$ be arbitrary; since θ is continuous, there is a neighborhood of u with some radius $\varepsilon > 0$, $N_\varepsilon(u)$ in $\mathscr{E}$ and $N_\varepsilon(u) \subset C$; for this ε, there is a natural number $n_0 \in \mathscr{N}$ such that $\|d_\theta(x_n, x)\| < \varepsilon$ for every $n \geq n_0$, and since $d_\theta(x_n, x) = u - [u - d_\theta(x_n, x)]$, we see that $\|u - [u - d_\theta(x_n, x)]\| < \varepsilon$ for every $n \geq n_0$; hence, $[u - d_\theta(x_n, x)] \in N_\varepsilon(u)$, and consequently, $u - d_\theta(x_n, x) \in C$, that is, $d_\theta(x_n, x) < u$. This proves that $x_n \xrightarrow{d_\theta}_{n \longrightarrow \infty} x$. □

Lemma 3. *The limit of any sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, C, d_θ) is unique.*

Proof. Suppose that $x_n \xrightarrow{d_\theta}_{n \longrightarrow \infty} x$ and $x_n \xrightarrow{d_\theta}_{n \longrightarrow \infty} y$; we show that $x = y$. Using Lemma 2, we have $d_\theta(x_n, x) \longrightarrow_{n \longrightarrow \infty} \Theta$ and $d_\theta(x_n, y) \longrightarrow_{n \longrightarrow \infty} \Theta$, and using the definition of d_θ and the continuity of θ, we have

$$\Theta < d_\theta(x, y) < \theta(d_\theta(x_n, x), d_\theta(x_n, y)) \xrightarrow{n \longrightarrow \infty} \theta(\Theta, \Theta) = \Theta. \tag{16}$$

Hence, $d_\theta(x, y) = \Theta$, that is, $x = y$. □

Lemma 4. *Let (X, C, d_θ) be a θ-cone-metric space, $x_n \xrightarrow{d_\theta}_{n \longrightarrow \infty} x$, and $y_n \xrightarrow{d_\theta}_{n \longrightarrow \infty} y$. Then,*

$$d_\theta(x_n, y_n) \xrightarrow{n \longrightarrow \infty} d_\theta(x, y). \tag{17}$$

Proof. Using Lemma 4, we have $d_\theta(x_n, x) \longrightarrow_{n \longrightarrow \infty} \Theta$ and $d_\theta(y_n, y) \longrightarrow_{n \longrightarrow \infty} \Theta$ since $d_\theta(x, y_n) < \theta(d_\theta(x, y), d_\theta(y, y_n))$ and θ is continuous; then, conditions (2) and (4) of θ imply the following:

$$d_\theta(x, y_n) < \theta(d_\theta(x, y), d_\theta(y, y_n))$$
$$\xrightarrow{n \longrightarrow \infty} \theta(d_\theta(x, y), \Theta) < d_\theta(x, y), \tag{18}$$

and consequently,

$$d_\theta(x_n, y_n) \preccurlyeq \theta(d_\theta(y_n, x), d_\theta(x, x_n)) \text{ use the continuity of } \theta$$
$$\xrightarrow{n \longrightarrow \infty} \theta(d_\theta(x, y), \Theta) \text{ use condition (4) of } \theta$$
$$\preccurlyeq d_\theta(x, y), \tag{19}$$

Using the continuity of θ once more gives

$$d_\theta(x, y) \prec \theta(d_\theta(x, x_n), \theta(d_\theta(x_n, y_n), d_\theta(y_n, y)))$$
$$\underset{n \longrightarrow \infty}{\longrightarrow} \lim_{n \longrightarrow \infty} \theta(d_\theta(x, x_n), \theta(d_\theta(x_n, y_n), d_\theta(y_n, y)))$$
$$\prec \theta\left(\lim_{n \longrightarrow \infty} d_\theta(x, x_n), \lim_{n \longrightarrow \infty} \theta(d_\theta(x_n, y_n), d_\theta(y_n, y))\right)$$
$$\prec \theta\left(\lim_{n \longrightarrow \infty} d_\theta(x, x_n), \theta\left(\lim_{n \longrightarrow \infty} d_\theta(x_n, y_n), \lim_{n \longrightarrow \infty} d_\theta(y_n, y)\right)\right)$$
$$\prec \theta\left(\Theta, \theta\left(\lim_{n \longrightarrow \infty} d_\theta(x_n, y_n), \Theta\right)\right) \prec \theta\left(\lim_{n \longrightarrow \infty} d_\theta(x_n, y_n), \Theta\right)$$
$$\prec \lim_{n \longrightarrow \infty} d_\theta(x_n, y_n).$$

(20)

Using inequalities (19) and (20) gives $d_\theta(x_n, y_n) \underset{n \longrightarrow \infty}{\longrightarrow} d_\theta(x, y)$ and completes the proof. □

Lemma 5. *A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (X, C, d_θ) is Cauchy if and only if*

$$d_\theta(x_n, x_m) \underset{n,m \longrightarrow \infty}{\longrightarrow} \Theta. \tag{21}$$

Proof. Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy, and let $\varepsilon > 0$. Then, choose $u \in \text{Im}(\theta)$, $\Theta \prec u$ with $M\|u\| < \varepsilon$, where M is the normal constant of C; for this u, there is $n_0 \in \mathbb{N}$ such that $d_\theta(x_n, x_m) < u$ for all $n, m \geq n_0$; hence, $\|d_\theta(x_n, x_m)\| < M\|u\|$ for all $n, m \geq n_0$, and accordingly, we have $\|d_\theta(x_n, x_m)\| < \varepsilon$ for all $n, m \geq n_0$. This proves that $d_\theta(x_n, x_m) \longrightarrow_{n,m \longrightarrow \infty} \Theta$.

Conversely, let $d_\theta(x_n, x_m) \longrightarrow_{n,m \longrightarrow \infty} \Theta$ $[\|d_\theta(x_n, x_m)\| \longrightarrow_{n,m \longrightarrow \infty} 0]$; we show that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy. Let $u \in \text{Im}(\theta)$ be arbitrary; since θ is continuous, there is a neighborhood of u with some radius $\varepsilon > 0$, $N_\varepsilon(u)$ in $\mathscr{E}$ such that $N_\varepsilon(u) \subset C$; for this ε, there is a natural number $n_0 \in \mathscr{N}$ such that $\|d_\theta(x_n, x_m)\| < \varepsilon$ for every $n, m \geq n_0$, and since $d_\theta(x_n, x_m) = u - [u - d_\theta(x_n, x_m)]$, we see that $\|u - [u - d_\theta(x_n, x_m)]\| < \varepsilon$ for every $n, m \geq n_0$. Hence, $[u - d_\theta(x_n, x_m)] \in N_\varepsilon(u)$, and consequently, $u - d_\theta(x_n, x_m) \in C$, that is, $d_\theta(x_n, x_m) \prec u$ for every $n, m \geq n_0$. This proves that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy. □

Lemma 6. *Every convergent sequence in the θ-cone-metric space is Cauchy.*

Proof. Let $\{x_n\}_{n \in \mathbb{N}}$ in (X, C, d_θ) converge to x, and let $m, n \in \mathbb{N}$ with $m \geq n$. Then, the continuity of θ insures the following:

$$d_\theta(x_n, x_m) \prec \theta(d_\theta(x, x_n), d_\theta(x, x_m)) \underset{n,m \longrightarrow \infty}{\longrightarrow} \theta(\Theta, \Theta) = \Theta. \tag{22}$$

The following lemmas are mainly used in the upcoming generalizations of fixed point theorems. □

Lemma 7. *Let (X, C, d_θ) be a θ-cone-metric space and $\{x_n\}_{n \in \mathbb{N}}$ be in X such that $\{d_\theta(x_n, x_{n+1})\}_{n \in \mathbb{N}}$ converges to Θ. Then, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence.*

Proof. Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is not Cauchy; then, there exist $u \in \text{Im}(\theta)$, $\Theta \prec u$, and sequences $\{J(n)\}_{n \in \mathbb{N}}$ and $\{k(n)\}_{n \in \mathbb{N}}$ of natural numbers such that, for any $J(n) > k(n) > n$,

$$u \prec d_\theta\left(x_{J(n)}, x_{k(n)}\right) \text{ and } d_\theta\left(x_{J(n)-1}, x_{k(n)}\right) < u \quad \forall n \in \mathbb{N}. \tag{23}$$

Using the definition and the continuity of θ and the fact that any subsequence of $\{d_\theta(x_n, x_{n+1})\}_{n \in \mathbb{N}}$ converges to Θ, we have the following contradiction:

$$u \prec d_\theta\left(x_{J(n)}, x_{k(n)}\right)$$
$$\prec \theta\left(d_\theta\left(x_{J(n)-1}, x_{k(n)}\right), d_\theta\left(x_{J(n)}, x_{j(n)-1}\right)\right)$$
$$< \theta\left(u, d_\theta\left(x_{J(n)}, x_{j(n)-1}\right)\right) \tag{24}$$
$$< \theta\left(u, \lim_{n \longrightarrow \infty} d_\theta\left(x_{J(n)}, x_{j(n)-1}\right)\right)$$
$$= \theta(u, \Theta) < u.$$

□

Lemma 8. *Let (X, C, d_θ) be a θ-cone-metric space and $\{x_n\}_{n \in N}$ be in X such that*

$$d_\theta(x_{n+2}, x_{n+1}) \prec t_0 d_\theta(x_{n+1}, x_n), \quad n = 0, 1, 2, \ldots, \tag{25}$$

for some real number t_0, $0 \leq t_0 < 1$. Then, $\{x_n\}_{n \in N}$ is Cauchy.

Proof. Taking into account the conditions $\alpha C \subset C$ for every nonnegative real number α, $C + C \subset C$, and $d_\theta(x_{n+1}, x_n) \prec t_0 d_\theta(x_n, x_{n-1})$ imply

$$t_0 d_\theta(x_n, x_{n-1}) - d_\theta(x_{n+1}, x_n) \in C. \tag{26}$$

Hence,

$$t_0[t_0 d_\theta(x_n, x_{n-1}) - d_\theta(x_{n+1}, x_n)] \in C, \tag{27}$$

that is,

$$t_0^2 d_\theta(x_n, x_{n-1}) - t_0 d_\theta(x_{n+1}, x_n) \in C. \tag{28}$$

Additionally, $d_\theta(x_{n+2}, x_{n+1}) \prec t_0 d_\theta(x_{n+1}, x_n)$ gives

$$t_0 d_\theta(x_{n+1}, x_n) - d_\theta(x_{n+2}, x_{n+1}) \in C. \tag{29}$$

Inclusions (28) and (29) yield the following:

$$\left[t_0^2 d_\theta(x_n, x_{n-1}) - t_0 d_\theta(x_{n+1}, x_n)\right] + \left[t_0 d_\theta(x_{n+1}, x_n) - d_\theta(x_{n+2}, x_{n+1})\right] \in C. \tag{30}$$

Thus,

$$\left[t_0^2 d_\theta(x_n, x_{n-1}) - d_\theta(x_{n+2}, x_{n+1})\right] \in C, \tag{31}$$

and hence,

$$d_\theta(x_{n+2}, x_{n+1}) \prec t_0 d_\theta(x_{n+1}, x_n) \prec t_0^2 d_\theta(x_n, x_{n-1}). \tag{32}$$

This implies successively

$$d_\theta(x_n, x_{n+1}) \prec t_0 d_\theta(x_{n-1}, x_n)$$
$$\prec t_0^2 d_\theta(x_{n-2}, x_{n-1})$$
$$\cdots \tag{33}$$
$$\prec t_0^n d_\theta(x_0, x_1).$$

Since $0 \le t_0 < 1$, the sequence $\{t_0^n\}_{n \in N}$ is convergent to 0, and taking the limit as $n \longrightarrow \infty$ of the two sides of (33) shows that $t_0^n d_\theta(x_1, x_0) \longrightarrow \Theta$ as $n \longrightarrow \infty$; hence, $\lim_{n \to \infty} d_\theta(x_n, x_{n+1}) = \Theta$. Using Lemma 7 shows that $\{x_n\}_{n \in N}$ is Cauchy and completes the proof. □　　□

2.2. Fixed Point Theorem for Contraction Types of Mappings.
We have the following definition.

Definition 5. A θ-cone-metric space (X, C, d_θ) is complete whenever every Cauchy sequence converges to an element belonging to it.

Banach's contraction principle [11] is one of the pivotal results of nonlinear analysis and its applications, which establishes that every contraction mapping defined on a complete metric space has a unique fixed point. In this paper, we use the concept of θ-cone-metric spaces to generalize the Banach contraction principle as follows.

Theorem 1. *Let (X, C, d_θ) be a complete θ-cone-metric space on a normed space $\mathscr{E}$ and $S: X \longrightarrow X$ be a mapping that satisfies the following:*

$$d_\theta(S(x), S(y)) \prec t_0 d_\theta(x, y), \quad \forall x, y \in X, \tag{34}$$

for some real number t_0, $0 \le t_0 < 1$. Then, S has a unique fixed point.

Proof. Let $x_0 \in X$ be a given element; then, the iterated sequence $\{S^n(x_0)\}_{n \in \mathbb{N}}$, $x_n = S^n(x)$, satisfies the following:

$$\begin{aligned} d_\theta(x_{n+2}, x_{n+1}) &= d_\theta(S(x_{n+1}), S(x_n)) \\ &\prec t_0 d_\theta(x_{n+1}, x_n), \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{35}$$

Using Lemma 8, the sequence of iterates is a Cauchy sequence. According to the completeness of X, there exists $x \in X$ such that $x_n \xrightarrow[n \to \infty]{d_\theta} x$; we claim that x is a fixed point of S. In fact, we have

$$\begin{aligned} d_\theta(x_{n+1}, S(x)) &= d_\theta(S(x_n), S(x)), \quad n = 0, 1, 2, \ldots \\ &\prec t_0 d_\theta(x_n, x) \xrightarrow[n \to \infty]{} \Theta. \end{aligned} \tag{36}$$

It means that $x_n \xrightarrow[n \to \infty]{d_\theta} S(x)$; using Lemma 3 proves that $S(x) = x$. Finally, we prove that x is the unique fixed point of S. Suppose, on the contrary, that x and y are two distinct fixed points of S; then, $d_\theta(x, y) \ne \Theta$, and on the other side, $(1 - t_0) d_\theta(x, y) \in C$ because $(1 - t_0) > 0$, that is, $t_0 d_\theta(x, y) \prec d_\theta(x, y)$. So, we get the following contradiction:

$$d_\theta(x, y) = d_\theta(S(x), S(y)) \prec t_0 d_\theta(x, y) \prec d_\theta(x, y). \tag{37}$$

One of beautiful generalizations of the Banach contraction principle was found by Reich [12] and further generalized by Hardy and Rogers [13]. In this paper, in case of the above introduced θ-cone-metric space setting, we give the following further generalization. □

Theorem 2. *Let (X, C, d_θ) be a complete θ-cone-metric space on a normed space $\mathscr{E}$ and $S: X \longrightarrow X$ be a mapping that satisfies the following:*

$$\begin{aligned} d_\theta(S(x), S(y)) &\prec a\, d_\theta(x, y) + b\, d_\theta(x, S(x)) \\ &\quad + c\, d_\theta(y, S(y)), \quad \forall x, y \in X, \end{aligned} \tag{38}$$

for some real numbers a, b, c, $0 < a + b + c < 1$. Then, S has a unique fixed point.

Proof. Let $x_0 \in X$ be a given element. Then, the iterated sequence $\{S^n(x_0)\}_{n \in \mathbb{N}}$, $x_n = S^n(x)$, satisfies the following:

$$\begin{aligned} d_\theta(x_{n+2}, x_{n+1}) &= d_\theta(S(x_{n+1}), S(x_n)), \quad n = 0, 1, 2, \ldots \\ &\prec a\, d_\theta(x_{n+1}, x_n) + b\, d_\theta(x_{n+1}, S(x_{n+1})) \\ &\quad + c\, d_\theta(x_n, S(x_n)) \\ &\prec [a + c] d_\theta(x_{n+1}, x_n) + b\, d_\theta(x_{n+1}, x_{n+2}), \end{aligned} \tag{39}$$

and consequently, $[a + c] d_\theta(x_{n+1}, x_n) - [1 - b]\ d_\theta(x_{n+2}, x_{n+1}) \in C$; hence, $[1/1 - b]\,([a + c] d_\theta(x_{n+1}, x_n) - [1 - b] d_\theta(x_{n+2}, x_{n+1})) \in C$, that is, $[a + c]/[1 - b] d_\theta(x_{n+1}, x_n) - d_\theta(x_{n+2}, x_{n+1}) \in C$. Therefore, we have

$$d_\theta(x_{n+2}, x_{n+1}) \prec \frac{[a + c]}{[1 - b]} d_\theta(x_{n+1}, x_n), \quad n = 0, 1, 2, \ldots. \tag{40}$$

Since $[a + c]/[1 - b] < 1$, $d_\theta(x_{n+1}, x_n) \xrightarrow[n \to \infty]{d_\theta} \Theta$, and using Lemma 7 proves that the sequence of iterates is a Cauchy sequence. According to the completeness of X, there exists $x \in X$ such that $x_n \xrightarrow[n \to \infty]{d_\theta} x$ and $\lim_{n \to \infty} d_\theta(x_n, x) = \Theta$; we show that x is a fixed point of S. In fact, we have

$$\begin{aligned} d_\theta(x_{n+1}, S(x)) &= d_\theta(S(x_n), S(x)), \quad n = 0, 1, 2, \ldots \\ &\prec a\, d_\theta(x_n, x) + b\, d_\theta(x, S(x)) + c\, d_\theta(x_n, S(x_n)) \\ &\prec a\, d_\theta(x_n, x) + b\, d_\theta(x, S(x)) + c\, d_\theta(x_n, x_{n+1}) \\ &\xrightarrow[n \to \infty]{} b\, d_\theta(x, S(x)), \end{aligned} \tag{41}$$

$$\begin{aligned} d_\theta(x, S(x)) &\prec \theta\big(d_\theta(x, x_n), d_\theta(x_n, S(x))\big) \\ &\prec \theta\big(d_\theta(x, x_n), \theta\big(d_\theta(x_n, x_{n+1}), d_\theta(x_{n+1}, S(x))\big)\big) \\ &\xrightarrow[n \to \infty]{} \theta\Big(\Theta, \theta\Big(\Theta, \lim_{n \to \infty} d_\theta(x_{n+1}, S(x))\Big)\Big) \\ &\prec \theta\Big(\Theta, \lim_{n \to \infty} d_\theta(x_{n+1}, S(x))\Big). \\ &\prec \lim_{n \to \infty} d_\theta(x_{n+1}, S(x)). \end{aligned} \tag{42}$$

Using inequalities (41) and (42) proves that $d_\theta(x, S(x)) \prec b\, d_\theta(x, S(x))$; hence, $(b - 1) d_\theta(x, S(x)) \in C$, and consequently, $d_\theta(x, S(x)) = \Theta$ because $C \cap -C = \{\Theta\}$, that is, $S(x) = x$. Finally; we prove that x is the unique fixed point of S. Suppose, on the contrary, that x and y are two distinct fixed points of S. So, we get the following:

$$d_\theta(x, y) = d_\theta(S(x), S(y)) \prec a\, d_\theta(x, y) + b\, d_\theta(x, S(x))$$
$$+ c\, d_\theta(y, S(y)) = a\, d_\theta(x, y).$$

(43)

Hence, $[a\, d_\theta(x, y) - d_\theta(x, y)] \in C$ and $[a - 1] d_\theta(x, y) \in C$, but $[a - 1] < 0$; this implies that $d_\theta(x, y) = \Theta$ because $C \cap - C = \{\Theta\}$, that is, $x = y$. $\square$

3. Conclusion

In this paper, we generalize the concept of θ-metric space to the concept of the θ-cone-metric space as a generalization of a metric by replacing the triangle inequality with a more generalized inequality using some θ-action functions and replacing the usual ordered relation of real numbers by ordered relation induced by a cone in a normed space. We investigate the convergence and Cauchyness in such a θ-cone-metric space. Furthermore, we prove some lemmas and use them to give further generalizations of some well-known fixed point theorems in this context [14].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

Authors' Contributions

The sole author contributed 100% to the article. The author read and approved the final manuscript.

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