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Research Article

Some Results on Strongly Cesáro Ideal Convergent Sequence Spaces

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Some algebraic properties of Cesáro ideal convergent sequence spaces, C^I and C_0^I, are studied in this article and some inclusion relations on these spaces are established.

1. Introduction

Consider the space $\omega = \{x = (x_k): x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$ of all real and complex sequences, where $\mathbb{R}$ and $\mathbb{C}$ are, respectively, the sets of all real and complex numbers.

Suppose that ℓ_∞, c, and c_0 are the linear spaces of bounded, convergent, and null sequences, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|, \quad \text{where}, k \in N, \tag{1}$$

$\mathbb{N}$ being the set of all natural numbers.

A sequence space $x = (x_k)$ of complex numbers is said to be $(C, 1)$ summable to $L \in C$ if for $\rho_k = 1/k \sum_{i=1}^{k} x_i$, $\lim_k \rho_k = L$. The sequence $(C, 1)$ is also called Cesáro summable sequence of complex numbers over C. Let us denote by C_1 the linear space of all $(C, 1)$ summable sequences of complex numbers over C, i.e.,

$$C_1 = \left\{ x = (x_k) \in \omega: \frac{1}{k} \sum_{i=1}^{k} x_i \in c \right\}. \tag{2}$$

Hardy and Littlewood [1] initiated the notion of strong Cesáro convergence for real numbers which is defined as follows.

A sequence (x_k) on a normed space $(X, \|\cdot\|)$ is said to be strongly Cesáro convergent to L if

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k - L\| = 0. \tag{3}$$

In [2–6], the authors have extended the notion of strong Cesáro convergence in various fields. In 1951, Fast [7] introduced the term statistical convergence, while Steinhaus [8] independently introduced the term "ordinary and asymptotic convergences."

Later on, Fridy [9, 10] also studied the statistical convergence and he linked it with the summability theory. Kostyrko et al. [11] gave the concept of ideal convergence (I-convergence) which was indeed a generalization of statistical convergence. Salat et al. [12] studied some properties of I-convergence, and further investigations in this field are done by Khan [13], Tripathy and Esi [14], Tripathy and Hazarika [15], and many others.

In this article, further interesting properties of Cesáro Ideal Convergent Sequences are established and a few inclusion relations are also proved.

2. Definitions of the Terms Used

Let us first present some definitions and notions that are required in the sequel.

(1) A family of subsets I of $\mathbb{N}$ is called an *ideal set* in $\mathbb{N}$

(i) If $\phi /\in I$

(ii) If the sets A, $B \in I$, then $A \cup B \in I$
(iii) If $B \subseteq A$ and $A \in I$, then $B \in I$

(2) A nontrivial ideal set I is said to be *admissible* if

$\{\{n\}: n \in \mathbb{N}\}\} \subset$ I

(3) A nonempty set $F \in 2^{\mathbb{N}}$ is known as a *filter* in $\mathbb{N}$ if

(a) $\phi / \notin F$
(b) A, $B \in F \Rightarrow A \cap B \in F$
(c) $A \in F$ with $A \subseteq AB \Rightarrow B \in F$

Remark 1. For every ideal I, there is a filter $F(I)$ (associated with I) defined as follows:

$$F(I) = \left\{ P \subseteq \mathbb{N}: \frac{\mathbb{N}}{p \varepsilon I} \right\}. \tag{4}$$

A sequence $(x_k) \in X$ is said to be *I-convergent* to a number L if, for every $\epsilon > 0$, the set $\{x = (x_k) \in X: \{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \in I\}$. In this case, we write $I - \lim x_k = L$. If $L = 0$, then it is called *I-null.*

A sequence $x_k \in \omega$ is said to be *I-Cauchy* if, for every $\varepsilon > 0$, there exists a number $m = m(\varepsilon)$ such that

$$\left\{ x = x_k \in X: \left\{ \left| x_n - x_m \right| \geq \varepsilon \right\} \in I \right\}. \tag{5}$$

Let *If* be the class of all finite subsets of $\mathbb{N}$. If $I = I_f$If, then I is *admissible ideal set* in $\mathbb{N}$.

A sequence space X is said to be *solid (normal)* if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ and (α_k) is a sequence of scalars with $|\alpha k| \leq 1$, for all $k \in \mathbb{N}$.

A sequence space X is a *Sequence Algebra* if, for every (x_k), $(y_k) \in X$, $(s_k, y_k) \in X$.

Let $K = \{k_1 < k_2 < k_3, \ldots.\} \subset \mathbb{N}$ and X be a sequence space. A *K-step space* of X is a sequence space $\lambda_K^X = \left\{ (x_{k_n}) \in \omega: (x_k) \in X \right\}$.

A *canonical preimage* of a sequence $(x_{kn}) \in \lambda_K^X$is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

A sequence space is *monotone* if it contains the canonical preimages of its step spaces.

3. Result

A canonical preimage of a step space λ_K^X is a set of preimages of all elements in λ_K^X, i.e., y is in the canonical preimage of λ_K^Xif and only if y is the canonical preimage of some $x \in \lambda_K^X$.

Let X and Y be two normed linear spaces. An operator T: $X \longrightarrow Y$ is known as a *compact linear operator* if [16].

(a) T is linear

(b) If, for every bounded subset D of X, the image $M(D)$ is relatively compact, i.e, the closure $\overline{T(D)}$is compact

Lemma 1 (see [12]). Every solid space is monotone.

Lemma 2 (see [12]). Let $K \in F(I)$ and $M \subseteq \mathbb{N}$. If $M \notin$ I, then $M \cap$ K $\notin$ I.

Lemma 3 (see [11]). Let $I \subset 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin$ I, then $M \cap$ K $\notin$ I.

4. Main Results

Let us first define C^I, the space of all Cesáro ideal convergent sequences and C_0^I, the space of all Cesáro ideal null sequences which are given as follows:

$$\begin{aligned} C^I &= \left\{ x = (x_k) \in \omega: I - \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \left\| x_k - L \right\| \right. \\ &\quad = 0, \text{ for some } L \in C\}, \\ C_0^I &= \left\{ x = (x_k) \in \omega: I - \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \left\| x_k \right\| = 0, \right\}. \end{aligned} \tag{7}$$

Theorem 1. *The sequence spaces C^I and C_0^I are linear.*

Proof. Assume that $x = (x_k)$, $y = (y_k) \in C^I$. Then, one has

$$\begin{aligned} I - \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \left\| x_k - L_1 \right\| &= 0, \quad \text{for some } L_1 \in C, \\ I - \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \left\| y_k - L_2 \right\| &= 0, \quad \text{for some } L_2 \in C. \end{aligned} \tag{8}$$

Let

$$A_1 = \left\{ k \in N: \frac{1}{n} \sum_{k=1}^{n} \left\| x_k - L_1 \right\| \right\}, \tag{9}$$

$$A_2 = \left\{ k \in N: \frac{1}{n} \sum_{k=1}^{n} \left\| y_k - L_2 \right\| \right\}. \tag{10}$$

Let α and β be some scalers.

By using the properties of norm, one can easily see that

$$\begin{aligned} &\lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \left\| (a x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \right\| \\ &\leq \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} |\alpha| \left\| x_k - L_1 \right\| + \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} |\beta| \left\| y_k - L_2) \right\|. \end{aligned} \tag{11}$$

Then, from (9) and (10), we have for each $\varepsilon > 0$,

$$\left\{k \in N: \lim_{n\longrightarrow\infty} \frac{1}{n}\sum_{k=1}^{n}\left\|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)\right\| > \varepsilon\right\}$$
$$\subset A_1 \cup A_2. \tag{12}$$

Therefore, $(\alpha x_k + \beta y_k) \in C^I$, for all scalars α, β and (x_k), $(y_k)\in C^I$.

Hence, C^I is a linear space.

On the similar manner, one can prove that $C_0^I$is also linear. □

Theorem 2. *Let $x = (x_k) \in \omega$ be any sequence Then, $C_0^I \subset C^I$.*

Proof. It can be easily observed. □

Theorem 3. *A sequence $x = (x_k) \in C^I$ is I-convergent if and only if, for every $\varepsilon > 0$, there exists $l = l(\varepsilon)\in\mathbb{N}$ such that*

$$\left\{k \in N: \frac{1}{n}\sum_{k=1}^{n}\left\|x_k - x_l\right\| < \varepsilon\right\} \in F(I). \tag{13}$$

Proof. Suppose that $x = (x_k) \in C^I$. Therefore, $I - \lim_{n\longrightarrow\infty} 1/n \sum_{k=1}^{n} \|x_k - L\| = 0$. Then, for all ε>0 the set

$$C_\varepsilon = \left\{k \in N: \frac{1}{n}\sum_{k=1}^{n}\left\|x_k - L\right\| < \frac{\varepsilon}{2}\right\} \in F(I). \tag{14}$$

Fix an $l(\varepsilon)\in\in C_\varepsilon$. Then, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left\|x_k - x_l\right\| \le \frac{1}{n}\sum_{k=1}^{n}\left\|x_k - L\right\| + \frac{1}{n}\sum_{k=1}^{n}\left\|x_l - L\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{15}$$

which holds for all $k \in\in C_\varepsilon$. Hence,

$$\left\{k \in N: \frac{1}{n}\sum_{k=1}^{n}\left\|x_k - x_l\right\| < \varepsilon\right\} \in F(I). \tag{16}$$

Conversely, suppose that, for all $\varepsilon > 0$, the set

$$\left\{k \in N: \frac{1}{n}\sum_{k=1}^{n}\left\|x_k - x_l\right\| < \varepsilon\right\} \in F(I). \tag{17}$$

Then, for every $\varepsilon > 0$, we have

$$B_\varepsilon = \left\{k \in N: \frac{1}{n}\sum_{k=1}^{n}\left\|x_k\right\| \in \left[\frac{1}{n}\sum_{k=1}^{n}\left\|x_l\right\| - \varepsilon, \frac{1}{n}\sum_{k=1}^{n}\left\|x_l\right\| + \varepsilon\right]\right\} \in F(I), \tag{18}$$

$$\text{Let, } P_\varepsilon = \left[\frac{1}{n}\sum_{k=1}^{n}\left\|x_l\right\| - \varepsilon, \frac{1}{n}\sum_{k=1}^{n}\left\|x_l\right\| + \varepsilon\right].$$

For fixed $\varepsilon > 0$,one has $B_\varepsilon \in F(I)$as well as $B_{\varepsilon/2}\in F(I)$. Hence, $B_\varepsilon \cap B_{\varepsilon/2} \in F(I)$.

This implies that $B_\varepsilon \cap B_{\varepsilon/2} \neq \phi$, that is,

$$\left\{k \in N: \frac{1}{n}\sum_{k=1}^{n}\left\|x_k\right\| \in P\right\} \in F(I). \tag{19}$$

That is diam $P \le$ diam P_ε, where the diam P denotes the length of the interval of P.

In this way, by induction, one obtains the sequence of closed intervals:

$$P_\varepsilon = J_0 \supseteq J_1 \supseteq J_2, \cdots, \supseteq J_k \supseteq, \cdots, \tag{20}$$

with the property that diam $J_k \le 1/2$diam J_{k-1} for $k = 1, 2, 3, \ldots$, and

$$\left\{k \in N: \frac{1}{n}\sum_{k=1}^{n}\left\|x_k\right\| \in J_k\right\} \in F(I), \tag{21}$$

for $k = 1,2,3, \ldots$,. Then, there exists a $L \in \cap J_k$ such that $L = I - \lim_{n\longrightarrow\infty} 1/n \sum_{k=1}^{n} \|x_k\|$showing that $x = (x_k)\in C^I$ is I-convergent. Hence, the result holds. □

Theorem 4. *The space $C_0^I$is solid and monotone.*

Proof. Let $(x_k) \in C_0^I$be any element. Then, one has

$$\left\{k \in N: I - \lim_{n\longrightarrow\infty} \frac{1}{n}\sum_{k=1}^{n}\left\|x_k\right\| = 0\right\}. \tag{22}$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \le 1$, for all $k\in\mathbb{N}$, and hence $1/n \sum_{k=1}^{n} |a_k| \le 1$.

Then, the result (that C_0^I is solid) follows from the above equation and inequality:

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left\|\alpha_k x_k\right\| &= \frac{1}{n}\sum_{k=1}^{n}\left|\alpha_k\right|\left\|x_k\right\| = \frac{1}{n}\sum_{k=1}^{n}\left|\alpha_k\right| \frac{1}{n}\sum_{k=1}^{n}\left\|x_k\right\| \\ &\le \frac{1}{n}\sum_{k=1}^{n}\left\|x_k\right\|,\end{aligned} \tag{23}$$

for all $k\in\mathbb{N}$.

The space $C_0^I$is monotone which follows from Lemma 1.

Hence, $C_0^I$is solid and monotone. □

Theorem 5. *The space C^I is neither solid nor monotone.*

Proof. For this theorem, we provide a counter example for the proof. □

5. Counter Example

Let $I = I_f$, and consider the k-step χ_k of χ defined as follows.

Let $(x_k)\in\chi$ and let $(y_k) \in\chi_k$ be such that

$$y_k = \begin{vmatrix} x_k, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{vmatrix} \tag{24}$$

Let us consider the sequence (x_k) defined by $x_k = 1$ for all $k \in\mathbb{N}$. Then, $(x_k)\in C^I$, but its K-step preimages do not belong to C^I. Thus, $(x_k) \in C^I$ is not monotone.

Hence, $(x_k) \in C^I$ is not solid.

Theorem 6. *Let $x=(x_k)$ and $y=(y_k)$ be two sequences in such a way that $T(x\cdot y)=T(x)T(y)$. Then, the space C^I and C_0^I are sequence algebra.*

Proof. Let $x=(x_k)$ and $y=(y_k)$ be two elements of C^I with

$$T(x\cdot y)=T(x)T(y). \tag{25}$$

For every $\varepsilon>0$ select $\beta>0$ in such a way that $\varepsilon<\beta$, then

$$\left\{k\in N\colon \frac{1}{n}\sum_{k=1}^{n}\left\|T(x_k)-L_1\right\|<\frac{\varepsilon}{2\beta}\right\}\in F(I),$$
$$\left\{k\in N\colon \frac{1}{n}\sum_{k=1}^{n}\left\|T(y_k)-L_2\right\|<\frac{\varepsilon}{2\left|L_1\right|}\right\}\in F(I). \tag{26}$$

Using the above and the property of norm, one obtains

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left\|T(x_k.y_k)-L_1L_2\right\|&=\frac{1}{n}\sum_{k=1}^{n}\left\|T(x_k)T(y_k)-L_1L_2\right\|\\&=\frac{1}{n}\sum_{k=1}^{n}\left\|T(x_k)T(y_k)-L_1T(y_k)+L_1T(y_k)-L_1L_2\right\|\\&\le\frac{1}{n}\sum_{k=1}^{n}\left\|T(y_k)\right\|\frac{1}{n}\sum_{k=1}^{n}\left\|T(x_k)-L_1\right\|+\left|L_1\right|\frac{1}{n}\sum_{k=1}^{n}\left\|T(y_k)-L_2\right\|<\frac{\varepsilon^2}{2\beta}+\left|L_1\right|\frac{\varepsilon}{2\left|L_1\right|}<\varepsilon.\end{aligned} \tag{27}$$

Therefore, the set

$$\left\{k\in N\colon \frac{1}{n}\sum_{k=1}^{n}\left\|T(x_k.y_k)-L_1L_2\right\|\ge\varepsilon\right\}\in I. \tag{28}$$

Thus, $(x_k).(y_k)\in^I$Ces. Hence, C^I is a sequence algebra.

On the similar manner, one can prove that space C_0^I is also sequence algebra. □

Data Availability

The data used to support the findings of the study are obtained from the author upon request.

Conflicts of Interest

The author declares no conflicts of interest.

Authors' Contributions

The author read and approved the final manuscript.

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