Some Results on Strongly Cesàro Ideal Convergent Sequence Spaces

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Some algebraic properties of Cesàro ideal convergent sequence spaces, $C$ and $C_0$, are studied in this article and some inclusion relations on these spaces are established.

1. Introduction

Consider the space $\omega = \{x = (x_k): x_k \in \mathbb{R}$ or $\mathbb{C}\}$ of all real and complex sequences, where $\mathbb{R}$ and $\mathbb{C}$ are, respectively, the sets of all real and complex numbers.

Suppose that $\ell_\infty$, $c$, and $c_0$ are the linear spaces of bounded, convergent, and null sequences, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|,$$

where $k \in \mathbb{N}$.

A sequence $(x_k)$ of complex numbers is said to be $(C, 1)$ summable to $L \in \mathbb{C}$ if for $\rho_k = 1/k \sum_{i=1}^{k} x_i$, $\lim_{n \to \infty} \rho_k = L$. The sequence $(C, 1)$ is also called Cesàro summable sequence of complex numbers over $\mathbb{C}$. Let us denote by $C_1$ the linear space of all $(C, 1)$ summable sequences of complex numbers over $\mathbb{C}$, i.e.,

$$C_1 = \left\{ x = (x_k) \in \omega: \lim_{k} \frac{1}{k} \sum_{i=1}^{k} x_i \in \mathbb{C} \right\}.$$  

Hardy and Littlewood [1] initiated the notion of strong Cesàro convergence for real numbers which is defined as follows.

A sequence $(x_k)$ on a normed space $(X, \| \cdot \|)$ is said to be strongly Cesàro convergent to $L$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k - L\| = 0.$$  

In [2–6], the authors have extended the notion of strong Cesàro convergence in various fields. In 1951, Fast [7] introduced the term statistical convergence, while Steinhaus [8] independently introduced the term “ordinary and asymptotic convergences.” Later on, Fridy [9, 10] also studied the statistical convergence and linked it with the summability theory. Kostyrko et al. [11] gave the concept of ideal convergence (I-convergence) which was indeed a generalization of statistical convergence. Salat et al. [12] studied some properties of I-convergence, and further investigations in this field are done by Khan [13], Tripathy and Esi [14], Tripathy and Hazarika [15], and many others.

In this article, further interesting properties of Cesàro Ideal Convergent Sequences are established and a few inclusion relations are also proved.

2. Definitions of the Terms Used

Let us first present some definitions and notions that are required in the sequel.

(i) A family of subsets $I$ of $\mathbb{N}$ is called an ideal set in $\mathbb{N}$
(ii) If the sets $A, B \in I$, then $A \cup B \in I$
(iii) If $B \subseteq A$ and $A \in I$, then $B \in I$

(2) A nontrivial ideal set $I$ is said to be admissible if
\[ \{|n|: n \in \mathbb{N}\} \subseteq I \]
(3) A nonempty set $F \in 2^{\mathbb{N}}$ is known as a filter in $\mathbb{N}$ if
(a) $\phi \not\in F$
(b) $A, B \in F \Rightarrow A \cap B \in F$
(c) $A \in F$ with $A \subseteq AB \Rightarrow Be F$

**Remark 1.** For every ideal $I$, there is a filter $F(I)$ (associated with $I$) defined as follows:
\[
F(I) = \left\{ \frac{N \subseteq \mathbb{N}}{n \in I} \right\}.
\]

A sequence $(x_k) \in X$ is said to be $I$-convergent to a number $L$ if, for every $\varepsilon > 0$, the set $\{x = (x_k) \in X: |x_k - L| \leq \varepsilon \} \in I$. In this case, we write $I$-lim $x_k = L$. If $L = 0$, then it is called $I$-null.

A sequence $(x_k) \in X$ is said to be $I$-Cauchy if, for every $\varepsilon > 0$, there exists a number $m = m(\varepsilon)$ such that
\[
\{x = x_k \in X: \{|x_n - x_m| \geq \varepsilon \} \in I\}.
\]

Let $I$ be the class of all finite subsets of $\mathbb{N}$. If $I = I, I$, then $I$ is admissible ideal set in $\mathbb{N}$.

A sequence space $X$ is said to be solid (normal) if $(a, x, y) \in X$ whenever $(x, y) \in X$ and $(a)$ is a sequence of scalars with $|a| \leq 1$, for all $k \in \mathbb{N}$.

A sequence space $X$ is a Sequence Algebra if, for every $(x_k), (y_k) \in X, (\sigma, y_k) \in X$.

Let $K = \{k_1 < k_2 < k_3, \ldots \} \in \mathbb{N}$ and $X$ be a sequence space. A $K$-step space of $X$ is a sequence space
\[
\lambda_K^X = \{(x_k) \in \omega: (x_k) \in X\}.
\]

A canonical preimage of a sequence $(x_k) \in \lambda_K^X$ is a sequence $(y_k) \in \omega$ defined by
\[
y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise}, \end{cases}
\]

A sequence space is monotone if it contains the canonical preimages of its step spaces.

**3. Result**

A canonical preimage of a step space $\lambda_K^X$ is a set of preimages of all elements in $\lambda_K^X$, i.e., $y$ is in the canonical preimage of $\lambda_K^X$ if and only if $y$ is the canonical preimage of some $x \in \lambda_K^X$.

Let $X$ and $Y$ be two normed linear spaces. An operator $T: X \rightarrow Y$ is known as a compact linear operator if [16].

(a) $T$ is linear

(b) If, for every bounded subset $D$ of $X$, the image $M(D)$ is relatively compact, i.e., the closure $T(D)$ is compact

**Lemma 1** (see [12]). Every solid space is monotone.

**Lemma 2** (see [12]). Let $K \in F(I)$ and $M \subseteq \mathbb{N}$. If $M \not\in I$, then $M \cap K \not\in I$.

**Lemma 3** (see [11]). Let $I \subseteq 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \not\in I$, then $M \cap K \not\in I$.

**4. Main Results**

Let us first define $C_I$, the space of all Cesàro ideal null sequences and $C_0^I$, the space of all Cesàro ideal null sequences which are given as follows:

\[
C_I = \{x = (x_k) \in \omega: I \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k - L\| = 0, \text{ for some } L \in C\},
\]

\[
C_0^I = \{x = (x_k) \in \omega: I \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k\| = 0\}.
\]

**Theorem 1.** The sequence spaces $C_I$ and $C_0^I$ are linear.

**Proof.** Assume that $x = (x_k), y = (y_k) \in C_I$. Then, one has

\[
I \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k - L_1\| = 0, \text{ for some } L_1 \in C,
\]

\[
I \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \|y_k - L_2\| = 0, \text{ for some } L_2 \in C.
\]

Let
\[
A_1 = \left\{ k \in N: \frac{1}{n} \sum_{k=1}^{n} \|x_k - L_1\| \right\},
\]

\[
A_2 = \left\{ k \in N: \frac{1}{n} \sum_{k=1}^{n} \|y_k - L_2\| \right\}.
\]

Let $a$ and $\beta$ be some scalers.

By using the properties of norm, one can easily see that

\[
I \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \|ax_k + \beta y_k\| - (aL_1 + \beta L_2) \leq I \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} |a| \|x_k - L_1\| + \frac{1}{n} \sum_{k=1}^{n} |\beta| \|y_k - L_2\|.
\]

Then, from (9) and (10), we have for each $\varepsilon > 0$,
\[
\left\{ k \in \mathbb{N} : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \| (ax_k + \beta y_k) - (\alpha L_1 + \beta L_2) \| > \varepsilon \right\} \\
\subset A_1 \cup A_2.
\]

(12)

Therefore, \((ax_k + \beta y_k) \in C^I\), for all scalars \(a, \beta\) and \((x_k), (y_k) \in C^I\).

Hence, \(C^I\) is a linear space.

On the similar manner, one can prove that \(C^I_0\) is also linear.

\[\Box\]

**Theorem 2.** Let \(x = (x_k) \in \omega \) be any sequence Then, \(C^I_0 \subset C^I\).

**Proof.** It can be easily observed.

\[\Box\]

**Theorem 3.** A sequence \(x = (x_k) \in C^I\) is I-convergent if and only if, for every \(\varepsilon > 0\), there exists \(l = l(\varepsilon) \in \mathbb{N}\) such that

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k - x_l \| < \varepsilon \right\} \in F(I).
\]

(13)

**Proof.** Suppose that \(x = (x_k) \in C^I\). Therefore, \(I - \lim_{n \to \infty} 1/n \sum_{k=1}^{n} \| x_k - L \| = 0\). Then, for all \(\varepsilon > 0\) the set

\[
C_{\varepsilon} = \left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k - L \| < \frac{\varepsilon}{2} \right\} \subset F(I).
\]

(14)

Fix an \(l(\varepsilon) \in C_{\varepsilon}\). Then, we have

\[
\frac{1}{n} \sum_{k=1}^{n} \| x_k - x_l \| \leq \frac{1}{n} \sum_{k=1}^{n} \| x_k - L \| + \frac{1}{n} \sum_{k=1}^{n} \| x_l - L \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

which holds for all \(k \in C_{\varepsilon}\). Hence,

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k - x_l \| < \varepsilon \right\} \subset F(I).
\]

(15)

Conversely, suppose that, for all \(\varepsilon > 0\), the set

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k - x_l \| < \varepsilon \right\} \subset F(I).
\]

(16)

Then, for every \(\varepsilon > 0\), we have

\[
B_{\varepsilon} = \left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \in \left[ \frac{1}{n} \sum_{k=1}^{n} \| x_l \| - \varepsilon, \frac{1}{n} \sum_{k=1}^{n} \| x_l \| + \varepsilon \right] \right\} \subset F(I),
\]

(17)

Let, \(P_{\varepsilon} = \left[ \frac{1}{n} \sum_{k=1}^{n} \| x_k \| - \varepsilon, \frac{1}{n} \sum_{k=1}^{n} \| x_l \| + \varepsilon \right] \). For fixed \(\varepsilon > 0\), one has \(B_{\varepsilon} \in F(I)\) as well as \(B_{\varepsilon} \cap P_{\varepsilon} \in F(I)\). Hence, \(B_{\varepsilon} \cap B_{\varepsilon/2} \in F(I)\).

This implies that \(B_{\varepsilon} \cap B_{\varepsilon/2} \neq \phi\), that is,

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \in P \right\} \in F(I).
\]

(19)

That is \(\text{diam } P \leq \text{diam } P_s\), where the \(\text{diam } P\) denotes the length of the interval of \(P\).

In this way, by induction, one obtains the sequence of closed intervals:

\[
P_s = (0, 2, 3, \ldots),
\]

(20)

with the property that \(\text{diam } J_k \leq 1/2 \text{diam } J_{k-1}\) for \(k = 1, 2, 3, \ldots\), and

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \in J_k \right\} \subset F(I),
\]

(21)

for \(k = 1, 2, 3, \ldots\). Then, there exists a \(L \in J_\omega\), such that

\[
L = I - \lim_{n \to \infty} 1/n \sum_{k=1}^{n} \| x_k \|,
\]

showing that \(x = (x_k) \in C^I\) is I-convergent. Hence, the result holds.

\[\Box\]

**Theorem 4.** The space \(C^I_0\) is solid and monotone.

**Proof.** Let \((x_k) \in C^I_0\) be any element. Then, one has

\[
\left\{ k \in \mathbb{N} : I - \lim_{n \to \infty} 1/n \sum_{k=1}^{n} \| x_k \| = 0 \right\}.
\]

(22)

Let \((\alpha_k)\) be a sequence of scalars such that \(|\alpha_k| \leq 1\), for all \(k \in \mathbb{N}\), and hence \(1/n \sum_{k=1}^{n} \| x_k \| < 1\).

Then, the result (that \(C^I_0\) is solid) follows from the above equation and inequality:

\[
\frac{1}{n} \sum_{k=1}^{n} \| \alpha_k x_k \| = \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \leq \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \leq \frac{1}{n} \sum_{k=1}^{n} |\alpha_k| \sum_{k=1}^{n} \| x_k \|
\]

(23)

for all \(k \in \mathbb{N}\).

The space \(C^I_0\) is monotone which follows from Lemma 1. Hence, \(C^I_0\) is solid and monotone.

\[\Box\]

**Theorem 5.** The space \(C^I\) is neither solid nor monotone.

**Proof.** For this theorem, we provide a counter example for the proof.

\[\Box\]

**5. Counter Example**

Let \(I = I_b\) and consider the \(k\)-step \(\chi_k\) of \(\chi\) defined as follows.

Let \((x_k) \in \chi\) and let \((y_k) \in \chi_k\) be such that

\[
y_k = \begin{cases} x_k, & \text{if } k \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}
\]

(24)

Let us consider the sequence \((x_k)\) defined by \(x_k = 1\) for all \(k \in \mathbb{N}\). Then, \((x_k) \in C^I\), but its \(K\)-step preimages do not belong to \(C^I\). Thus, \((x_k) \in C^I\) is not monotone.

Hence, \((x_k) \in C^I\) is not solid.
Theorem 6. Let \( x = (x_k) \) and \( y = (y_k) \) be two sequences in such a way that \( T(x \cdot y) = T(x)T(y) \). Then, the space \( C' \) and \( C'_0 \) are sequence algebras.

Proof. Let \( x = (x_k) \) and \( y = (y_k) \) be two elements of \( C' \) with 
\[
T(x \cdot y) = T(x)T(y). \tag{25}
\]

For every \( \varepsilon > 0 \) select \( \beta > 0 \) in such a way that \( \varepsilon < \beta \), then

\[
\frac{1}{n} \sum_{k=1}^{n} \|T(x_k, y_k) - L_1L_2\| = \frac{1}{n} \sum_{k=1}^{n} \|T(x_k)T(y_k) - L_1L_2\| 
\]
\[
= \frac{1}{n} \sum_{k=1}^{n} \|T(x_k)T(y_k) - L_1T(y_k) + L_1T(y_k) - L_1L_2\| 
\]
\[
\leq \frac{1}{n} \sum_{k=1}^{n} \|T(x_k)\| \cdot \frac{1}{n} \sum_{k=1}^{n} \|T(y_k)\| + |L_1| \cdot \frac{1}{n} \sum_{k=1}^{n} \|T(y_k) - L_2\| < \frac{\varepsilon^2}{\beta^2} + |L_1| \cdot \frac{\varepsilon^2}{2|L_1|} \leq \varepsilon.
\]

Therefore, the set
\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \|T(x_k, y_k) - L_1L_2\| \geq \varepsilon \right\} \in I. \tag{28}
\]

Thus, \( (x_k, y_k) \in \ell^1 \) Ces. Hence, \( C' \) is a sequence algebra. On the similar manner, one can prove that space \( C'_0 \) is also sequence algebra. \( \square \)

Data Availability

The data used to support the findings of the study are obtained from the author upon request.

Conflicts of Interest

The author declares no conflicts of interest.

Authors’ Contributions

The author read and approved the final manuscript.

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