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Research Article

On the Set Version of Selectively Star-CCC Spaces

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A space X is said to be set selectively star-ccc if for each nonempty subset B of X, for each collection $\mathscr{U}$ of open sets in X such that $\overline{B} \subset \cup\mathscr{U}$, and for each sequence $(\mathscr{A}_n: n \in \mathbb{N})$ of maximal cellular open families in X, there is a sequence $(A_n: n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $A_n \in \mathscr{A}_n$ and $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{U})$. In this paper, we introduce set selectively star-ccc spaces and investigate the relationship between set selectively star-ccc and other related spaces. We also study the topological properties of set selectively star-ccc spaces. Some open problems are posed.

1. Introduction

In 1996, Scheepers [1] initiated the systematic study of selection principle in topology and their relations to game theory and Ramsey theory (also see [2]). After this study, it becomes one of the most active areas in set theoretic topology. Kočinac [3, 4] applied the star operator to these selection principles and introduced and studied the new selection principles called star selection principles. It should be noted that classical selection principles have been used to define and characterize various covering properties such as Rothberger [5], Menger [6], and star-Menger [3]. In this paper, we use the following selection principle of the Scheepers type from [1]. Let $\mathscr{A}$ and $\mathscr{B}$ be families of sets.

Then, $S_1(\mathscr{A}, \mathscr{B})$ denotes, for each sequence $(\mathscr{A}_n: n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(A_n: n \in \mathbb{N})$ such that, for each n, $A_n \in \mathscr{A}_n$ and $\{A_n: n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

If $\mathscr{O}$ is the family of all open covers of a space X, then $S_1(\mathscr{O}, \mathscr{O})$ is the *Rothberger covering property*.

On the contrary, Arhangel'skii [7] defined a cardinal function sL and spaces X such that $sL(X) = \omega$; we call s-Lindelöf: a space X is *s-Lindelöf* if, for each nonempty subset A of X and each open cover $\mathscr{U}$ of $\overline{A}$ by sets open in X, there is a countable set $\mathscr{V} \subset \mathscr{U}$ such that $A \subset \overline{\bigcup\mathscr{V}}$. Following this idea and modifying it, Kočinac and Konca [8] considered new types of selective covering properties called set-covering properties. A space X is said to have the *set-Menger property* [8, 9] if, for each nonempty subset A of X and each sequence $(\mathscr{U}_n: n \in \mathbb{N})$ of collections of sets open in X such that $\overline{A} \subset \bigcup\mathscr{U}_n$, there is a sequence $(\mathscr{V}_n: n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $A \subset \bigcup_{n\in\mathbb{N}} \bigcup \mathscr{V}_n$. Recently, Kočinac, Konca, and Singh initiated a study of star versions of the set-Menger covering property.

In [10], Aurichi introduced the class of selectively ccc spaces. Bal and Kočinac [11] introduced and studied the star version of selectively ccc spaces called selectively star-ccc spaces.

The purpose of this paper is to introduce the set selectively star-ccc spaces, the class which lies between Lindelöf spaces and selectively star-ccc spaces. We investigate the relationship between set selectively star-ccc and other related spaces and study the topological properties of set selectively star-ccc spaces. Some open problems are posed.

2. Preliminaries

By "a space" we mean "a topological space." Throughout the paper, an open cover $\mathscr{U}$ of a subset $A \subset X$ means elements of

$\mathcal{U}$ are open in X such that $A \subset \bigcup \mathcal{U} = \bigcup\{U\colon U \in \mathcal{U}\}$, unless otherwise stated.

If A is a subset of a space X and $\mathcal{U}$ is a collection of subsets of X, then

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon U \cap A \neq \varnothing\}. \tag{1}$$

We usually write $\mathrm{St}(x,\mathcal{U}) = \mathrm{St}(\{x\},\mathcal{U})$. One defines $\mathrm{St}^0(A,\mathcal{U}) = A$, and for $k \geq 1$, $\mathrm{St}^{k+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^k(A,\mathcal{U}),\mathcal{U})$.

A family of pairwise disjoint open sets in a topological space X is called a *cellular open family*. A space X is said to be a *ccc space* if every cellular open family in X is countable.

Definition 1 (see [10]). A space X is said to be selectively ccc space if for each sequence $(\mathcal{A}_n\colon n \in \mathbb{N})$ of maximal cellular open families in X, there is a sequence $(A_n\colon n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $A_n \in \mathcal{A}_n$ and $\bigcup_{n\in\mathbb{N}} A_n$ is dense in X.

Definition 2 (see [12–14]). A space X is said to be R-separable if, for each sequence $(D_n\colon n \in \mathbb{N})$ of dense subsets of X, there is a sequence $(x_n\colon n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $x_n \in D_n$ and $\{x_n\colon n \in \mathbb{N}\}$ is dense in X.

Every R-separable space is a selectively ccc, and every selectively ccc space is ccc.

Definition 3 (see [11]). A space X is said to be selectively star-ccc if, for each open cover $\mathcal{U}$ of X and for each sequence $(\mathcal{A}_n\colon n \in \mathbb{N})$ of maximal cellular open families in X, there is a sequence $(A_n\colon n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $A_n \in \mathcal{A}_n$ and $X = \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n,\mathcal{U})$.

Definition 4 (see [15]). A space X is said to be strongly star-Lindelöf if, for each open cover $\mathcal{U}$ of X, there is a countable subset F of X such that $X = \mathrm{St}(F,\mathcal{U})$.

Note that strongly star-Lindelöf spaces are also called star countable in [16].

In a similar way, Kočinac, Konca, and Singh defined the following.

Definition 5. Let $k \in \mathbb{N}$. A space X is said to be set strongly k-starcompact (resp., set strongly k-star-Lindelöf) if, for each nonempty subset A of X and for each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subset \bigcup\mathcal{U}$, there is a finite (resp., countable) subset F of X such that $A \subset \mathrm{St}^k(F,\mathcal{U})$.

We say set strongly starcompact and set strongly star-Lindelöf instead of set strongly 1-starcompact and set strongly 1-star-Lindelöf, respectively. It is clear, by the definitions, that every set strongly star-Lindelöf space is strongly star-Lindelöf and every set strongly k-starcompact space is set strongly k-star-Lindelöf, $k \geq 1$.

Definition 6 (see [17, 18]). A space X is said to be absolutely countably compact (shortly, acc) if, for each open cover $\mathcal{U}$ of X and for each dense subset Y of X, there is a finite subset F of Y such that $X = \mathrm{St}(F,\mathcal{U})$.

Clearly, a compact space is acc, and an acc Hausdroff space is countably compact (see [17]). In a similar way, we define the following.

Definition 7. A space X is said to be set absolutely countably compact (shortly, set-acc) if, for each nonempty subset B of X, for each collection $\mathcal{U}$ of open sets in X such that $\overline{B} \subset \bigcup\mathcal{U}$, and for every dense subset Y of X, there is a finite subset F of Y such that $B \subset \mathrm{St}(F,\mathcal{U})$.

Lemma 1. *Every compact space is set-acc.*

Proof. Let B be any nonempty subset of X, $\mathcal{U}$ be any collection of open sets in X such that $\overline{B} \subset \bigcup\mathcal{U}$, and Y be any dense subset of X. Since a closed subset of a compact space is compact, $\overline{B}$ is compact; hence, there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\overline{B} \subset \bigcup\mathcal{V}$. For each $V \in \mathcal{V}$, $V \cap Y \neq \varnothing$. Take $x_V \in V \cap Y$. Put $F = \{x_V \in Y\colon V \in \mathcal{V}\}$. Then, F is a finite subset of Y and $B \subset \overline{B} \subset \bigcup\mathcal{V} \subset \mathrm{St}(F,\mathcal{U})$. Thus, X is set-acc.

It is clear from the definitions that every set-acc space is acc and thus we have the following corollary. □

Corollary 1. *Every Hausdorff set-acc space is countably compact.*

The following is an open question.

Problem 1. Does an acc space which is not set-acc exist?

Recall that a subspace Y of a space X is *ω-dense* in X if, for every $a \in X$, there is a countable $A \subset Y$ such that $a \in \overline{A}$.

Lemma 2. *If X is countably compact and every dense subspace of X is ω-dense in X, then X is set-acc.*

Proof. Let B be any nonempty subset of X, $\mathcal{U}$ be any collection of open sets in X such that $\overline{B} \subset \bigcup\mathcal{U}$, and Y be any dense subset of X. Since a closed subset of a countably compact space is countably compact, $\overline{B}$ is countably compact, and so there exists a finite subset $A \subset \overline{B}(\subset X)$ such that $\overline{B} \subset \mathrm{St}(A,\mathcal{U})$. Since Y is ω-dense in X, for every $a \in A$, pick a countable $B_a \subset Y$ such that $a \in \overline{B_a}$. Set $B' = \cup\{B_a\colon a \in A\}$. Then, B' is countable, and for each $a \in A$, $\mathrm{St}(a,\mathcal{U}) \subset \bigcup\{\mathrm{St}(b,\mathcal{U})\colon b \in B_a\}$. So, $\{\mathrm{St}(b,\mathcal{U})\colon b \in B'\}$ is a countable cover of $\overline{B}$. By countable compactness of $\overline{B}$, there is a finite subset $F \subset B' \subset Y$ such that $B \subset \overline{B} \subset \mathrm{St}(F,\mathcal{U})$.

By Lemma 2, we have the following result. □

Theorem 1. *If a countably compact space X has a countable tightness, then X is set-acc.*

Recall that a collection $\mathcal{A}$ of infinite subsets of ω is said to be *almost disjoint* if the sets $A \cap B$ are finite for all distinct elements $A, B \in \mathcal{A}$. For an almost disjoint family $\mathcal{A}$, put $\psi(\mathcal{A}) = \mathcal{A} \bigcup \omega$ and topologize $\psi(\mathcal{A})$ as follows: for each element $A \in \mathcal{A}$ and each finite set $F \subset \omega$, $\{A\} \bigcup (AF)$ is a basic open neighborhood of A; each $n \in \omega$ is isolated. The spaces of this type are called *Isbell-Mrówka ψ-spaces* [19] or $\psi(\mathcal{A})$ spaces.

Throughout the paper, the cardinality of a set is denoted by $|A|$. Let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, and $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. Other notation and terminology follow [20].

A sub set A of X is said to be *regular open* if $A = \mathrm{Int}(\mathrm{Cl}(A))$. A subset B is said to be *regular closed* if its complement is regular open or, equivalently, if $B = \mathrm{Cl}(\mathrm{Int}(B))$.

3. Set Selectively Star-CCC and Related Spaces

In this section, we give some examples showing the relationships between set selectively star-ccc spaces and other related spaces.

Definition 8. A space X is said to be set selectively star-ccc if, for each nonempty subset B of X, for each collection $\mathscr{U}$ of open sets in X such that $\overline{B} \subset \cup\mathscr{U}$, and for each sequence $(\mathscr{A}_n\colon n \in \mathbb{N})$ of maximal cellular open families in X, there is a sequence $(A_n\colon n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $A_n \in \mathscr{A}_n$ and $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{U})$.

It is clear by the definitions that every set selectively star-ccc space is selectively star-ccc.

Theorem 2. *Every Lindelöf space is set selectively star-ccc.*

Proof. Let B be any nonempty subset of a Lindelöf space X, $\mathscr{U}$ be any collection of open sets in X such that $\overline{B} \subset \bigcup\mathscr{U}$, and $(\mathscr{A}_n\colon n \in \mathbb{N})$ be a sequence of maximal cellular open families in X. Since space X is Lindelöf, closed subset $\overline{B}$ is also Lindelöf. Thus, there exists a countable subset $\mathscr{V} = \{U_1, U_2, \ldots\}$ of $\mathscr{U}$ such that $B \subset \overline{B} \subset \bigcup\mathscr{V}$. For each $n \in \mathbb{N}$, $\mathscr{A}_n$ is maximal cellular open family in X, and thus there exists $A_n \in \mathscr{A}_n$ such that $A_n \cap U_n \neq \varnothing$. Thus,

$$B \subset \overline{B} \subset \bigcup\mathscr{V} \subset \mathrm{St}\left(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{U}\right). \qquad (2)$$

Therefore, X is a set selectively star-ccc space.

The following example shows that the converse of Theorem 2 need not be true. □

Example 1

(1) Let the ordinal space $X = [0, \omega_1)$ be equipped with the usual order topology. Then, X is a countably compact space of countable tightness, and by Theorem 1, X is set-acc. By Theorem 5, X is a set selectively star-ccc space. However, X is not Lindelöf.

(2) Let X be any Tychonoff space such that the function space $C_p(X)$ is not Lindelöf. For a Tychonoff space X the function space $C_p(X)$ is selectively ccc; thus, by Theorem 3, $C_p(X)$ is set selectively star-ccc.

Theorem 3. *Every selectively ccc space is set selectively star-ccc.*

Proof. Let B be any nonempty subset of a selectively ccc space X, $\mathscr{U}$ be any collection of open sets in X such that $\overline{B} \subset \bigcup\mathscr{U}$, and $(\mathscr{A}_n\colon n \in \mathbb{N})$ be any sequence of maximal cellular open families in X. Since the space X is selectively ccc, there exists a sequence $(A_n\colon n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $A_n \in \mathscr{A}_n$ and $A = \bigcup_{n\in\mathbb{N}} A_n$ is dense in X. Therefore, $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{U})$, which shows that X is a set selectively star-ccc space.

The following example shows that converse of Theorem 3 is not true. □

Example 2. Let $D(\mathfrak{c})$ be the discrete space of cardinality $\mathfrak{c}$ and let $L(\mathfrak{c}) = D(\mathfrak{c}) \cup \{\infty\}$, where $\infty \notin D(\mathfrak{c})$, be the one-point Lindelöfication of $D(\mathfrak{c})$. The topology on $L(\mathfrak{c})$ is defined as follows: for each $\alpha < \mathfrak{c}$, $\{\alpha\}$ is isolated and a set U containing ∞ is open if and only if $L(\mathfrak{c})U$ is countable. Then, $L(\mathfrak{c})$ is a Tychonoff Lindelöf space, and thus it is a set selectively star-ccc space. However, the collection $\{\{\alpha\}\colon \alpha < \mathfrak{c}\}$ is a maximal cellular open family in $L(\mathfrak{c})$ which is not countable. Thus, $L(\mathfrak{c})$ is not a ccc space; hence, not selectively ccc (since every selectively ccc space is ccc).

Corollary 2. *Every R-separable space is set selectively star-ccc.*

Example 3

(1) In Example 2, the space $L(\mathfrak{c})$ is set selectively star-ccc but not separable, and thus not R-separable, since every R-separable space is separable.

(2) The space $2^{\mathrm{cov}\{\mathscr{M}\}}$ contains a dense countable subspace X which is not R-separable (see Theorem 50 in [13]]), but being countable X is set selectively star-ccc.

(3) The ordinal space $[0, \omega_1)$ is set selectively star-ccc but not separable, and thus not R-separable.

Theorem 4. *Let X be a space which has a dense subset Y of isolated points. If, for any nonempty set $B \subset X$ and for each collection $\mathscr{U}$ of open sets in X such that $\overline{B} \subset \cup\mathscr{U}$, there is a countable set $D \subset Y$ such that $B \subset St(D, \mathscr{U})$; then, X is a set selectively star-ccc space.*

Proof. Let B be any nonempty subset of X, $\mathscr{U}$ be any collection of open sets in X such that $\overline{B} \subset \bigcup\mathscr{U}$, and $(\mathscr{A}_n\colon n \in \mathbb{N})$ be any sequence of maximal cellular open families in X. It follows from hypothesis that there is a countable set $D = \{d_n\colon n \in \mathbb{N}\} \subset Y$ such that $B \subset \mathrm{St}(D, \mathscr{U})$. For each $n \in \mathbb{N}$, $\mathscr{A}_n$ is a maximal cellular open family in X and d_n is an isolated point of X; hence, there exists $A_n \in \mathscr{A}_n$ such that $d_n \in A_n$ and thus $\mathrm{St}(d_n, \mathscr{U}) \subset \mathrm{St}(A_n, \mathscr{U})$. Therefore,

$$B \subset \mathrm{St}(D, \mathscr{U}) \subset \mathrm{St}\left(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{U}\right), \qquad (3)$$

which proves that X is a set-selectively star-ccc space. □

Corollary 3. *If X is a space with a countable dense subset of isolated points, then it is set selectively star-ccc.*

The following theorem follows from the definition of set-acc spaces and from Theorem 4.

Theorem 5. *Every set-acc space is set selectively star-ccc.*

The converse of Theorem 5 is not true.

Example 4. Any Lindelöf space which is not countably compact is such an example. Such spaces are the Sorgenfrey line and the real line $\mathbb{R}$ endowed with the open-minus-countable topology τ: $V \subset \mathbb{R}$ belongs to τ if it is of the form $U \backslash C$, where U is open in the usual metric topology on $\mathbb{R}$, and C is a countable subset of $\mathbb{R}$.

Theorem 6. *Every continuous open image of a set selectively star-ccc space is set selectively star-ccc.*

Proof. Let X be a set-selectively star-ccc space and $f\colon X \longrightarrow Y$ be a continuous open mapping from X onto Y. Let B be any subset of Y, $\mathscr{V}$ be a collection of open sets in Y such that $\overline{B} \subset \bigcup \mathscr{V}$, and $(\mathscr{A}_n\colon n \in \mathbb{N})$ be any sequence of maximal cellular open families in Y. Let $A = f^{\leftarrow}(B)$. Then, $\mathscr{U} = \{f^{\leftarrow}(V)\colon V \in \mathscr{V}\}$ is the collection of open sets in X with

$$\overline{A} = \overline{f^{\leftarrow}(B)} \subset f^{\leftarrow}(\overline{B}) \subset f^{\leftarrow}\left(\bigcup \mathscr{V}\right) = \bigcup \mathscr{U}, \tag{4}$$

and because f is open, for every $n \in \mathbb{N}$, $\mathscr{B}_n = \{f^{\leftarrow}(A)\colon A \in \mathscr{A}_n\}$ is a maximal cellular open family in X. Since X is a set selectively star-ccc space, there is a sequence $(B_n\colon n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $B_n \in \mathscr{B}_n$ and $A \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} B_n, \mathscr{U})$.

For each n, let $A_n \in \mathscr{A}_n$ be such that $f^{\leftarrow}(A_n) = B_n$. Now, we have to show that $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{V})$.

Let $y \in B$. There exists $x \in A$ such that $f(x) = y$. Thus, $x \in \mathrm{St}(\bigcup_{n\in\mathbb{N}} B_n, \mathscr{U})$. Choose $U_x \in \mathscr{U}$ such that $x \in U_x$ and $(\bigcup_{n\in\mathbb{N}} B_n) \cap U_x \neq \varnothing$. Thus, $U_x \cap B_m \neq \varnothing$ for some $m \in \mathbb{N}$. Let $x' \in U_x \cap B_m$. Then, $x' \in B_m$, which implies $f(x') \in f(B_m) = A_m \subset \bigcup_{n\in\mathbb{N}} A_n$. Also, there exists a $V_y \in \mathscr{V}$ such that $f(U_x) = V_y$ and $f(x') \in V_y$. Thus, $\bigcup_{n\in\mathbb{N}} A_n \cap V_y \neq \varnothing$, $V_y \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{V})$. Since $x \in U_x$, $y = f(x) \in V_y \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{V})$. Therefore, Y is a set selectively star-ccc space.

Now, we discuss the nature of set selectively star-ccc property on subspaces. The following example shows that the set selectively star-ccc property is not preserved under open subspaces. □

Example 5. There exists a Tychonoff set selectively star-ccc space having open subspace which is not set selectively star-ccc.

Let $L(\mathfrak{c})$ be the one-point Lindelöfication of the discrete space $D(\mathfrak{c})$ (Example 2). Then, $L(\mathfrak{c})$ is set selectively star-ccc space. However, its open (discrete) uncountable subspace $D(\mathfrak{c})$ is not set selectively star-ccc.

The following example shows that the set selectively star-ccc property is not preserved under closed subspaces.

Example 6. There exists a Tychonoff pseudocompact set selectively star-ccc space having closed subspace which is not set selectively star-ccc.

Proof. Let $X = \mathscr{M} \cup \omega$ be the Isbell-Mrówka space with $|\mathscr{M}| = \mathfrak{c}$, where $\mathscr{M}$ is a maximal almost disjoint family of infinite subsets of ω. Then, X is a Tychonoff pseudocompact space. Since ω is a countable dense subset of X containing isolated points, by Corollary 3, X is set selectively star-ccc space. On the contrary, $\mathscr{M}$ is a closed uncountable discrete subspace of X; thus, $\mathscr{M}$ is not set-selectively star-ccc.

The following example shows that the set selectively star-ccc property is not preserved under regular closed subspaces. □

Example 7. There exists a Tychonoff set selectively star-ccc space having regular closed subspace which is not set selectively star-ccc.

Proof. Let $\mathscr{M}$ be the maximal almost disjoint family of infinite subsets of ω with $|\mathscr{M}| = \mathfrak{c}$. Define

$$Y = \mathscr{M} \cup (\mathfrak{c} \times \omega) \tag{5}$$

and topologize Y as follows: $\mathfrak{c} \times \omega$ has the usual product topology and is an open subspace of Y; for $m \in \mathscr{M}$, a basic neighbourhood is of the form $\{m\} \cup ((\alpha, \mathfrak{c}) \times (m \backslash F))$, $\alpha < \mathfrak{c}$, F a finite subset of ω. Let $Z = \mathscr{M} \cup \omega$ be the Isbell–Mrówka space.

Let X be the quotient space of the disjoint sum $Y \oplus Z$ by identifying the subspace $\mathscr{M}$ of Y with the subspace $\mathscr{M}$ of Z and let $\varphi\colon Y \oplus Z \longrightarrow X$ be the quotient map. Notice that $\varphi(Y)$ is a regular closed subspace of X. □

Claim 1. X is a set selectively star-ccc space.

Let B be any nonempty subset of X, $\mathscr{U}$ be any collection of open sets in X such that $\overline{B} \subset \bigcup \mathscr{U}$, and $(\mathscr{A}_n\colon n \in \mathbb{N})$ be any sequence of maximal cellular open families in X. There are three cases:

Case (i): $B \subset \varphi(Z)$.

Note first that the Tychonoff space Z is set selectively star-ccc by Corollary 3. Since ω is a countable dense subset of isolated points of Z and $\phi(Z)$ is homeomorphic to Z, $\varphi(Z)$ is also set selectively star-ccc. Thus, there are $A_n \in \mathscr{A}_n$, $n \in \mathbb{N}$, such that $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{U})$.

Case (ii): $B \subset \mathfrak{c} \times \omega$.

The space $\mathfrak{c} = [0, \mathfrak{c})$ is countably compact; hence, set-acc, and, by Theorem 5, set selectively star-ccc. For each $n \in \omega$, $\mathfrak{c} \times \{n\}$ is homeomorphic to $\mathfrak{c}$ and thus $\mathfrak{c} \times \{n\}$ is set selectively star-ccc. It is easy to conclude that it follows from here that $\mathfrak{c} \times \omega$ is also set selectively star-ccc. Therefore, there is a sequence $(A_n\colon n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $A_n \in \mathscr{A}_n$ and $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathscr{U})$.

Case (iii): $B = B_1 \cup B_2$, where $B_1 \subset \varphi(Z)$ and $B_2 \subset \mathfrak{c} \times \omega$.

Let $\mathbb{N} = N_1 \cup N_2$ be a partition of $\mathbb{N}$ into two disjoint infinite subsets. By Case (i) and Case (ii), we have

$B_1 \subset \mathrm{St}(\bigcup_{n\in N_1} A_n, \mathcal{U})$ and $B_2 \subset \mathrm{St}(\bigcup_{n\in N_2} A_n, \mathcal{U})$. This implies that $\{A_n: n \in N_1\} \cup \{A_n: n \in N_2\}$ is a sequence witnessing that $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathcal{U})$.

These three cases show that X is set selectively star-ccc.

Claim 2. $\varphi(Y)$ is not set selectively star-ccc.

Song and Xuan (Example 1 in [21]) proved that the Tychonoff space Y is not selectively star-ccc. Thus, Y is not set selectively star-ccc (since every set selectively star-ccc space is selectively star-ccc). Then, $\varphi(Y)$, which is homeomorphic to Y, is not set selectively star-ccc.

However, we have the following result on subspaces of set selectively star-ccc spaces.

Theorem 7. *Every clopen subspace of a set selectively star-ccc space is also set selectively star-ccc.*

Proof. Let X be a set selectively star-ccc space and $Y \subset X$ be an open and closed set. Let B be any subset of Y, $\mathcal{U}$ be any collection of open sets in (Y, τ_Y) such that $\mathrm{Cl}_Y(B) \subset \bigcup \mathcal{U}$, and $(\mathcal{A}_n: n \in \mathbb{N})$ be any sequence of maximal cellular open families in Y. Since Y is open, then $\mathcal{U}$ is a collection of open sets in X. For each $n \in \mathbb{N}$ set $C_n = \mathcal{A}_n \cup \{X\backslash Y\}$. Then, $(C_n: n \in \mathbb{N})$ is a sequence of maximal cellular open families in X. Since Y is closed, $\mathrm{Cl}_Y(B) = \mathrm{Cl}_X(B)$. Since X is set-selectively star-ccc space, there is a sequence $(C_n: n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $C_n \in C_n$ and $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} C_n, \mathcal{U})$. Set $A_n = C_n$ if $C_n \in \mathcal{A}_n$, and A_n = an arbitrary element from $\mathcal{A}_n$ if $C_n = X\backslash Y$. Hence, $B \subset \mathrm{St}(\bigcup_{n\in\mathbb{N}} A_n, \mathcal{U})$, which shows that Y is a set selectively star-ccc space.

Observe that the set selectively star-ccc property is not finitely productive. The following example shows that the product of a set selectively star-ccc space and a Lindelöf (hence, set electively star-ccc) space is not set selectively star-ccc. □

Example 8. The ordinal space $X = [0, \omega_1)$ (with the usual order topology) is set selectively star-ccc, and the one-point Lindelöfication $L(\omega_1)$ of the discrete space $D(\omega_1)$ is Lindelöf. Then, $X \times Y$ is not set selectively star-ccc because in Example 4 [22] it was shown that this product is not selectively star-ccc.

We now give a positive result about the set selectively star-ccc property in the product of topological spaces.

Example 9. The space $\mathfrak{c} \times \omega$ is set selectively star-ccc.

Proof. Let B be any nonempty subset of $\mathfrak{c} \times \omega$, $\mathcal{U}$ be any collection of open sets in $\mathfrak{c} \times \omega$ such that $\overline{B} \subset \bigcup \mathcal{U}$, and $(\mathcal{A}_n: n \in \mathbb{N})$ be any sequence of maximal cellular open families in $\mathfrak{c} \times \omega$. Consider a partition of $\mathbb{N}$ into pairwise disjoint infinite subsets N_i: $\mathbb{N} = N_1 \cup N_2 \cup \ldots$. For each $n \in \omega$, $\mathfrak{c} \times \{n\}$ is set selectively star-ccc because it is homeomorphic to $\mathfrak{c}$, and $\mathfrak{c}$ is countably compact, hence set-acc, and thus, by Theorem 5, set selectively star-ccc. For each $n \in \mathbb{N}$ and for each $i \in N_n$, let $\mathcal{C}_i = \{A \cap (\mathfrak{c} \times \{n\}): A \in \mathcal{A}_i\}$ and $B_n = B \cap (\mathfrak{c} \times \{n\})$. Then, $(\mathcal{C}_i: i \in N_n)$ is a sequence of maximal cellular families of $\mathfrak{c} \times \{n\}$. Therefore, there is a sequence $(C_i: i \in N_n)$ such that, for each $i \in N_n$, $C_i \in \mathcal{C}_i$ and $B_n \subset \mathrm{St}(\bigcup_{i\in N_n} C_i), \mathcal{U})$,which implies

$$B \subset \mathrm{St}\Big(\cup_{n\in\mathbb{N}} \Big(\cup_{i\in N_n} A_i \Big), \mathcal{U} \Big). \tag{6}$$

So, $\mathfrak{c} \times \omega$ is set selectively star-ccc. □

4. Set Selectively k-Star-CCC Spaces

Definition 9. Let $k \in \mathbb{N}$. A space X is said to be set selectively k-star-ccc if, for each nonempty subset B of X, for each collection $\mathcal{U}$ of open sets in X such that $\overline{B} \subset \cup \mathcal{U}$, and for each sequence $(\mathcal{A}_n: n \in \mathbb{N})$ of maximal cellular open families in X, there is a sequence $(A_n: n \in \mathbb{N})$ such that, for each $n \in \mathbb{N}$, $A_n \in \mathcal{A}_n$ and $B \subset \mathrm{St}^k(\bigcup_{n\in\mathbb{N}} A_n, \mathcal{U})$.

The following lemma follows from the definitions.

Lemma 3. *For a space X, the following statements hold:*

(1) *Every set selectively k-star-ccc space is set selectively $(k+1)$-star-ccc.*

(2) *Every set selectively k-star-ccc space is selectively k-star-ccc.*

Theorem 8. *Every set strongly star-Lindelöf space is set selectively 2-star-ccc.*

Proof. Let B be any nonempty subset of a set strongly star-Lindelöf space X, $\mathcal{U}$ be any collection of open sets in X such that $\overline{B} \subset \bigcup \mathcal{U}$, and $(\mathcal{A}_n: n \in \mathbb{N})$ be any sequence of maximal cellular open families in X. Since X is a set strongly star-Lindelöf space, there is a countable subset $F = \{x_1, x_2, \ldots\}$ of X such that $B \subset \mathrm{St}(F, \mathcal{U})$. Then, $\{\mathrm{St}(x_n, \mathcal{U}): n \in \mathbb{N}\}$ is an open cover of B. For each $n \in \mathbb{N}$, there exists $U_n \in \mathcal{U}$ such that $x_n \in U_n \subset \mathrm{St}(x_n, \mathcal{U})$, that is, $U_n \subset \mathrm{St}(x_n, \mathcal{U}) \subset \mathrm{St}(U_n, \mathcal{U})$. For each $n \in \mathbb{N}$, $\bigcup \mathcal{A}_n$ is dense in X. Thus, for each $n \in \mathbb{N}$, $U_n \cap (\bigcup \mathcal{A}_n) \neq \varnothing$, which implies that, for each $n \in \mathbb{N}$, there exists $A_n \in \mathcal{A}_n$ such that $U_n \cap A_n \neq \varnothing$. Therefore, for each $n \in \mathbb{N}$,

$$\mathrm{St}(U_n, \mathcal{U}) \subset \mathrm{St}(\mathrm{St}(A_n, \mathcal{U}), \mathcal{U}) = \mathrm{St}^2(A_n, \mathcal{U}). \tag{7}$$

Hence, $B \subset \mathrm{St}(F, \mathcal{U}) \subset \mathrm{St}^2(\cup_{n\in\mathbb{N}} A_n, \mathcal{U})$, which shows that X is set selectively 2-star-ccc space. □

Corollary 4. *For a space X, the following statements hold:*

(1) *Every set strongly k-star-Lindelöf space X is set selectively $(k+1)$-star-ccc*

(2) *Every set strongly k-starcompact space X is set selectively $(k+1)$-star-ccc*

Lemma 4. *If a space X has a dense set selectively star-ccc subspace, then X is set selectively 2-star-ccc.*

Proof. The proof follows from the definitions. □

Example 10. There exists a Tychonoff set selectively 2-star-ccc space which is neither set strongly star-Lindelöf nor set selectively star-ccc.

Proof. Let $Y = \mathscr{M} \cup (\mathfrak{c} \times \omega)$ be the space from Example 7. Song and Xuan (Example 1 in [21]) showed that the Y is a Tychonoff space which is neither strongly star-Lindelöf nor selectively star-ccc. Thus, Y is neither set strongly star-Lindelöf nor set selectively star-ccc (since every set selectively star-ccc space is selectively star-ccc and every set strongly star-Lindelöf space is strongly star-Lindelöf).

Now, we prove that Y is a set selectively 2-star-ccc space. By Example 9, the space $\mathfrak{c} \times \omega$ is set selectively star-ccc. On the other side, $\mathfrak{c} \times \omega$ is a dense subset of Y. By Lemma 4, Y is set selectively 2-star-ccc.

Example 2 shows that there exists a Tychonoff set selectively 2-star-ccc space which is not ccc. Now, a natural question arises: is a ccc space set selectively 2-star-ccc? Song and Xuan [22] gave some sufficient conditions under which a ccc space is selectively 2-star-ccc. We use these results and give some conditions under which a ccc space is set selectively 2-star-ccc. Our results improve the corresponding results in [22]. □

Theorem 9. *If X is a ccc space which has a dense paracompact subspace Y, then X is set selectively 2-star-ccc.*

Proof. Since X is a ccc space, thus Y has to be ccc. If we prove that Y is Lindelöf, then by Theorem 2 and Lemma 4, X is set selectively 2-star-ccc. Since every paracompact space with countable extent is Lindelöf, so if Y is not Lindelöf, then Y must have an uncountable closed discrete subset D. Using the collection-wise normality of Y, D has an uncountable disjoint expansion, which contradicts the fact that Y is ccc. Thus, Y is Lindelöf, and hence X is set selectively 2-star-ccc. □

Corollary 5. *If X is Čech-complete ccc space, then X is set selectively 2-star-ccc.*

Proof. By a well-known result of Šapirovskij [23], X contains a dense paracompact Čech-complete subspace. Thus, by Theorem 9, X is set selectively 2-star-ccc. □

Corollary 6. *If X is ccc space which has a dense metrizable subspace, then X is set selectively 2-star-ccc.*

Proof. Since metrizable subspace is paracompact, thus by Theorem 9, X is set selectively 2-star-ccc. □

Theorem 10. *If X is a ccc space which has a monotonically normal dense subspace (hence, a dense GO-space), then X is selectively 2-star-ccc.*

Proof. Since every ccc monotonically normal space is (hereditary) Lindelöf, rest of the proof is similar to the proof of Theorem 9. □

5. Open Problems

We finish the paper by the following questions which we could not answer while working on this paper.

Problem 2. Does there exist a Tychonoff selectively star-ccc space which is not set selectively star-ccc? Do there exist similar examples for $k > 1$?

Since every set selectively 2-star-ccc spaces is selectively 2-star-ccc, the following problem is an improved version of the Problem 4.9 in [11].

Problem 3. Are ccc spaces set selectively 2-star-ccc?

Problem 4. Does there exist in ZFC a normal set selectively 2-star-ccc space which is neither set strongly star-Lindelöf nor set selectively star-ccc?

Let us notice that, under assumption $2^{\aleph_0} = 2^{\aleph_1}$, there is such an example. It is the space $S(X, \omega) = L \cup (\omega_1 \times \omega)$, $|L| = \aleph_1$, $L \cap \omega = \varnothing$, in (Example 2.2 in[24]). This space is set selectively 2-star-ccc because it contains the set selectively star-ccc space $L \cap \omega = \varnothing$ as a dense subspace. On the contrary, this space is not set strongly star-Lindelöf (because it is not star-Lindelöf), and it is not set selectively star-ccc as it was shown in (Example 3.4 in [21]).

In [25], Scheepers gave a game-theoretic characterization of selectively ccc spaces.

Problem 5. Do there exist game-theoretic characterizations of set selectively star-ccc and set selectively k-star-ccc spaces?

Song and Xuan (Theorem 3.6 in [21]) showed that an open F_σ-subset of selectively star-ccc space is selectively star-ccc.

Problem 6. Is open F_σ-subset of a set selectively star-ccc space also set selectively star-ccc?

6. Conclusion

Set-selective properties of topological spaces show how the subsets of a space are located in the space. We used this idea and the method of stars to study the set version of an important class of selectively star-ccc spaces. It is proved that the class of set selectively star-ccc spaces contains Lindelöf spaces, countably compact spaces of countable tightness, and Rothberger separable spaces. On the contrary, the class of set selectively star-ccc spaces is different from the class of Lindelöf spaces and some other classes of spaces which are set selectively star-ccc. A few open problems are posed to suggest a further research in this field. In particular, it would be interesting to investigate set selectively k-star-ccc spaces, $k \geq 2$, defined by the iteration of the star operator.

Data Availability

The data used to support the findings of this study are cited at relevant places within the text as references and are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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