Hindawi
Journal of Mathematics
Volume 2020, Article ID 9292171, 8 pages
https://doi.org/10.1155/2020/9292171

Research Article

On Quasi *S*-Propermutable Subgroups of Finite Groups

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Received 26 February 2020; Accepted 22 June 2020; Published 1 August 2020

Academic Editor: Ji Gao

A subgroup H of a finite group G is said to be quasi S-propermutable in G if $K \lhd G$ such that HK is S-permutable in G and $H \cap K \leq H_{qsG}$, where H_{qsG} is the subgroup formed by all those subgroups of H which are S-propermutable in G. In this paper, we give some generalizations of finite group G by using the properties and effects of quasi S-propermutable subgroups.

1. Introduction

A finite group is a group, of which the underlying set contains a finite number of elements. Throughout this paper, all groups are finite and G always denotes a finite group. Moreover Sylow subgroups are denoted by $\mathrm{Syl}(G)$ and the set of primes is denoted by $\pi(G)$, if order of G is divisible by some prime. For any $q \in \pi(G)$ implies G_q is a $\mathrm{Syl}_q(G)$. Furthermore, supersolvable groups are denoted by $\mathcal{U}$ here. Other notions that are used and not defined in this paper are taken from [1, 2]. A solvable group (also called as soluble group) can be constructed from the abelian groups by using extensions.

The term S-propermutable was introduced by Yi and Skiba in [3]. Recall that a subgroup H is S-quasinormal if $H \leq G$ and is S-permutable, if it commutes with all Sylow subgroups $\mathrm{syl}(G)$ of G [4]. For interesting properties of S-permutable, we refer the readers to [5, 6]. The c-normal subgroups were introduced by Wang [7] as follows: a subgroup S of G is called c-normal if $G = ST$ with $S \cap T \leq S_G$, where S contains the largest normal subgroup S_G with $T \trianglelefteq G$. A subgroup K of G, where $G = N_G(K)A$, is called S-propermutable in G if K is S-permutable in A [3]. The structure of finite groups in which permutability is a transitive relation is discussed in [8] by Robinson in 2001. In 2002, Ballester-Bolinches and Esteban-Romero discussed the Sylow permutable subnormal subgroups of G, and weakly s-permutable subgroups of G were studied by Skiba in 2007. Beidleman and Ragland in [9] studied some properties of subnormal, permutable, and embedded subgroups in G. In 2012, Zhang and Wang studied the influence of s-semipermutable subgroups of G. Some generalizations of permutability and S-permutability are given in [10]. For details, we refer the readers to [11–13]. In this paper, we aim to study some interesting properties of quasi S-propermutable subgroups of G.

Definition 1 (quasi S-propermutable subgroup). A subgroup H of a finite group G is said to be quasi S-propermutable in G if $K \lhd G$, such that HK is S-permutable in G and $H \cap K \leq H_{qsG}$, where H_{qsG} is the subgroup formed by all subgroups of H which are S-propermutable in G.

In it clear from the definition of quasi S-propermutable subgroup that both the ideas of c-normal subgroups and S-propermutable subgroups are covered by quasi S-propermutable subgroups. But converse is not true (see Examples 1 and 2).

Example 1. Suppose $G = S_4$ and $K = \langle(14)\rangle$. If $G = A_4K$ and $K \cap A_4 = 1$, then K is quasi S-propermutable in G, but K does not commute with all Sylow subgroups of A_4, so K is not S-propermutable in G.

Example 2. Suppose $G = S_4$ and M is normal subgroup of G of order four. If K is $\mathrm{Syl}_3(G)$, then $KM = A_4 \trianglelefteq G$ and $K \cap M = 1$. This implies that K is not c-normal but K is quasi S-propermutable.

In the theory of propermutable subgroups, our contributions are the following theorems.

Theorem 1. *Let G be a Sylow q-group, where q is a prime and divides $|G|$ and $(|G|, q-1) = 1$. Then, any $Q_1 \leq Q$, which is quasi S-propermutable in G, does not have a q-nilpotent supplement in G and is hence solvable.*

Theorem 2. *Let us consider a Sylow q-subgroup Q of M, where $M \trianglelefteq G$ and q is a prime divisor of $|M|$ satisfying $(|M|, q-1) = 1$. If every largest subgroup Q_1 of Q is quasi S-propermutable in G such that Q_1 does not have a q-supersolvable supplement in G, then each chief factor of G between M and $O_{q'}(M)$ is cyclic.*

Theorem 3. *Let $Syl_q(G)$ be contained in G , where q is a prime division of $|G|$ and $(|G|, q-1) = 1$. Then, G is q-nilpotent if every largest subgroup Q_1 of Q is quasi S-propermutable in G such that Q_1 does not have a q-nilpotent supplement in G.*

To prove our main contribution, somehow, we used same methodology as we used in [14].

2. Preliminaries

In this section, we present some lemmas that will be helpful to prove Theorems 1–3.

In the following lemma, the sufficient conditions for S-subgroup to be S-permutable are given.

Lemma 1 (see [4, 15]). *Let X be S- subgroup of G; then, the following statements hold:*

(1) If $X \leq Y \leq G$, then X is S-permutable in Y.
(2) If $M \trianglelefteq G$, then XM is S-permutable in G and (XM/M) is S-permutable in (G/M).
(3) If $Y \leq G$, then $X \cap Y$ is S-permutable in Y.
(4) X is subnormal in G.
(5) If Y is S-permutable subgroup in G, then $X \cap Y$ is S-permutable in G.

In the following lemma, some interesting properties of S-propermutable and normal subgroups are given.

Lemma 2 (see [3], Lemma 2.3). *Let X and M be S-propermutable and normal in G, respectively. Then, the following statements hold:*

(1) (XM/M) is S-propermutable in (G/M).
(2) For a prime divisor q of $|G|$, X commutes with some Sylow q-subgroup of G.
(3) If G is π-solvable, then X commutes with some Hall π-subgroup of G.

In the following lemma, we give equivalent statements for a q-subgroup of a group G.

Lemma 3. *Let K be a q-subgroup of a group G. Then, the following statements are equivalent:*

(1) K is S-permutable in G.
(2) K is S-propermutable in G and $K \leq O_q(G)$.

Proof

(1) $\longrightarrow$ (2) Suppose K is S-permutable in G. Since K is S-permutable in G, K is also subnormal in G [4], that is, $K \leq O_q(G)$. As an S-permutable subgroup is also S-propermutable, (2) holds.

(2) $\longrightarrow$ (1) Suppose K is S-propermutable in G. By definition, there will be $D \leq G$ such that

$$\begin{aligned} G &= N_G(K)D, \\ KY &= YK, \quad \forall Y \in \mathrm{Syl}(D). \end{aligned} \tag{1}$$

Particularly, if $Y = S \in \mathrm{Syl}_p(D)$, $p \neq q$, then $KS = SK$ and

$$K = O_q(G) \cap KS \triangleleft KS. \tag{2}$$

Therefore, $O_q(G)$ is a subgroup of $N_G(K)$, and hence K is S-permutable [16].

Some properties of S-propermutable subgroups are given in the following lemma. □

Lemma 4. *Let X be S-propermutable and suppose that $T \trianglelefteq G$ and $Y \leq G$. Then, we have the following statements:*

(1) If $X \leq Y$, then X is S-propermutable in Y.
(2) If A is any S-propermutable subgroup in G, then (AT/T) is S-propermutable in (G/T).
(3) If Y is S-propermutable in G, then XY is S-propermutable in G.
(4) If $T \leq Y$ and (Y/T) is S-propermutable in (G/T), then Y is S-propermutable in G.

Proof

(1) Suppose D is a supplement of X such that

$$\begin{aligned} G &= N_G(X)D, \\ XY &= YX, \quad \forall Y \in \mathrm{Syl}(D). \end{aligned} \tag{3}$$

By using Dedekind identity, we have

$$\begin{aligned} Y &= (N_G(X)D) \cap Y \\ &= N_G(X)(D \cap Y) \\ &= N_G(X)D_1. \end{aligned} \tag{4}$$

So, D_1 be the supplement of X in Y.

Furthermore, there exists $S \in \mathrm{Syl}(D)$ for any $A \in \mathrm{Syl}(D_1)$ such that $A \leq S$. So, we have $XS = SX$, and hence

$$\begin{aligned}(XS) \cap Y &= X(S \cap Y) = XA, \\ Y \cap (SX) &= (Y \cap S)X = AX.\end{aligned} \tag{5}$$

Therefore,

$$XA = AX, \quad \forall A \in \mathrm{Syl}(D_1). \tag{6}$$

Hence, X is S-propermutable in Y.

(2) This follows immediately from Lemma 2 (1).

(3) Suppose $D \leq G$, such that $G = N_G(X)D$ and $XV = VX, \forall V \in \mathrm{Syl}(D)$. Since Y is S-propermutable in G, $XY \leq G$ and D is a supplement of $N_G(XY)$ in G. Thus,

$$\begin{aligned}(XY)V &= X(YV) \\ &= X(VY), \\ X(VY) &= (XV)Y \\ &= (VX)Y \\ &= V(XY).\end{aligned} \tag{7}$$

Hence, XY is S-propermutable in G.

(4) Let (Y/T) be S-propermutable in (G/T). Then, by definition, there exists (D/T), which is supplement of $N_G(Y/T)$ to (G/T). As D is also a supplement of $N_G(Y)$ to G, (RT/T) is a $\mathrm{Syl}(D/T)$, for any $R \in \mathrm{Syl}(D)$. This implies

$$\frac{Y}{T} \cdot \frac{RT}{T} = \frac{RT}{T} \cdot \frac{Y}{T}. \tag{8}$$

So,

$$YRT = RTY. \tag{9}$$

Furthermore, T is contained in Y. Thus,

$$YR = RY, \quad \forall R \in \mathrm{Syl}(D). \tag{10}$$

Hence, Y is S-propermutable in G.

The following two lemmas are about the basic properties of subgroups of group G. □

Lemma 5. *Suppose $X \leq Y \leq G$. Then, we have the following statements:*

(1) $X_{qsG} \leq X_{qsK}$.

(2) Let Y be p-group and $X \trianglelefteq G$. Then, $(Y_{qsG}/X) \leq (Y/X)_{qs(G/H)}$.

(3) $(D_{qsG}X/X) \leq (DX/X)_{qs(G/X)}$, where $(|X|, |D|) = 1$ and $X \trianglelefteq G$.

Proof. These results can be easily proved by using Lemmas 3 and 4. □

Lemma 6. *Suppose that $X \leq G$. Then, we have the following statements:*

(1) If X is quasi S-propermutable and $X \leq Y \leq G$, then X is quasi S-propermutable in Y.

(2) If $C \trianglelefteq G$ such that $B \leq X$ and X is q-group and quasi S-propermutable, where q is a prime, then (X/C) is quasi S-propermutable in (G/C).

(3) If C is a normal $q\prime$-subgroup of G and X is a q-subgroup and quasi S-propermutable in G, then (XC/C) is quasi S-propermutable in (G/C).

(4) If X is quasi S-propermutable in G, such that $X \leq Y \trianglelefteq G$, then $B \trianglelefteq G$ such that XB is S-permutable in G with $X \cap B \leq X_{qsG}$ and $XB \leq Y$.

Proof

(1) Let $X \leq Y \leq G$ and $B \trianglelefteq G$ such that XB is S-permutable in G and $X \cap B \leq X_{qsG}$. Then, $Y \cap B$ is normal and

$$X(Y \cap B) = XB \cap Y, \tag{11}$$

is S-permutable in Y. Using Lemma 1 (3), we have

$$X \cap (Y \cap B) = X \cap B \leq X_{qsG} \leq X_{qsK}. \tag{12}$$

Hence, X is quasi S-propermutable in Y.

(2) Let X be a quasi S-propermutable in G, so we have $Y \trianglelefteq G$ such that XY is S-permutable in G and $X \cap Y \leq X_{qsG}$.

By Lemma 1 (2), we have $(YC/C) \trianglelefteq (G/C)$ and $(X/C)(YC/C) = (XY/C)$ is S-permutable in (G/C). Using Lemma 5 (2), we have

$$\begin{aligned}\frac{X}{C} \cap \frac{YC}{C} &= \frac{(X \cap Y)C}{C} \leq \frac{X_{qsG}C}{C} \\ &= \frac{X_{qsG}}{C} \leq \left(\frac{X}{C}\right)_{qs(G/C)}.\end{aligned} \tag{13}$$

Thus, (X/C) is quasi S-propermutable in (G/C).

(3) Let X be quasi S-propermutable in G, so we have $Y \trianglelefteq G$ such that XY is S-permutable and $X \cap Y \leq X_{qsG}$. Obviously, $(YC/C) \trianglelefteq G$ and

$$\left(\frac{X}{C}\right)\left(\frac{YC}{C}\right) = \frac{XYC}{C}, \tag{14}$$

are S-permutable in (G/C). Now by Lemma 1 (2) and $(|XC\colon X|, |XC\colon C|) = 1$, we have

$$\begin{aligned}\frac{XC}{C} \cap \frac{YC}{C} &= \frac{(XC \cap Y)C}{C} \\ &= \frac{(X \cap Y)(C \cap Y)C}{C} \\ &= \frac{(X \cap Y)C}{C} \leq \frac{X_{qsG}C}{C}.\end{aligned} \tag{15}$$

Using Lemma 5 (3), we have

$$\frac{X_{qsG}C}{C} \leq \left(\frac{XC}{C}\right)_{qs(G/C)}. \quad (16)$$

Hence, (XC/C) is quasi S-propermutable in (G/C).

(4) Let X be quasi S-propermutable in G. So, we have $Y \trianglelefteq G$ such that XY is S-permutable in G and $X \cap Y \leq X_{qsG}$.

Now if $S = B \cap Y$, then S will be normal in G, and

$$BS = X(B \cap Y) = XB \cap Y, \quad (17)$$

is S-permutable. Now, using Lemma 1 (5), we have $XS \leq Y$ and

$$X \cap S = X \cap B \cap Y = X \cap B \leq X_{qsG}. \quad (18)$$

Hence, the desired result is proved.

The relation between q-supersoluble, q-nilpotent, cyclic Sylow q-subgroup, and normal subgroups is given in the following lemma. □

Lemma 7. *Let q be a prime divisor of $|G|$ such that $(|G|, q-1) = 1$. Then,*

(1) If G is q-supersoluble, then G is q-nilpotent.

(2) If G has cyclic Sylow q-subgroup, then G is q-nilpotent.

(3) If $|G\colon X| = q$ and $X \leq G$, then X is normal in G.

(4) If $|M| = q$ and $M \triangleleft G$, then N lies in $Z(G)$.

Proof. One can prove (1) by using the approach of [14]. Proofs of (2)–(4) are obvious and can be seen in ([17], Theorem 2.8).

Now, we give some known lemmas that are very important to prove our main theorems. □

Lemma 8 (see [18]). *Let $Y \trianglelefteq G$. Then, $(Y/\Phi(Y)) \leq Z_{\mathcal{U}}(G/\Phi(Y))$ if and only if $Y \leq Z_{\mathcal{U}}(G)$.*

Lemma 9 (see [16], Theorem A). *If Q is an S-permutable q-subgroup of G, then $N_G(Q) \geq O^q(G)$.*

Lemma 10 (see [2], VI, 4.10). *Let $C, D \leq G$ such that $G \neq C\,D$. Then, a nontrivial normal subgroup of G contains either C or D satisfying $CD^g = D^gC$, for any $g \in G$.*

Lemma 11 (see [19], Lemma 2.12). *Let q be a prime divisor of $|G|$ such that $(|G|, q-1) = 1$ and Q be a Sylow q-subgroup of G. Then, G is q-nilpotent if every largest subgroup of Q has a q-nilpotent supplement in G.*

Lemma 12 (see [20], Lemma 2.11). *Suppose M is elementary abelian normal subgroup of G. Let $A \leq M$ satisfying $1 < |A| < |M|$ and $K \leq M$ such that $|K| = |A|$ is S-permutable in G. Then, G contains largest normal subgroup of M.*

Lemma 13. *Suppose $X \triangleleft G$ is q-subgroup, where q is a prime. Then, we have $A \trianglelefteq G$ where A is the largest subgroup of X which is also quasi S-propermutable.*

Proof. If order of X is q, then the result holds.

If $Y \leq X$ is a normal q-subgroup and $X \neq Y$, then by using Lemma 6 (2), we can easily obtain the required result.

If the subgroup (L/Y) of $(X/Y) \triangleleft (G/Y)$, then obviously $L \leq X$ and $L \trianglelefteq G$ and the result holds.

If $X = Y$ and L is any largest subgroup of X, then there will be $E \trianglelefteq G$ such that LE is S-permutable and $L \cap E \leq L_{qsG}$. Let $L \neq L_{qs}G$. Then, $LE \neq L$ and $Y \neq 1$. If $X \leq LE$, then

$$X = X \cap LE = L(X \cap E). \quad (19)$$

Hence, $X \leq E$, which shows that $L = L \cap E = L_{qsG}$. Thus, it is a contradiction.

Now, if $X \nleq LE$, then

$$L = L(E \cap X). \quad (20)$$

So using Lemma 1 (5), $LE \cap X$ is S-permutable, which is again a contradiction. Thus, $L = L_{qsG}$. So, using Lemma 3, L is S-permutable in G. Consequently, we have largest subgroup X such that $X \triangleleft G$, and by using Lemma 12, the result is proved. □

3. Proofs of Main Theorems

In this section, we prove our main theorems.

Proof of Theorem 1. We divide our proofs into 6 steps.

Step 1. First we prove that $O_q(G) = 1$.

Let

$$O_q(G) \neq 1, \quad (21)$$

and take

$$O_q(G) = C. \quad (22)$$

Then obviously, (Q/C) is a $\mathrm{Syl}_q(G/C)$. Suppose (Q_1/C) is the largest subgroup of (Q/C). Clearly Q_1 will be the largest subgroup of Q and (Q_1/C) has a q-nilpotent supplement (AC/C) in (G/C) provided Q_1 has supplement of A in G which is q-nilpotent.

If Q_1 is quasi S-propermutable, then by using Lemma 6 (2), (Q_1/C) is quasi S-propermutable in (G/C). Now, as G is smallest, so (G/C) is solvable. Hence, our supposition is wrong, and thus $O_q(G) = 1$.

Step 2. In this step, we prove that $O_{q'}(G) = 1$.

Suppose on contrary that

$$O_{q'}(G) \neq 1. \quad (23)$$

If

$$O_{q'}(G) = V, \quad (24)$$

then clearly (QV/V) is a Sylow q-subgroup of (G/V). Consider (J/V) to be the largest subgroup of (QV/V). So, there will be a largest subgroup Q_1 of Q such that $J = Q_1V$. Then, (J/V) has a q-nilpotent supplement (BV/V) in (G/V). If Q_1 has a q-nilpotent supplement B

in G, then by using Lemma 6 (3), we obtain that (J/V) is quasi S-propermutable in (G/V) provided Q_1 is quasi S-propermutable in G. Since G has smallest order, (G/V) is solvable and by using the Feit–Thompson theorem, V is solvable. It follows that G is solvable, which contradicts our supposition, and hence $O_{q'}(G) = 1$.

Step 3. Here, we prove that Q is not cyclic.

Let Q be a cyclic group; then, by Lemma 7, G is q-nilpotent. So, G is solvable, which is a contradiction to our supposition, and hence Q is not cyclic.

Step 4. Here, we prove that Y is not solvable if $Y \lhd G$ and $QY = G$.

Let Y be q-soluble; then, either

$$\begin{gathered} O_q(Y) \neq 1, \\ \text{or } O_{q'}(Y) \neq 1. \end{gathered} \tag{25}$$

As

$$O_q(Y) \leq O_q(G), \tag{26}$$

so

$$O_{q'}(Y) \leq O_{q'}(G). \tag{27}$$

Thus,

$$\begin{gathered} O_q(G) \neq 1, \\ \text{or } O_{q'}(G) \neq 1, \end{gathered} \tag{28}$$

which is a contradiction to step (1) or (2). So our supposition is wrong and Y is not solvable.

Now, we will prove the later part. For this, let

$$QY < G. \tag{29}$$

Then, by Lemma 6 (1), every largest subgroup Q_1 of Q is quasi S-propermutable in QY.

As Q_1 does not have a q-nilpotent supplement in QY, QY fulfills all the conditions of our theorem. Since G is of smallest order, this implies QY and Y are also solvable, which is a contradiction, and thus $QY = G$.

Step 5. Here, we prove that Y is unique and smallest subgroup of G, such that $Y \lhd G$.

Since by step 4, $QY = G$ for every $Y \trianglelefteq G$, (G/Y) is solvable. Hence, Y is smallest and unique, and $Y \lhd G$.

Step 6. $Q_1 \cap Y = (Q_1)_{qsG} \cap Y$.

By Lemma 11, G is q-nilpotent if every largest subgroup of Q has a q-nilpotent supplement in G, which shows that G is solvable, a contradiction. So, we can assume a largest subgroup Q_1 of Q such that Q_1 is quasi S-propermutable, so

$$N \trianglelefteq G, \tag{30}$$

as $Q_1 N$ is S-permutable and

$$Q_1 \cap N \leq (Q_1)_{qsG}. \tag{31}$$

Now if

$$N = 1, \tag{32}$$

then Q_1 is S-permutable in G.

Now by Lemma 9, we have

$$Q_1 \trianglelefteq QO^q(G) = G. \tag{33}$$

In view of step (5),

$$\begin{gathered} Q_1 = 1 \\ \text{or } Y \leq Q_1. \end{gathered} \tag{34}$$

By step (4), Y is not solvable. This implies

$$Q_1 = 1. \tag{35}$$

Hence, Q is cyclic, which is a contradiction to step (3). So,

$$\begin{gathered} N \neq 1, \\ Y \leq N. \end{gathered} \tag{36}$$

Consequently,

$$Q_1 \cap Y = (Q_1)_{qsG} \cap Y. \tag{37}$$

Now, for any Sylow p-subgroup Y_p of Y with $p \neq q$, we may write by using step (2)

$$(Q_1)_{qsG} Y_p = Y_p (Q_1)_{qsG}. \tag{38}$$

So,

$$\begin{aligned} (Q_1)_{qsG} Y_p \cap Y &= Y_p\big((Q_1)_{qsG} \cap Y\big) \\ &= Y_p(Q_1 \cap Y), \end{aligned} \tag{39}$$

that is,

$$Q_1 \cap Y, \tag{40}$$

is S-permutable in Y. Let

$$Y \cong Y_1 \times \cdots \times Y_k. \tag{41}$$

By Lemma 2 (1), $Q_1 \cap Y$ is S-permutable in $(Q_1 \cap Y)Y_1$. Hence,

$$\begin{aligned} (Q_1 \cap Y)\big(Y_{1p}\big)^{m1} \cap Y_1 &= \big(Y_{1p}\big)^{m1} (Q_1 \cap Y \cap Y_1) \\ &= \big(Y_{1p}\big)^{m1} (Q_1 \cap Y_1), \end{aligned} \tag{42}$$

for any $m_1 \in Y_1$, where Y_{1p} is a $\mathrm{Syl}_p(Y_1)$ with $p \neq q$. As $(Y_{1p})^{m1}\ (Q_1 \cap Y_1) \neq Y_1$, so by Lemma 10, Y_1 is not simple, which is a contradiction.

Hence the desired result is proved. □

Proof of Theorem 2. Here, we use the contradiction method to prove this theorem. There are seven steps.

Step 1. Firstly, we will prove that C is q-nilpotent.

Suppose that Q_1 is the largest subgroup of Q and Q_1 has a q-supersolvable supplement $X \cap C$ in C provided Q_1 has a q-supersolvable supplement X in G. Because

$$(|C|, q-1) = 1, \tag{43}$$

this implies $X \cap C$ is q-nilpotent by Lemma 7 (1). If Q_1 is quasi S-propermutable in G, then Q_1 is also quasi S-propermutable in C by Lemma 6 (1). Also, Q_1 does not have any q-nilpotent supplement in C. So, by Theorem 1, C is q-nilpotent.

Step 2. In this step, we show that $Q = C$.

Using step (1), $O_{q'}(C)$ is the normal Hall q'-subgroup of C.

Let $O_{q'}(C) \neq 1$. We can check it easily that our theorem is true for $(G/O_{q'}(C, (C/O_{q'}(C))))$. Using induction, we can see that every chief factor of $(G/O_{q'}(C))$, between 1 and $(C/O_{q'}(C))$, is cyclic, which implies that each factor between C and $O_{q'}(C)$ is cyclic, so $O_{q'}(C) = 1$, and hence $Q = C$.

Step 3. Here, we prove that $\Phi(Q) = 1$.

First, we let $\Phi(Q) \neq 1$; then, by Lemma 3 (2), our theorem holds for $((G/\Phi(Q)), (Q/\Phi(Q)))$. Every chief factor of $(G/\Phi(Q))$ under $(Q/\Phi(Q))$ is cyclic by our selection of (G, C) by Lemma 8, which is a contradiction.

Step 4. Here, we prove that every largest subgroup of Q is quasi S-propermutable in G.

Consider Q_1 is the largest subgroup of Q such that J is q-supersolvable supplement of Q_1 in G. Thus,

$$QJ = G, \tag{44}$$

with $Q \cap J \neq 1$. Because

$$Q \cap J \trianglelefteq J, \tag{45}$$

we suppose that $Q \cap J$ contains a smallest normal subgroup Y of J. Here, obviously $|Y| = q$.

Since Q is elementary abelian and $G = QJ$, this implies

$$Y \trianglelefteq G. \tag{46}$$

Here, we can check that our theorem holds for $((G/Y), (Q/Y))$. By our selection of (G, C), we can see that every chief factor of (G/Y) under (Q/Y) is cyclic. As a consequence, every chief factor of G under Q is cyclic, which is a contradiction, and hence (4) holds.

Step 5. Now, we prove that G does not have a smallest normal subgroup Q.

Let $Q \triangleleft G$, so by Lemma 13, G contains some largest normal subgroup of Q, which cannot be true because Q is of smallest order.

Step 6. Let $Y \triangleleft Q$ of G; then,

$$\begin{gathered} \frac{Q}{Y} \leq Z_{\mathscr{U}}\left(\frac{G}{Y}\right), \\ |Y| > q. \end{gathered} \tag{47}$$

Moreover, using Lemma 3 (2), our theorem is satisfied $((G/Y), (Q/Y))$. Thus, from our selection of $(G, C) = (G, Q)$, every chief factor of (G/Y) under (Q/Y) is cyclic.

If $|Y| = q$, then Y is a cyclic group, which contradiction of our supposition. Now if Q contains two smallest normal subgroups S and Y of G, then

$$\frac{YS}{S} \leq \frac{Q}{S}, \tag{48}$$

and from the isomorphism

$$\frac{YS}{S} \cong Y, \tag{49}$$

it follows that

$$|Y| = q, \tag{50}$$

a contradiction again. Thus, step (6) is true.

Step 7. Finally, to prove our theorem, we need the following contradiction.

Suppose that $y \triangleleft Q$ of G and Y_1 is a largest subgroup of Y. To show Y_1 is S-permutable, we may suppose that B is a complement of Y in Q, as Q is an elementary abelian q-group.

Also, take $W = Y_1 B$. Clearly, W is a largest subgroup of Q, so by step (4), W is quasi S-propermutable in G, and by Lemma 6 (4), there will be $S \trianglelefteq G$ satisfying the condition

$$\begin{gathered} W \cap S \leq W_{qsG}, \\ WS \leq Q, \end{gathered} \tag{51}$$

and WS is S-permutable in G. So by virtue of Lemma 3, W_{qsG} is an S-permutable subgroup in G.

Now, if $S = Q$, then $W = W_{qsG}$ is S-permutable; by Lemma 1 (5),

$$W \cap Y = Y_1 C \cap Y = Y_1 (C \cap Y) = Y_1, \tag{52}$$

is S-permutable.

If $S = 1$, this gives $W = WS$ is S-permutable. As a result, Y_1 is S-permutable. Consider $1 < S < Q$; then, $Y \leq S$ by step (6). So, by Lemma 1 (5),

$$Y_1 = W \cap Y = W_{qsG} \cap Y, \tag{53}$$

is S-permutable. This implies $|Y| = q$, which contradicts step (6).

This completes the proof of our Theorem 2. □

Proof of Theorem 3. Consider q-nilpotent group G, so G contains a normal Hall q'-subgroup $G_{q'}$. Suppose that the largest $Q_1 \leq Q$; then,

$$\left|G \colon Q_1 G_{q'}\right| = q. \tag{54}$$

Using Lemma 7 (3), we obtain

$$Q_1 G_{q'} \trianglelefteq G. \tag{55}$$

Clearly,

$$Q_1 \cap G_{q'} = 1. \tag{56}$$

Thus, Q_1 is quasi S-propermutable in G.

For sufficient condition, we suppose that hypothesis is wrong. So, our proof consists of the following seven steps.

Step 1. Firstly, we need to prove that G is solvable, which can be proved easily by Theorem 1.

Step 2. Here, we show that (G/Y) is q-nilpotent provided Y is the smallest unique normal subgroup.

Let $Y \lhd G$, which is smallest. By step (1), G is solvable; this implies that Y is an elementary abelian. Hence, in light of Lemma 6, (G/Y) satisfies our theorem. Following this, (G/Y) is q-nilpotent as G is of smallest order, which is the required result.

Step 3. Here, we need to show that $\Phi(G) = 1$, which is clear from step (2).

Step 4. Now, we show that Q is not cyclic.

Let Q be cyclic; then, by Lemma 7 (2), G will be q-nilpotent, which is against our supposition. Thus, Q is not cyclic.

Step 5. Now, it is obvious that $O_{q'}(G) = 1$.

Step 6. In this step, we prove that G contained the q-nilpotent supplement of every largest subgroup of Q.

Obviously,

$$Y \leq O_q(G). \tag{57}$$

So by step (3), we can select a largest K of G satisfying

$$\begin{gathered} G = YK, \\ \frac{G}{Y} \cong K. \end{gathered} \tag{58}$$

Let Q_1 be the largest subgroup of Q. So, we need to show that G contains a q-nilpotent supplement of Q_1. As Y has the q-nilpotent supplement K, we will show $Y \leq Q_1$, where Q_1 is quasi S-propermutable in G. For this, suppose that

$$\begin{gathered} L \trianglelefteq G, \\ Q_1 \cap L \leq (Q_1)_{qsG}, \end{gathered} \tag{59}$$

and $Q_1 L$ is S-permutable in G. There are two possibilities.

(i) If $L = 1$.

It follows that Q_1 is S-permutable. Also, by Lemma 9,

$$Q_1 \trianglelefteq QO^q(G) = G. \tag{60}$$

In view of step (3) and Lemma 7 (2), we have

$$Q_1 \neq 1. \tag{61}$$

So by step (2), we have

$$Y \leq Q_1. \tag{62}$$

(ii) If $L \neq 1$, then

$$Y \leq L. \tag{63}$$

This implies that

$$Q_1 \cap Y = (Q_1)_{qsG} \cap Y. \tag{64}$$

By using step (4), we obtain

$$(Q_1)_{qsG} G_p = G_p (Q_1)_{qsG}, \tag{65}$$

where G_p is any $\mathrm{Syl}_p(G)\,(q \neq p)$.

Then,

$$(Q_1)_{qsG} \cap Y = (Q_1)_{qsG} G_p \cap Y \trianglelefteq (Q_1)_{qsG} G_p. \tag{66}$$

Obviously,

$$Q_1 \cap Y \trianglelefteq Q. \tag{67}$$

That is why

$$Q_1 \cap Y \trianglelefteq G. \tag{68}$$

Since Y is smallest subgroup, it follows

$$\begin{gathered} Q_1 \cap Y = 1, \\ Q_1 \cap Y = Y. \end{gathered} \tag{69}$$

If

$$Q_1 \cap Y = 1, \tag{70}$$

then

$$|Y| = q, \tag{71}$$

because largest subgroup

$$Q_1 \cap Y \leq Y. \tag{72}$$

As a result, G is q-nilpotent by Lemma 7 (4) and step (2). Hence,

$$\begin{gathered} Q_1 \cap Y = Y, \\ Y \leq Q_1. \end{gathered} \tag{73}$$

Step 7. Finally, we prove the contradiction.

By step (6), G contained the q-nilpotent supplement of every largest subgroup of Q, so by Lemma 11, G is q-nilpotent, hence a contradiction.

This completes the proof of Theorem 3. □

4. Concluding Remarks

In this paper, we gave some properties of quasi S-propermutable subgroups of a finite group. We relate quasi S-propermutable subgroups with solvable subgroups and cyclic subgroups. Lastly, we gave necessary and sufficient condition for quasi S-propermutable subgroups. The following theorem can be obtained immediately from our results.

Theorem 4. *Suppose that a saturated formation is denoted by $\mathcal{F}$, having all the supersolvable groups and $Y \trianglelefteq G$ such that $(G/Y) \in \mathcal{F}$. Then, $G \in \mathcal{F}$ provided every noncyclic Sylow subgroup Q of $F^*(Y)$ is quasi S-propermutable in G such that every largest subgroup of Q does not have any supersolvable supplement in G.*

Data Availability

All data required for this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study.

Acknowledgments

This research was supported by HEC Pakistan.

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