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Research Article

Best Proximity Point for the Sum of Two Non-Self-Operators

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In the present paper, we focus our attention on the existence of the fixed point for the sum of the cyclic contraction and the noncyclic accretive operator. Also, we study the best proximity point for the sum of two non-self-mappings. Moreover, we provide the existence of the best proximity point for the cyclic contraction through the notion of the nonlinear $\mathscr{D}$-set contraction. Finally, we give the existence of the best proximity point for the sum of the nonlinear $\mathscr{D}$-set contraction mapping and partially completely continuous mapping in the setting of the partially ordered complete normed linear space.

1. Introduction

Fixed point theory plays an important role in the area of nonlinear functional analysis, and it has many applications in the study of nonlinear differential and integral equations. The study of nonlinear equations of the form $\Gamma_1\eta + \Gamma_2\eta = \eta$, where $\Gamma_1, \Gamma_2\colon \mathscr{B} \longrightarrow \mathscr{B}$ are mappings on the Banach space $\mathscr{B}$, helps to solve many physical nonlinear real-life problems. For example, Dhage and Otrocol [1] gave the existence and approximation of solutions to the following hybrid differential equation:

$$\begin{cases} x'(t) = f(t, x(t)) + g\left(t, \max_{a\le\xi\le t} x(\xi)\right), \\ x(a) = \alpha_0 \in \mathbb{R}, \end{cases} \tag{1}$$

for all $t \in J = [a, b]$ and $f, g\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. Also, Banaś and Amar [2] obtained the existence of the solution to the nonlinear integral equation of the form

$$x(t) = a(t) + \int_0^1 k(t,s) f(s, x(s))\mathrm{d}s + \int_0^1 u(t, s, x(s))\mathrm{d}s, \tag{2}$$

for $t \in J = [a, b]$ and $a \in L^1(J)$, $f\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$, $k\colon J \times J \longrightarrow \mathbb{R}^+$, and $u\colon J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$. So, the researchers involved in finding the solution of the equation $\Gamma_1\eta + \Gamma_2\eta = \eta$, which is clearly the problem of finding the sufficient condition for the existence of fixed point for the sum of two mappings.

In the sequel, in 1955, Krasnoselskii gave an existence of the solution for the equation $\Gamma_1\eta + \Gamma_2\eta = \eta$ in the Banach space setting, where Γ_1 is the contraction and Γ_2 is the compact operator. Later, many researchers extended Krasnoselskii's theorem in different directions (see [3–7] and the references therein). Vijayaraju [7] proved the theorems pointing the existence of the fixed point for a sum of nonexpansive and continuous mappings and also a sum of asymptotically nonexpansive and continuous mappings in the setting of locally convex spaces. O'Regan [5] established the fixed point theorem for the sum of two operators $\Gamma_1 + \Gamma_2$ if Γ_1 is compact and Γ_2 is nonexpansive. Moreover, the results were used to prove the existence of the solution for second-order boundary value problem. O'Regan and Taoudi [6] proved the fixed point theorems for the sum of two weakly sequentially continuous mappings in the Banach space. Dhage [3] proved the fixed point result by combining two fixed point theorems of Krasnoselskii and Dhage and also derived the existence result for the product of two operators in Banach algebra. Dhage [4] found the local

version of fixed point theorems of Krasnoselskii and Nashed et al. By using this result, he provided the application to nonlinear functional integral equations.

Agarwal et al. [8] obtained the existence of the fixed point for two mappings, compact mapping and nonexpansive mapping, in the setting of both the weak and the strong topology of a Banach space. Ben Amar and Garcia-Falset [9] proved the existence of fixed point theorems for different kinds of contractions such as nonlinear weakly condensing, 1-set weakly contractive, and pseudo-contractive and nonexpansive operators defined on unbounded domains and provided application to generalized Hammerstein integral equations. Arunchai and Plubtieng [10] improved the Krasnoselskii theorem on fixed points for the sum of operators $\Gamma_1 + \Gamma_2$, where Γ_1 is the weakly-strongly continuous mapping and Γ_2 is the asymptotically nonexpansive mapping. Banaś and Amar [2] proved some new types of fixed point theorems for the sum of $\Gamma_1 + \Gamma_2$ on an unbounded closed convex subset of a Hausdorff topological vector space, which are used to provide the solution of integral equations in the Lebesgue space. Wang [11] derived fixed point results for the sum of two operators Γ_1 and Γ_2 if Γ_1 is contractive with respect to the measure of weak noncompactness and Γ_2 is the ϕ-nonlinear contraction in the setting of the Banach space. By having this result, he obtained the existence of solutions to a nonlinear Hammerstein integral equation in the L^1 space. Ben Amar et al. [12] improved Krasnoselskii-type fixed point results for the equation $(\Gamma_1 + \Gamma_2)x = x$, where the operator Γ_1 is (ws)-compact and Γ_2 is (ws)-compact and asymptotically Φ-nonexpansive operator on an unbounded closed convex subset of a Banach space.

Later, the people were interested in finding the fixed point in partially ordered metric spaces which are more general than metric spaces. First, Ran and Reurings [13] initiated the fixed point theorems for the contraction mappings in the partially ordered metric space, which is further improved by Nieto and Rodríguez-López [14] and by Petruşel and Rus [15], and used to give the existence of the solution for boundary value problems of nonlinear first-order ordinary differential equations.

Recently, the researchers show interest in finding the optimum solution for real-time modeling problems. In this direction, instead of studying about the fixed point (whenever the fixed point does not exist), the researchers are working on approximating the fixed point in some sense, known as the best proximity point. The existence of best proximity point theorems helps to obtain the optimum solution for different types of modelings. In the literature, there are more number of articles about the existence of the best proximity point for single operators. For example, Basha [16] derived best proximity point theorems for proximal contractions of the first and second kind in the setting of the metric space. Basha [17] proved the existence of the best proximity point for principal cyclic contractive mappings, proximal cyclic contractive mappings, and proximal contractive mappings which are defined on the metric space. In 2000, Eldred and Veeramani [18] proved the best proximity point result for contractive type mappings in the metric space. Al-Thagafi and Shahzad [19] obtained convergence and existence results of the best proximity points for cyclic ϕ-contraction maps in the metric space. For more existence of the best proximity point results, we refer the reader to [20–25].

In the light of the above literature survey, we want to find the approximate solution for the fixed point equation of the form $(\Gamma_1 + \Gamma_2)x = x$, where Γ_1, Γ_2 are non-self-mappings. So, in this work, we initiate to study the best proximity point for the sum of two non-self-operators, and we provide the existence of the best proximity point for the sum of two operators using best proximity point theorems for the single operator. Additionally, we prove an existence result of the fixed point for the sum of cyclic and noncyclic operators, which involves the concept of accretive operators. Finally, we discuss some notions of the ordered normed linear space, and we find sufficient conditions for the existence of the best proximity point in this space.

2. Preliminaries

First, we collect some notions from [26]. Throughout the paper, we denote $\mathscr{B}, \mathscr{X}$ as the Banach space and metric space, respectively, and let

$$\mathscr{B}^* = \{f : \mathscr{B} \longrightarrow \mathbb{R} \mid f \text{ is continuous linear transformation}\} \tag{3}$$

be its dual. For each $\eta \in \mathscr{B}$, we associate the set

$$J(\eta) = \left\{f \in \mathscr{B}^* : \langle \eta, f \rangle = \|\eta\|^2 = \|f\|^2\right\}, \tag{4}$$

where $\langle \eta, f \rangle$ denotes $f(\eta)$. The multivalued operator $J : \mathscr{B} \longrightarrow \mathscr{B}^*$ is called the duality mapping of $\mathscr{B}$. Suppose Γ is an operator from $\mathscr{B}$ to $\mathscr{B}$. Then, the operator $-\Gamma : \mathscr{B} \longrightarrow \mathscr{B}$ is defined by $-\Gamma(\eta) = -\Gamma\eta$.

Here, we give the definition for weak accretive via the accretive operator in [26].

Definition 1. An operator Γ: $\mathscr{B} \longrightarrow \mathscr{B}$ is said to be weak accretive if for $(\eta_1, \Gamma\eta_1), (\eta_2, \Gamma\eta_2) \in \mathscr{B} \times \mathscr{B}$, there exists $\varphi \in J(\eta_1 - \eta_2)$ such that $\langle \Gamma\eta_1 - \Gamma\eta_2, \varphi \rangle \geq 0$, where J is as in (4).

The following lemma is helpful for our one of the results.

Lemma 1. *If Γ is weak accretive on $\mathscr{B}$, then $\|\eta_1 - \eta_2\| \leq \|\eta_1 - \eta_2 + \lambda(\Gamma\eta_1 - \Gamma\eta_2)\|$, for all $\lambda \geq 0$, and $\eta_1, \eta_2 \in \mathscr{B}$.*

Proof. Let $\eta_1, \eta_2 \in \mathscr{B}$ with $\eta_1 \neq \eta_2$. Since Γ is weak accretive, for $(\eta_1, \Gamma\eta_1), (\eta_2, \Gamma\eta_2) \in \mathscr{B} \times \mathscr{B}$, there exists $\varphi \in J(\eta_1 - \eta_2)$ such that $\langle \Gamma\eta_1 - \Gamma\eta_2, \varphi \rangle \geq 0$. Now, we have

$$\begin{aligned}
\|\eta_1 - \eta_2\|^2 &= \langle \eta_1 - \eta_2, \varphi \rangle \\
&\leq \langle \eta_1 - \eta_2, \varphi \rangle + \lambda \langle \Gamma\eta_1 - \Gamma\eta_2, \varphi \rangle \\
&= \langle \eta_1 - \eta_2 + \lambda(\Gamma\eta_1 - \Gamma\eta_2), \varphi \rangle \\
&\leq \|\eta_1 - \eta_2 + \lambda(\Gamma\eta_1 - \Gamma\eta_2)\| \|\varphi\| \\
&= \|\eta_1 - \eta_2 + \lambda(\Gamma\eta_1 - \Gamma\eta_2)\| \|\eta_1 - \eta_2\|.
\end{aligned} \tag{5}$$

□

Theorem 1 (see [27]). *Let $\mathcal{M}, \mathcal{N}$ be nonempty subsets of $\mathcal{X}$ which are complete. Suppose Γ: $\mathcal{M} \cup \mathcal{N} \longrightarrow \mathcal{M} \cup \mathcal{N}$ satisfies*

(1) $\Gamma(\mathcal{M}) \subseteq \mathcal{N}$ and $\Gamma(\mathcal{N}) \subseteq \mathcal{M}$.

(2) $d(\Gamma\eta, \Gamma\xi) \leq k\, d(\eta, \xi)$ for $\eta \in \mathcal{M}, \xi \in \mathcal{N}$,

where $0 < k < 1$. Then, there exists $\tau \in \mathcal{M} \cap \mathcal{N}$ such that $\Gamma(\tau) = \tau$.

3. Main Results

Theorem 2. *Let $\mathcal{M}, \mathcal{N}$ be subsets of $\mathcal{B}$ such that $\mathcal{M} \cup \mathcal{N}$ is the subspace. Let Γ_1, Γ_2: $\mathcal{M} \cup \mathcal{N} \longrightarrow \mathcal{M} \cup \mathcal{N}$, where Γ_1 and Γ_2 are cyclic and noncyclic mappings which satisfy the following:*

(1) $-\Gamma_2$ is the weak accretive operator, and Γ_1 is onto

(2) $(I - \Gamma_2)$ is invertible on $\Gamma_1(\mathcal{M} \cup \mathcal{N})$, where I is the identity operator

(3) $\eta = \Gamma_2\eta + \Gamma_1\xi, \xi \in \mathcal{M} \Longrightarrow \eta \in \mathcal{N}$ and $\eta = \Gamma_2\eta + \Gamma_1\xi, \xi \in \mathcal{N} \Longrightarrow \eta \in \mathcal{M}$

(4) $\|\Gamma_1\eta - \Gamma_1\xi\| \leq k\|\eta - \xi\|, k \in (0, 1)$, for $\eta \in \mathcal{M}$ and $\xi \in \mathcal{N}$

Then, there exists $\tau \in \mathcal{M} \cup \mathcal{N}$ such that $(\Gamma_1 + \Gamma_2)\tau = \tau$.

Proof. First, we note that assumption (3) is valid only if $\mathcal{M} \cup \mathcal{N}$ is the subspace. Since Γ_1 is onto, $\Gamma_1(\mathcal{M} \cup \mathcal{N}) = \mathcal{M} \cup \mathcal{N}$. Since $(I - \Gamma_2)$ is invertible on $\Gamma_1(\mathcal{M} \cup \mathcal{N})$, we can define

$$(I - \Gamma_2)^{-1}\Gamma_1 : \mathcal{M} \cup \mathcal{N} \longrightarrow \mathcal{M} \cup \mathcal{N}. \tag{6}$$

Now, we show that $(I - \Gamma_2)^{-1}\Gamma_1$ is the cyclic mapping. Let $\xi \in \mathcal{M}$; then, $\Gamma_1\xi \in \mathcal{N}$. Since $(I - \Gamma_2)$ is invertible on $\Gamma_1(\mathcal{M} \cup \mathcal{N})$, there exists unique η such that $(I - \Gamma_2)^{-1}\Gamma_1\xi = \eta$. Then, the equation $\eta = \Gamma_2\eta + \Gamma_1\xi$ has a unique solution η. Then, by (3), we obtain $(I - \Gamma_2)^{-1}\Gamma_1\xi \in \mathcal{N}$. Then, $(I - \Gamma_2)^{-1}\Gamma_1(\mathcal{M}) \subseteq \mathcal{N}$. Similarly, we can prove $(I - \Gamma_2)^{-1}\Gamma_1(\mathcal{N}) \subseteq \mathcal{M}$. Next, we prove that $(I - \Gamma_2)^{-1}$ is nonexpansive on $\Gamma_1(\mathcal{M} \cup \mathcal{N})$. Since $(I - \Gamma_2)$ is invertible on $\Gamma_1(\mathcal{M} \cup \mathcal{N})$, it implies that $R(I - \Gamma_2) = \Gamma_1(\mathcal{M} \cup \mathcal{N})$, where $R(I - \Gamma_2)$ is the range of $I - \Gamma_2$. Let $\eta, \xi \in \Gamma_1(\mathcal{M} \cup \mathcal{N})$; then, there exist $z_1, z_2 \in \Gamma_1(\mathcal{M} \cup \mathcal{N})$ such that $(I - \Gamma_2)^{-1}\eta = z_1$, $(I - \Gamma_2)^{-1}\xi = z_2$. Using Lemma 1, we get

$$\begin{aligned}
\|\eta - \xi\| &= \|z_1 - \Gamma_2 z_1 - (z_2 - \Gamma_2 z_2)\| \\
&= \|z_1 + (-\Gamma_2 z_1) - (z_2 + (-\Gamma_2 z_2))\| \\
&= \|z_1 - z_2 + ((-\Gamma_2 z_1) - (-\Gamma_2 z_2))\| \\
&\geq \|z_1 - z_2\| \\
&= \|(I - \Gamma_2)^{-1}\eta - (I - \Gamma_2)^{-1}\xi\|.
\end{aligned} \tag{7}$$

Therefore, for $\eta \in \mathcal{M}, \xi \in \mathcal{N}$, we obtain

$$\begin{aligned}
\|(I - \Gamma_2)^{-1}\Gamma_1\eta - (I - \Gamma_2)^{-1}\Gamma_1\xi\| &\leq \|\Gamma_1\eta - \Gamma_1\xi\| \\
&\leq k\|\eta - \xi\|.
\end{aligned} \tag{8}$$

Then, the operator $(I - \Gamma_2)^{-1}\Gamma_1$ agrees with the hypothesis of Theorem 1, and then there is $\tau \in \mathcal{M} \cap \mathcal{N}$ such that $(I - \Gamma_2)^{-1}\Gamma_1\tau = \tau$, which implies $(\Gamma_2 + \Gamma_1)\tau = \tau$. □

4. Best Proximity Point Theorems

The following notions are used in this article: let $\mathcal{M}, \mathcal{N}$ be nonempty subsets of $\mathcal{X}$.

$$\begin{aligned}
d(\mathcal{M}, \mathcal{N}) &= \inf\{d(\eta, \xi) : \eta \in \mathcal{M}, \xi \in \mathcal{N}\}, \\
\mathcal{M}_0 &= \left\{\eta \in \mathcal{M} : d(\eta, \xi\prime) = d(\mathcal{M}, \mathcal{N}) \text{ for some } \xi\prime \in \mathcal{N}\right\}, \\
\mathcal{N}_0 &= \left\{\xi \in \mathcal{N} : d(\eta\prime, \xi) = d(\mathcal{M}, \mathcal{N}) \text{ for some } \eta\prime \in \mathcal{M}\right\}.
\end{aligned} \tag{9}$$

The pair $(\mathcal{M}, \mathcal{N})$ is said to have P-property if for $\eta_1, \eta_2 \in \mathcal{M}$ and $\xi_1, \xi_2 \in \mathcal{N}$,

$$\begin{cases} d(\eta_1, \xi_1) = d(\mathcal{M}, \mathcal{N}), \\ d(\eta_2, \xi_2) = d(\mathcal{M}, \mathcal{N}), \end{cases} \Longrightarrow d(\eta_1, \eta_2) = d(\xi_1, \xi_2). \tag{10}$$

Definition 2. A function Γ: $\mathcal{X} \longrightarrow \mathcal{X}$ is called Lipschitzian if there exists a real constant $\kappa \geq 0$ such that, for all $\eta, \xi \in \mathcal{X}$,

$$d(\Gamma(\eta), \Gamma(\xi)) \leq \kappa\, d(\eta, \xi). \tag{11}$$

Definition 3 (see [20]). Let $\mathcal{M}, \mathcal{N}$ be nonempty subsets of $\mathcal{X}$. A map Γ: $\mathcal{M} \longrightarrow \mathcal{N}$ is said to be a weakly contractive mapping if

$$d(\Gamma\eta, \Gamma\xi) \leq d(\eta, \xi) - \chi(d(\eta, \xi)), \tag{12}$$

for all $\eta, \xi \in \mathcal{M}$, where χ: $[0, \infty) \longrightarrow [0, \infty)$ is a nondecreasing and continuous function such that χ is positive on $(0, \infty)$, $\chi(0) = 0$ and $\lim_{t \longrightarrow \infty}\chi(t) = \infty$.

Theorem 3 (see [20]). *Let $\mathcal{X}$ be complete and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be nonempty, closed sets such that $\mathcal{M}_0$ is nonempty. Let Γ: $\mathcal{M} \longrightarrow \mathcal{N}$ be a weakly contractive mapping such that $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$. Assume the pair $(\mathcal{M}, \mathcal{N})$ has the P-property. Then, there exists unique η^* in $\mathcal{M}$ such that $d(\eta^*, \Gamma\eta^*) = d(\mathcal{M}, \mathcal{N})$.*

The following theorem tells that the sum of two non-self-operators has the best proximity point in Banach space settings. The notion $(1/2)\mathcal{N} = \{\eta : \eta = (b/2), b \in \mathcal{N}\}$, where $\mathcal{N}$ is the subset of $\mathcal{B}$, is used in the following theorem.

Theorem 4. *Let $\mathcal{M}, \mathcal{N}$ be two nonempty convex, closed subsets of $\mathcal{B}$. Assume $\mathcal{M}_0$ is nonempty. Let Γ_1, Γ_2: $\mathcal{M} \longrightarrow \mathcal{N}$ be two mappings which satisfy*

(1) If $\eta \in \mathcal{M}$, then $\Gamma_1\eta + \Gamma_2\eta \in \mathcal{N}$

(2) Γ_1 is $(k/2)$-Lipschitzian, where $k \in (0, 1]$

(3) $\|\Gamma_2\eta - \Gamma_2\xi\| \leq (k/2)\|\eta - \xi\| - \chi(\|\eta - \xi\|)$, for all $\eta, \xi \in \mathcal{M}$, where χ is as in Definition 3 and $k \in (0, 1]$

(4) $\Gamma_1(\mathcal{M}_0) \subseteq (1/2)\mathcal{N}_0$ and $\Gamma_2(\mathcal{M}_0) \subseteq (1/2)\mathcal{N}_0$

If the pair $(\mathcal{M}, \mathcal{N})$ has the P-property, then there exists unique η^* in $\mathcal{M}$ such that $\|\eta^* - (\Gamma_1 + \Gamma_2)\eta^*\| = d(\mathcal{M}, \mathcal{N})$.

Proof. Because of (1), we define $\Gamma_1 + \Gamma_2 \colon \mathscr{M} \longrightarrow \mathscr{N}$ by $(\Gamma_1 + \Gamma_2)\eta = \Gamma_1\eta + \Gamma_2\eta$. First, we show that $\Gamma_1 + \Gamma_2$ satisfies $(\Gamma_1 + \Gamma_2)(\mathscr{M}_0) \subseteq \mathscr{N}_0$. For $\eta \in \mathscr{M}_0$, we have $\Gamma_1(\eta) \in (1/2)\mathscr{N}_0$ and $\Gamma_2(\eta) \in (1/2)\mathscr{N}_0$. Therefore, there exist $\xi_1, \xi_2 \in \mathscr{N}_0$ such that $\Gamma_1\eta = (\xi_1/2), \Gamma_2\eta = (\xi_2/2)$. Then, $(\Gamma_1 + \Gamma_2)(\eta) = \Gamma_1\eta + \Gamma_2\eta = (\xi_1/2) + (\xi_2/2)$. Since $\mathscr{N}_0$ is convex, $(\Gamma_1 + \Gamma_2)(\eta) \in \mathscr{N}_0$. This shows $(\Gamma_1 + \Gamma_2)(\mathscr{M}_0) \subseteq \mathscr{N}_0$. Now we prove the mapping $\Gamma_1 + \Gamma_2$ is weakly contractive. For $\eta, \xi \in \mathscr{M}_0$, we have

$$\begin{aligned}\|(\Gamma_1 + \Gamma_2)(\eta) - (\Gamma_1 + \Gamma_2)(\xi)\| &= \|\Gamma_1\eta - \Gamma_1\xi + \Gamma_2\eta - \Gamma_2\xi\| \\ &\le \|\Gamma_1\eta - \Gamma_1\xi\| + \|\Gamma_2\eta - \Gamma_2\xi\| \\ &\le \frac{k\|\eta - \xi\|}{2} + \frac{k\|\eta - \xi\|}{2} - \chi(\|\eta - \xi\|) \\ &= k\|\eta - \xi\| - \chi(\|\eta - \xi\|) \\ &\le \|\eta - \xi\| - \chi(\|\eta - \xi\|).\end{aligned} \tag{13}$$

This gives $\Gamma_1 + \Gamma_2$ is a weakly contractive mapping from $\mathscr{M}_0$ to $\mathscr{N}_0$. Clearly, the pair $(\mathscr{M}_0, \mathscr{N}_0)$ has the P-property, and $d(\mathscr{M}_0, \mathscr{N}_0) = d(\mathscr{M}, \mathscr{N})$. Then, the map $\Gamma_1 + \Gamma_2$ satisfies the requirements of Theorem 3, and there exists $\eta^* \in \mathscr{M}_0$ such that $\|\eta^* - (\Gamma_1 + \Gamma_2)(\eta^*)\| = d(\mathscr{M}, \mathscr{N})$. □

5. Best Proximity Point Theorems in the Ordered Normed Linear Space

In this section, first we extract some notions from [28] to obtain best proximity point results. Let $\mathscr{V}$ be the real vector space. The pair $(\mathscr{V}, \prec)$ is called the partially ordered linear space, where $\prec$ is the partial order. Two elements $\eta, \xi \in \mathscr{V}$ are called comparable if either $\eta \prec \xi$ or $\xi \prec \eta$ holds. A nonempty subset $\mathscr{C}$ of $\mathscr{V}$ is said to be a chain or totally ordered if any two elements of $\mathscr{C}$ are comparable. The space $(\mathscr{V}, \prec, \|\cdot\|)$ is called the partially ordered normed linear space, where $\|\cdot\|$ is the norm on $\mathscr{V}$.

Definition 4 (see [28]). A mapping $\Gamma \colon \mathscr{V} \longrightarrow \mathscr{V}$ is called monotone nondecreasing if $\eta \prec \xi$ implies $\Gamma\eta \prec \Gamma\xi$ for all $\eta, \xi \in \mathscr{V}$. Similarly, Γ is called monotone nonincreasing if $\eta \prec \xi$ implies $\Gamma\xi \prec \Gamma\eta$ for all $\eta, \xi \in \mathscr{V}$.

Definition 5 (see [28]). A subset $\mathscr{C}$ of $(\mathscr{V}, \prec, \|\cdot\|)$ is called partially bounded if every chain in $\mathscr{C}$ is bounded.

We denote by $P_{bd,ch}(\mathscr{V}), P_{rcp,ch}(\mathscr{V})$ the family of all bounded chains and relatively compact chains of $\mathscr{V}$, respectively.

Definition 6 (see [28]). A mapping $\mu^p \colon P_{bd,ch}(\mathscr{V}) \longrightarrow \mathbb{R}^+ = [0, \infty)$ is called a partial measure of noncompactness in $\mathscr{V}$ if it satisfies

(1) $\varnothing \ne (\mu^p)^{-1}(0) \subset P_{rcp,ch}(\mathscr{V})$.

(2) $\mu^p(\overline{D}) = \mu^p(D)$.

(3) μ^p is nondecreasing, i.e., if $D_1 \subset D_2 \Longrightarrow \mu^p(D_1) \le \mu^p(D_2)$.

(4) If sequence $\{D_n\}$ of closed chains in $P_{bd,ch}(\mathscr{V})$ with $D_{n+1} \subset D_n (n = 1, 2, \ldots)$ and if $\lim_{n\longrightarrow\infty}\mu^p(D_n) = 0$, then the set $\overline{D}_\infty = \cap_{n=1}^{\infty} D_n$ is nonempty.

The notion in (1) is known as the kernel of μ^p, that is,

$$\ker\mu^p = \left\{D \in P_{bd,ch}(\mathscr{V}) \colon \mu^p(D) = 0\right\}. \tag{14}$$

Clearly, $\ker\mu^p \subset P_{rcp,ch}(\mathscr{V})$. And so, $D_\infty \in \ker\mu^p$. Because of $\mu^p(D_\infty) \le \mu^p(D_n)$ for all n, we obtain $\mu^p(D_\infty) = 0$. Then, $D_\infty \in \ker\mu^p$. The measure μ^p is called sublinear if it satisfies

(5) $\mu^p(D_1 + D_2) \le \mu^p(D_1) + \mu^p(D_2)$ for all $D_1, D_2 \in P_{bd,ch}(\mathscr{V})$.

(6) $\mu^p(\lambda D) = |\lambda|\mu^p(D)$ for $\lambda \in \mathbb{R}$.

And μ^p has the maximum property if

(7) $\mu^p(D_1 + D_2) = \max\{\mu^p(D_1), \mu^p(D_2)\}$.

Finally, μ^p is said to be full if

(8) $\ker\mu^p = P_{rcp,ch}(\mathscr{V})$.

Definition 7 (see [28]). A mapping $\chi \colon [0, \infty) \longrightarrow [0, \infty)$ is called a $\mathscr{D}$-function if it is monotone nondecreasing and upper semicontinuous with $\chi(0) = 0$.

Definition 8 (see [28]). A nondecreasing mapping $\Gamma \colon \mathscr{V} \longrightarrow \mathscr{V}$ is said to be partially nonlinear $\mathscr{D}$-set-Lipschitz if there is a $\mathscr{D}$-function χ such that

$$\mu^p(\Gamma\mathscr{C}) \le \chi(\mu^p(\mathscr{C})), \tag{15}$$

for any bounded chain $\mathscr{C}$ in $\mathscr{V}$. If $\chi(s) < s$ for $s > 0$, then Γ is called a partially nonlinear $\mathscr{D}$-set-contraction.

Lemma 2 (see [28]). *If χ is a $\mathscr{D}$-function with $\chi(s) < s$ for $s > 0$, then $\lim_{n\longrightarrow\infty}\chi^n(s) = 0$ for all $s \in [0, \infty)$ and vice-versa.*

Definition 9 (see [27]). Let $\mathscr{M}, \mathscr{N}$ be nonempty subsets of $\mathscr{X}$. A map $\Gamma \colon \mathscr{M} \cup \mathscr{N} \longrightarrow \mathscr{M} \cup \mathscr{N}$ is a partially cyclic contraction map if it satisfies

(1) $\Gamma(\mathscr{M}) \subseteq \mathscr{N}$ and $\Gamma(\mathscr{N}) \subseteq \mathscr{M}$

(2) $d(\Gamma\eta, \Gamma\xi) \le k\,d(\eta, \xi) + (1 - k)d(\mathscr{M}, \mathscr{N})$, for some $k \in (0, 1)$ and for $\eta \in \mathscr{M}, \xi \in \mathscr{N}$, with η and ξ are comparable

From (2), one can easily verify that $d(\Gamma\eta, \Gamma\xi) \le d(\eta, \xi)$ holds.

Lemma 3 (see [18]). *Let $\mathscr{M}, \mathscr{N}$ be nonempty subsets of $\mathscr{X}$. Suppose $\Gamma \colon \mathscr{M} \cup \mathscr{N} \longrightarrow \mathscr{M} \cup \mathscr{N}$ is a cyclic contraction map. Then, for any η_0 in $\mathscr{M} \cup \mathscr{N}$, we have $d(\eta_n, \Gamma\eta_n) \longrightarrow d(\mathscr{M}, \mathscr{N})$, where $\eta_{n+1} = \Gamma\eta_n$, $n = 0, 1, 2, 3, \ldots$.*

First, we show the result for a single operator.

Theorem 5. *Let $\mathscr{M}, \mathscr{N}$ be two nonempty closed and partially bounded subsets of $(\mathscr{V}, \prec, \|\cdot\|)$. Let Γ: $\mathscr{M} \cup \mathscr{N} \longrightarrow \mathscr{M} \cup \mathscr{N}$ be nondecreasing, partially cyclic contraction, and partially nonlinear $\mathscr{D}$-set contraction in $\mathscr{M} \cup \mathscr{N}$. If there exists an element $\eta_0 \in \mathscr{M}$ such that $\eta_0 \underline{\prec} \Gamma\eta_0$, then Γ has a best proximity point.*

Proof. Since Γ is cyclic, for $\eta_0 \in \mathscr{M}$, then $\Gamma\eta_0 \in \mathscr{N}$. Define $\eta_1 = \Gamma\eta_0$. Now, again $\Gamma\eta_1 \in \mathscr{M}$, and define $\eta_2 = \Gamma\eta_1$. In the same way, we construct $\eta_{n+1} = \Gamma\eta_n$ such that $\{\eta_{2n}\} \subseteq \mathscr{M}$ and $\{\eta_{2n+1}\} \subseteq \mathscr{N}$, for $n = 0, 1, 2, \ldots$. Since $\eta_0 \underline{\prec} \Gamma\eta_0$ and Γ is nondecreasing, it implies that $\eta_1 \underline{\prec} \eta_2$. Again using Γ is nondecreasing, we obtain $\eta_0 \underline{\prec} \eta_1 \underline{\prec} \eta_2 \underline{\prec} \cdots$. Denote

$$D_{2n} = \{\eta_{2n}, \eta_{2n+2}, \eta_{2n+4}, \ldots\}, \tag{16}$$

for $n = 0, 1, 2, \ldots$. Here, each D_{2n} is a bounded chain in $\mathscr{M}$, and $D_{2n} = \Gamma^2(D_{2n-2})$, $n = 1, 2, \ldots$. From the construction of $\{D_{2n}\}$, we obtain

$$D_0 \supseteq D_2 \supseteq D_4 \supseteq \cdots. \tag{17}$$

Consequently,

$$\overline{D}_0 \supseteq \overline{D}_2 \supseteq \overline{D}_4 \supseteq \cdots. \tag{18}$$

Now,

$$\begin{aligned} \mu^p(D_{2n}) &= \mu^p\left(\Gamma^2(D_{2n-2})\right) \\ &\leq \chi\left(\mu^p(\Gamma D_{2n-2})\right) \\ &\leq \chi^2\left(\mu^p(D_{2n-2})\right) \\ &\leq \chi^4\left(\mu^p(D_{2n-4})\right) \\ &\vdots \\ &\leq \chi^{2n}\left(\mu^p(D_0)\right). \end{aligned} \tag{19}$$

Letting limit superior as $n \longrightarrow \infty$ and using Lemma 2, we obtain

$$\begin{aligned} \mu^p(\overline{D}_{2n}) &= \lim_{n\longrightarrow\infty} \mu^p(D_{2n}) \leq \lim_{n\longrightarrow\infty} \sup \chi^{2n}\left(\mu^p(D_0)\right) \\ &= \lim_{n\longrightarrow\infty} \chi^{2n}\left(\mu^p(D_0)\right) = 0. \end{aligned} \tag{20}$$

Therefore, by condition (4) of μ^p, we get

$$\overline{D}_\infty = \cap_{n=1}^{\infty} D_{2n} \neq \varnothing, \quad D_\infty \in P_{rcp,ch}. \tag{21}$$

Since $\lim_{n\longrightarrow\infty} \mu^p(D_{2n}) = 0$, then for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu^p(D_{2n}) < \varepsilon$, for $n \geq n_0$. This implies that $\overline{D}_{2n_0}$ and $\overline{D}_0$ are compact chains in $\mathscr{M}$. Hence, $\{\eta_{2n}\}$ has a convergent subsequence, that is, $\eta_{2n_k} \longrightarrow \eta \in \mathscr{M}$. Now,

$$d(\mathscr{M}, \mathscr{N}) \leq \left\|\eta - \eta_{2n_k-1}\right\| \leq \left\|\eta - \eta_{2n_k}\right\| + \left\|\eta_{2n_k} - \eta_{2n_k-1}\right\|. \tag{22}$$

As $k \longrightarrow \infty$ and by Lemma 3, we obtain

$$d(\mathscr{M}, \mathscr{N}) \leq \lim_{k\longrightarrow\infty} \left\|\eta - \eta_{2n_k-1}\right\| \leq d(\mathscr{M}, \mathscr{N}). \tag{23}$$

Therefore, $\lim_{k\longrightarrow\infty} \|\eta - \eta_{2n_k-1}\| = d(\mathscr{M}, \mathscr{N})$. Since Γ is a partially cyclic contraction, we have

$$d(\mathscr{M}, \mathscr{N}) \leq \left\|\eta_{2n_k} - \Gamma\eta\right\| \leq \left\|\eta_{2n_k-1} - \eta\right\|. \tag{24}$$

As $k \longrightarrow \infty$, we get

$$d(\mathscr{M}, \mathscr{N}) \leq \lim_{k\longrightarrow\infty} \left\|\eta_{2n_k} - \Gamma\eta\right\| \leq d(\mathscr{M}, \mathscr{N}), \tag{25}$$

which implies that $\|\eta - \Gamma\eta\| = d(\mathscr{M}, \mathscr{N})$.

Using the above theorem, here, we provide the result on the best proximity point for the sum of two operators. □

Theorem 6. *Let $\mathscr{M}, \mathscr{N}$ be a nonempty, closed, and partially bounded subset of $(\mathscr{V}, \prec, \|\cdot\|)$. Let Γ_1, Γ_2: $\mathscr{M} \cup \mathscr{N} \longrightarrow \mathscr{M} \cup \mathscr{N}$ be nondecreasing cyclic mappings which satisfy*

(1) $\eta \in \mathscr{M} \Longrightarrow \Gamma_1\eta + \Gamma_2\eta \in \mathscr{N}$ and $\eta \in \mathscr{N} \Longrightarrow \Gamma_1\eta + \Gamma_2\eta \in \mathscr{M}$.

(2) Γ_1 is partially completely continuous, and

$$\left\|\Gamma_1\eta - \Gamma_1\xi\right\| \leq \frac{k}{2}\|\eta - \xi\| + \frac{1}{2}d(\mathscr{M}, \mathscr{N}), \tag{26}$$

where $k \in (0, (1/2))$ and $\eta \in \mathscr{M}, \xi \in \mathscr{N}$ with η and ξ are comparable.

(3) Γ_2 is a nonlinear $\mathscr{D}$-set contraction on $\mathscr{M}$, and

$$\left\|\Gamma_2\eta - \Gamma_2\xi\right\| \leq \frac{k}{2}\|\eta - \xi\|, \tag{27}$$

where $k \in (0, (1/2))$ and $\eta \in \mathscr{M}, \xi \in \mathscr{N}$ with η and ξ are comparable.

(4) There exists an element $\eta_0 \in \mathscr{M}$ such that $\eta_0 \underline{\prec} \Gamma_1\eta_0 + \Gamma_2\eta_0$.

Then, there exists $\eta^* \in \mathscr{M}$ such that $\|\eta^* - (\Gamma_1 + \Gamma_2)\eta^*\| = d(\mathscr{M}, \mathscr{N})$.

Proof. Define $\Gamma_1 + \Gamma_2$: $\mathscr{M} \cup \mathscr{N} \longrightarrow \mathscr{M} \cup \mathscr{N}$ by $(\Gamma_1 + \Gamma_2)(\eta) = \Gamma_1(\eta) + \Gamma_2(\eta)$. Clearly, because of (1), $\Gamma_1 + \Gamma_2$ is a cyclic mapping. And one can easily prove $\Gamma_1 + \Gamma_2$ is nondecreasing on $\mathscr{M} \cup \mathscr{N}$. Since $k < (1/2)$ implies $(1/2) < 1 - k$, now, we show that $\Gamma_1 + \Gamma_2$ is a partial contraction. For $\eta \in \mathscr{M}$ and $\xi \in \mathscr{N}$ with η and ξ are comparable, we have

$$\begin{aligned} \left\|(\Gamma_1 + \Gamma_2)(\eta) - (\Gamma_1 + \Gamma_2)(\xi)\right\| &= \left\|\Gamma_1\eta + \Gamma_2\eta - \Gamma_1\xi - \Gamma_2\xi\right\| \\ &\leq \left\|\Gamma_1\eta - \Gamma_1\xi\right\| + \left\|\Gamma_2\eta - \Gamma_2\xi\right\| \\ &\leq \frac{k\|\eta - \xi\|}{2} + \frac{d(\mathscr{M}, \mathscr{N})}{2} + \frac{k\|\eta - \xi\|}{2} \\ &\leq k\|\eta - \xi\| + (1 - k)d(\mathscr{M}, \mathscr{N}), \end{aligned} \tag{28}$$

which show that $\Gamma_1 + \Gamma_2$ is a partial contraction. Next, we prove that $\Gamma_1 + \Gamma_2$ is a partially nonlinear $\mathscr{D}$-set contraction.

Let $\mathscr{C}$ be a bounded chain in $\mathscr{M}$. Therefore, $(\Gamma_1+\Gamma_2)(\mathscr{C})\subseteq\Gamma_1(\mathscr{C})+\Gamma_2(\mathscr{C})$. By conditions sublinearity and full of μ^p, we get

$$\begin{aligned}\mu^p\left((\Gamma_1+\Gamma_2)(\mathscr{C})\right)&\leq\mu^p\left(\Gamma_1(\mathscr{C})+\Gamma_2(\mathscr{C})\right)\\&\leq\mu^p\left(\Gamma_1(\mathscr{C})\right)+\mu^p\left(\Gamma_2(\mathscr{C})\right)\\&\leq\chi\left(\mu^p(\mathscr{C})\right).\end{aligned}\tag{29}$$

Therefore, the mapping $\Gamma_1+\Gamma_2$ follows Theorem 5, and there exists $\eta^*\in\mathscr{M}$ such that $\|\eta^*-(\Gamma_1+\Gamma_2)(\eta^*)\|=d(\mathscr{M},\mathscr{N})$. □

6. Conclusions

In the nonlinear functional analysis, many mathematical problems can be solved by the existence result of fixed points. The fixed point theorems provide sufficient conditions to ensure the fixed point equation $\Gamma x=x$, where Γ is the self-mapping, has a solution. In case of nonlinear problems, it is written as $\Gamma_1x+\Gamma_2x=x$, where Γ_1,Γ_2 are self-mappings, and then the fixed point theorems for the sum of two mappings help to obtain the solution for such an equation. Suppose the mappings Γ_1,Γ_2 are non-self-cases, then the fixed point equation $\Gamma_1x+\Gamma_2x=x$ does not possess a solution. In the literature, there are many research papers which deal with the existence of the best proximity point for the equation of the form $\Gamma x=x$, where Γ is the non-self-mapping. However, there is no single research work which gives the existence result of the best proximity point for the sum of mappings. So, we want to obtain an approximate solution via finding the best proximity point for such an equation in some sense. So, in this research article, we study the existence of the best proximity point for the sum of two non-self-mappings using best proximity point theorems for a single operator. Moreover, using the notion accretive operators, we prove an existence result of the fixed point for the sum of cyclic and noncyclic operators. Also, we study some notions of the ordered normed linear space, and we provide sufficient conditions for the existence of the best proximity point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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