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Research Article

n-Coherence Relative to a Hereditary Torsion Theory

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Let R be a ring, $\tau = (\mathcal{T}, \mathcal{F})$ a hereditary torsion theory of mod-R, and n a positive integer. Then, R is called right τ-n-coherent if every n-presented right R-module is $(\tau, n+1)$-presented. We present some characterizations of right τ-n-coherent rings, as corollaries, and some characterizations of right n-coherent rings and right τ-coherent rings are obtained.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules considered are unitary. For any R-module M, $M^+ = \text{Hom}(M, (\mathbb{Q}/\mathbb{Z}))$ will be the character module of M.

Recall that a *torsion theory* [1] $\tau = (\mathcal{T}, \mathcal{F})$ for the category of all right R-modules consists of two subclasses $\mathcal{T}$ and $\mathcal{F}$ such that

(1) $\text{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$

(2) If $\text{Hom}(T, F) = 0$ for all $F \in \mathcal{F}$, then $T \in \mathcal{T}$

(3) If $\text{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, then $F \in \mathcal{F}$

In this case, $\mathcal{T}$ is called a torsion class and its objects are called τ-torsion, $\mathcal{F}$ is called a torsion-free class, and its objects are called τ-torsion free. From, Proposition 2.1, Chap VI, in [9], a class $\mathcal{T}$ of right R-modules is a torsion class for some torsion theory if and only if $\mathcal{T}$ is closed under quotient modules, direct sums, and extensions. From Proposition 2.2, Chap VI in [9], a class $\mathcal{F}$ of right R-modules is a torsion-free class for some torsion theory if and only if $\mathcal{F}$ is closed under submodules, direct products, and extensions. A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is called hereditary if $\mathcal{T}$ is closed under submodules.

We recall also that a right R-module M is called *FP-injective* [2] or *absolutely pure* [3] if $\text{Ext}_R^1(A, M) = 0$ for every finitely presented right R-module A; a left R-module M is flat if and only if $\text{Tor}_1^R(A, M) = 0$ for every finitely presented right R-module A; a ring R is *right coherent* [4] if every finitely generated right ideal of R is finitely presented, or equivalently, if every finitely generated submodule of a projective right R-module is finitely presented. FP-injective modules, flat modules, coherent rings, and their generalizations have been studied extensively by many authors. For example, in 1994, Costa introduced the concept of *right n-coherent* rings in [5]. Following [5], a ring R is called right *n-coherent* in case every n-presented right R-module is $(n+1)$-presented, where a right R-module A is called n-presented in case there exists an exact sequence of right R-modules $F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$, in which every F_i is finitely generated free. It is easy to see that a ring R is right coherent if and only if R is right 1-coherent. In 1996, Chen and Ding introduced the concepts of *n-FP-injective* modules and *n-flat* modules in [6], using the two concepts and characterized right n-coherent rings. Following [6], a right R-module M is called *n-FP*-injective in case $\text{Ext}_R^n(A, M) = 0$ for every n-presented right R-module A; a left R-module M is called n-flat in case $\text{Tor}_n^R(A, M) = 0$ for every n-presented right R-module A.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a (hereditary) torsion theory for the category of all right R-modules. Then, according to [7], a right R-module M is called right *τ-finitely generated* (or *τ-FG* for short) if there exists a finitely generated submodule N such that $(M/N) \in \mathcal{T}$; a right R-module A is called *τ-finitely presented* (or *τ-FP* for short) if there exists an exact sequence of right R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with F finitely generated free and K τ-finitely generated; R is called *τ-coherent* if every finitely generated right ideal of R is τ-FP. In 1993, Nieves introduced the concept of *τ-n-presented* (or *τ-n-FP* for short) modules in [8]. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory for the category of all right R-modules; then, according to [8], a right

R-module A is called τ-n-presented in case there exists an exact sequence of right R-modules $0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$, where each F_i is finitely generated free and K_{n-1} is τ-finitely generated. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory. Then, by Theorem 3.3 in [4], it is easy to see that R is right τ-coherent if and only if every finitely presented right R-module is τ-2-FP.

In this article, we wish to extend the concepts of right τ-coherent rings and right n-coherent rings to right τ-n-coherent rings and give some characterizations of these rings (see Theorems 1 and 2), as corollaries; some characterizations of right n-coherent rings and right τ-coherent rings will be obtained (see Corollaries 2 and 4).

2. Characterizations of τ-$(n+1)$-Presented Modules and Right τ-n-Coherent Rings

We recall that a nonempty subclass $\mathcal{T}$ of right R-modules is called a weak torsion class [9] if $\mathcal{T}$ is closed under homomorphic images and extensions. Following [9], if a class $\mathcal{T}$ of right R-modules is a weak torsion class, then a right R-module M is called $\mathcal{T}$-finitely generated (or $\mathcal{T}$-FG for short) if there exists a finitely generated submodule N such that $(M/N) \in \mathcal{T}$; a right R-module A is called $\mathcal{T}$-finitely presented (or $\mathcal{T}$-FP for short) if there exists an exact sequence of right R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with F finitely generated free and K $\mathcal{T}$-finitely generated; a right R-module A is called $(\mathcal{T}, n)$-presented if there exists an exact sequence of right R-modules:

$$0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \tag{1}$$

such that $F_0, \cdots, F_{n-1}$ are finitely generated free and K_{n-1} is $\mathcal{T}$-finitely generated, where n is a positive integer. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory of mod-R. Then, it is easy to see that $\mathcal{T}$ is a weak torsion class. In this case, a right R-module A is τ-finitely generated if and only if it is $\mathcal{T}$-finitely generated, and a right R-module A is τ-n-presented if and only if it is $(\mathcal{T}, n)$-presented for any positive integer n. Let A be a right R-module, τ a torsion theory of mod-R, and n a positive integer. Then, form Proposition 3.4 in [11], the following conditions are equivalent:

(1) A is τ-n-presented.

(2) A is $(n-1)$-presented, and if there exists an exact sequence of right R-modules:

$$0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0, \tag{2}$$

such that $F_0, \cdots, F_{n-1}$ are finitely generated free; then, K_{n-1} is τ-finitely generated.

(3) There exists an exact sequence of right R-modules:

$$0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0, \tag{3}$$

such that F is finitely generated free and K is τ-$(n-1)$-presented.

If $n \geq 2$, then the above conditions are also equivalent to

(4) A is $(n-2)$-presented, and if there exists an exact sequence of right R-modules

$$0 \longrightarrow K_{n-2} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0, \tag{4}$$

such that $F_0, \cdots, F_{n-2}$ are finitely generated free; then, K_{n-2} is τ-finitely presented.

Now, we give the characterizations of τ-$(n+1)$-presented modules.

Theorem 1. *Let τ be a torsion theory of mod-R, n a nonnegative integer, and A an n-presented right R-module. Then, the following statements are equivalent for A:*

(1) A is τ-$(n+1)$-presented

(2) The canonical map $\varinjlim Ext_R^n(A, X_i) \longrightarrow Ext_R^n(A, \varinjlim X_i)$ is an isomorphism for each direct system $\{X_i\}_{i \in I}$ of τ-torsion-free modules

(3) $Tor_n^R(A, X^+) \cong Ext_R^n(A, X)^+$ for each τ-torsion-free module X

(4) $Tor_n^R(A, E^+) = 0$ for each τ-torsion-free injective module E

Proof

(1)⇒(2) Use induction on n. If $n = 0$, then the result holds by Proposition 2.5(3) in [4]. Assume that the result holds when $n = k$. Then, when $n = k+1$, suppose A is a τ-$(k+2)$-presented module. Let $0 \longrightarrow N \longrightarrow F \longrightarrow A \longrightarrow 0$ be an exact sequence of right R-modules, where F is finitely generated free and N is τ-$(k+1)$-presented. Then, we have a commutative diagram:

$$\begin{array}{ccccccc}
\varinjlim \mathrm{Ext}_R^k(F, X_i) & \longrightarrow & \varinjlim \mathrm{Ext}_R^k(N, X_i) & \longrightarrow & \varinjlim \mathrm{Ext}_R^{k+1}(A, X_i) & \longrightarrow & 0 \\
\downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \\
\mathrm{Ext}_R^k(F, \varinjlim X_i) & \longrightarrow & \mathrm{Ext}_R^k(N, \varinjlim X_i) & \longrightarrow & \mathrm{Ext}_R^{k+1}(A, \varinjlim X_i) & \longrightarrow & 0
\end{array} \tag{5}$$

with exact rows. Since ϕ_1 is an isomorphism and hence epic and ϕ_2 is an isomorphism by hypothesis, we have that ϕ_3 is also an isomorphism by the Five Lemma.

(2)⇒(1) Use induction on n. If $n = 0$, then the result holds by Proposition 2.5 (3) in [4]. Assume that the result holds when $n = k$. Then, when $n = k+1$, suppose A is a $(k+1)$-presented right R-module. Let $0 \longrightarrow N \longrightarrow F \longrightarrow A \longrightarrow 0$ be an exact sequence of right R-modules, where F is finitely generated free and N is k-presented. Then, for any direct system $\{X_i\}_{i \in I}$ of τ-torsion-free modules. If $k > 0$, then we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim \mathrm{Ext}_R^k(N, X_i) \arrow[r] \arrow[d, "\phi_2"] & \varinjlim \mathrm{Ext}_R^{k+1}(A, X_i) \arrow[r] \arrow[d, "\phi_3"] & 0 \\ 0 \arrow[r] & \mathrm{Ext}_R^k(N, \varinjlim X_i) \arrow[r] & \mathrm{Ext}_R^{k+1}(A, \varinjlim X_i) \arrow[r] & 0 \end{tikzcd} \quad (6)$$

with exact rows. Since ϕ_3 is an isomorphism by condition, we have that ϕ_2 is also an isomorphism. So, by hypothesis, N is τ-$(k+1)$-presented, and hence A is τ-$(k+2)$-presented. If $k = 0$, then we have a commutative diagram:

$$\begin{tikzcd} \varinjlim \mathrm{Hom}(A, X_i) \arrow[r] \arrow[d, "\phi_0"] & \varinjlim \mathrm{Hom}(F, X_i) \arrow[r] \arrow[d, "\phi_1"] & \varinjlim \mathrm{Hom}(N, X_i) \arrow[r] \arrow[d, "\phi_2"] & \varinjlim \mathrm{Ext}_R^1(A, X_i) \arrow[r] \arrow[d, "\phi_3"] & 0 \\ \mathrm{Hom}(A, \varinjlim X_i) \arrow[r] & \mathrm{Hom}(F, \varinjlim X_i) \arrow[r] & \mathrm{Hom}(N, \varinjlim X_i) \arrow[r] & \mathrm{Ext}_R^1(A, \varinjlim X_i) \arrow[r] & 0 \end{tikzcd} \quad (7)$$

with exact rows. From 25.4 (d) in [10], ϕ_0 is an isomorphism and hence epic, and ϕ_3 is an isomorphism by condition. Note that ϕ_1 is an isomorphism, so, by the Five Lemma, we have that ϕ_2 is also an isomorphism. So, N is τ-FP by Proposition 2.5 in [4], and it shows that A is τ-2-FP.

(1)$\Rightarrow$(3) In case, $n = 0$, then the result holds by Lemma 3.1 in [4]. In case, $n = 1$, then there is an exact sequence of right R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$, where F is finitely generated free and K is τ-FP. And then we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1^R(A, X^+) \arrow[r] \arrow[d, "f"] & K \otimes X^+ \arrow[r] \arrow[d, "g"] & F \otimes X^+ \arrow[d, "h"] \\ 0 \arrow[r] & \mathrm{Ext}_R^1(A, X)^+ \arrow[r] & \mathrm{Hom}(K, X)^+ \arrow[r] & \mathrm{Hom}(F, X)^+ \end{tikzcd} \quad (8)$$

with exact rows. By Lemma 3.1 [4], g and h are isomorphisms. So, by the Five Lemma, f is also an isomorphism. In case, $n > 1$, then we have an exact sequence of right R-modules $0 \longrightarrow K_{n-2} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$, where each F_i is finitely generated free and K_{n-2} is τ-2-FP, and hence we have $\mathrm{Tor}_n^R(A, X^+) \cong \mathrm{Tor}_1^R(K_{n-2}, X^+) \cong \mathrm{Ext}_R^1(K_{n-2}, X)^+ \cong \mathrm{Ext}_R^n(A, X)^+$, as required.

(3)$\Rightarrow$(4) It is obvious.

(4)$\Rightarrow$(1) Since A is n-FP, there exists an exact sequence of right R-modules $0 \longrightarrow K_{n-2} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$, where each F_i is finitely generated free and K_{n-2} is finitely presented. Thus, $\mathrm{Tor}_1^R(K_{n-2}, E^+) \cong \mathrm{Tor}_n^R(A, E^+) = 0$ for any τ-torsion-free injective module E by (4). It follows from Proposition 2 in [6] that K_{n-2} is τ-2-FP, and therefore A is τ-$(n+1)$-presented. □

Let M be a right R-module and n be a positive integer. If $\tau = (0, \mathrm{mod}-R)$, then it is easy to see that M is $(\tau, n+1)$-presented if and only if it is $(n+1)$-presented. If $\tau = (\mathrm{mod}-R, 0)$, then it is easy to see that M is $(\tau, n+1)$-presented if and only if it is n-presented.

Corollary 1. *Let n be a nonnegative integer and A an n-presented right R-module. Then, the following statements are equivalent:*

(1) A is $(n+1)$-presented

(2) The canonical map $\varinjlim \mathrm{Ext}_R^n(A, X_i) \longrightarrow \mathrm{Ext}_R^n(A, \varinjlim X_i)$ is an isomorphism for each direct system $\{X_i\}_{i\in I}$ of right R-modules

(3) $\mathrm{Tor}_n^R(A, X^+) \cong \mathrm{Ext}_R^n(A, X)^+$ for each right R-module X

(4) $\mathrm{Tor}_n^R(A, E^+) = 0$ for each injective right R-module E

Definition 1. Let τ be a torsion theory of mod-R. Then, the ring R is called right τ-n-coherent, if every n-presented right R-module is $(\tau, n+1)$-presented.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory of mod-R. Then, it is easy to see that R is right τ-n-coherent if and only if every n-presented right R-module is $(\mathcal{T}, n+1)$-presented.

Example 1

(1) Let $\tau = (0, \mathrm{mod}-R)$. Then, R is right τ-n-coherent if and only if R is right n-coherent.

(2) R is right τ-coherent if and only if R is right τ-1-coherent.

(3) Let $\tau = (\mathrm{mod}-R, 0)$. Then, R is right τ-n-coherent.

Proof. (1) and (3) are obvious. (2) follows from Theorem 3.3 (2) in [4]. □

Definition 2. Let τ be a torsion theory of mod-R and n a positive integer. Then, a right R-module M is said to be τ-n-FP-injective, if $\mathrm{Ext}_R^n(A, M) = 0$ for each τ-$(n+1)$-presented module M; a right R-module M is said to be τ-FP-injective if it is τ-1-FP-injective.

Clearly, each n-FP-injective module is τ-n-FP-injective. If $\tau = (\mathrm{mod}-R, 0)$, then it is easy to see that a right R-module M is τ-n-FP-injective if and only if it is n-FP-injective. Now, we give our characterization of right τ-n-coherent rings.

Theorem 2. *Let τ be a hereditary torsion theory of mod-R and n a positive integer. Then, the following statements are equivalent for the ring R:*

(1) *R is right τ-n-coherent*

(2) $\varinjlim \mathrm{Ext}_R^n(A, X_i) \cong \mathrm{Ext}_R^n(A, \varinjlim X_i)$ *for any n-presented right R-module A and direct system $\{X_i\}_{i\in I}$ of τ-torsion-free modules*

(3) $\mathrm{Tor}_n^R(A, X^+) \cong \mathrm{Ext}_R^n(A, X^+)$ *for any n-presented right R-module A and each τ-torsion-free module X*

(4) $\mathrm{Tor}_n^R(A, E^+) = 0$ *for any n-presented right R-module A and each τ-torsion-free injective module E*

(5) *If X is a τ-n-FP-injective module, then X is n-FP-injective*

(6) *Any direct limit of τ-torsion-free n-FP-injective modules is n-FP-injective*

(7) *Any direct limit of τ-torsion-free FP-injective modules is n-FP-injective*

(8) *Any direct limit of τ-torsion-free injective modules is n-FP-injective*

(9) *A τ-torsion-free module X is n-FP-injective if and only if X^+ is n-flat*

(10) *If Y is a pure submodule of a τ-torsion-free n-FP-injective module X, then X/Y is n-FP-injective.*

Proof

(1)$\Leftrightarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4) follows from Theorem 1.

(1)$\Rightarrow$(5), (6)$\Rightarrow$(7)$\Rightarrow$(8), and (3)$\Rightarrow$(9) are obvious.

(5)$\Rightarrow$(6) Let $X = \varinjlim X_i$, where each $\{X_i\}_{i\in I}$ is a τ-torsion-free n-FP-injective module. Then, for any τ-$(n+1)$-FP module A, by Theorem 1, we have that $\mathrm{Ext}_R^n(A, X) = \mathrm{Ext}_R^n(A, \varinjlim X_i) \cong \varinjlim \mathrm{Ext}_R^n(A, X_i) = 0$, so X is τ-n-FP-injective and thus it is n-FP-injective by (5).

(8)$\Rightarrow$(1). Let A be an n-presented right R-module with a finite n-presentation $F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} A \longrightarrow 0$. Write $K_{n-1} = \mathrm{Ker}(d_{n-1})$ and $K_{n-2} = \mathrm{Ker}(d_{n-2})$. Then, K_{n-1} is finitely generated, and we get an exact sequence of right R-modules $0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow K_{n-2} \longrightarrow 0$. Let $\{E_i\}_{i\in I}$ be any direct system of τ-torsion-free injective right R-modules (with I directed). Then, $\varinjlim E_i$ is n-FP-injective by (8), so $\mathrm{Ext}_R^n(A, \varinjlim E_i) = 0$ and hence $\mathrm{Ext}_R^1(K_{n-2}, \varinjlim E_i) = 0$. Thus, we have a commutative diagram:

$$\begin{tikzcd}
\varinjlim \mathrm{Hom}(K_{n-2}, E_i) \arrow[r] \arrow[d, "f"] & \varinjlim \mathrm{Hom}(F_{n-1}, E_i) \arrow[r] \arrow[d, "g"] & \varinjlim \mathrm{Hom}(K_{n-1}, E_i) \arrow[r] \arrow[d, "h"] & 0 \\
\mathrm{Hom}(K_{n-2}, \varinjlim E_i) \arrow[r] & \mathrm{Hom}(F_{n-1}, \varinjlim E_i) \arrow[r] & \mathrm{Hom}(K_{n-1}, \varinjlim E_i) \arrow[r] & 0
\end{tikzcd} \tag{9}$$

with exact rows. Since f and g are isomorphisms by 25.4(d) in [10], h is an isomorphism by the Five Lemma. Now, let $\{X_i\}_{i\in I}$ be any direct system of τ-torsion-free modules (with I directed). Then, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \mathrm{Hom}(K_{n-1}, X_i) \arrow[r] \arrow[d, "\phi_1"] & \varinjlim \mathrm{Hom}(K_{n-1}, E(X_i)) \arrow[r] \arrow[d, "\phi_2"] & \varinjlim \mathrm{Hom}(K_{n-1}, E(X_i)/X_i) \arrow[d, "\phi_3"] \\
0 \arrow[r] & \mathrm{Hom}(K_{n-1}, \varinjlim X_i) \arrow[r] & \mathrm{Hom}(K_{n-1}, \varinjlim E(X_i)) \arrow[r] & \mathrm{Hom}(K_{n-1}, \varinjlim E(X_i)/X_i)
\end{tikzcd} \tag{10}$$

where $E(X_i)$ is the injective hull of X_i. Since K_{n-1} is finitely generated, by 24.9 in [10], the maps ϕ_1, ϕ_2, and ϕ_3 are monic. Since τ is a hereditary torsion theory and X_i is τ-torsion-free, by Proposition 3.2, Chap VI in [9], $E(X_i)$ is τ-torsion-free. And so, by the above proof, ϕ_2 is an isomorphism. Hence, ϕ_1 is also an isomorphism by the Five Lemma again, and then K_{n-1} is τ-finitely presented by Proposition 2.5 (3) [4], and thus A is τ-$(n+1)$-presented. Therefore, R is right τ-n-coherent.

(9)$\Rightarrow$(10) Since Y is a pure submodule, the pure exact sequence $0 \longrightarrow Y \longrightarrow X \longrightarrow (X/Y) \longrightarrow 0$ induces a split exact sequence $0 \longrightarrow (X/Y)^+ \longrightarrow X^+ \longrightarrow Y^+ \longrightarrow 0$. Since X is τ-torsion-free and n-FP-injective, by (9), X^+ is n-flat, so $(X/Y)^+$ is also n-flat, and thus (X/Y) is n-FP-injective by Corollary 2.8 in [2].

(10)$\Rightarrow$(6) Let $\{X_i\}_{i\in I}$ be a direct system of τ-torsion-free n-FP-injective modules. Then by Proposition 1 in [7], we have a map-pure, and hence pure exact sequence $0 \longrightarrow K \longrightarrow \oplus_{i\in I} X_i \longrightarrow \varinjlim X_i \longrightarrow 0$. Observing that $\oplus_{i\in I} X_i$ is τ-torsion-free and n-FP-injective, by (10), we have that $\varinjlim X_i$ is n-FP-injective. $\square$

We call a right R-module X *weakly n-FP-injective* if $\mathrm{Ext}_R^n(A, X) = 0$ for any $(n+1)$-presented right R-module A. Let $\tau = (0, \mathrm{mod} - R)$. Then, we have the following results.

Corollary 2. *Let n be a positive integer. Then, the following statements are equivalent for a ring R:*

(1) R is right n-coherent

(2) $\varinjlim Ext_R^n(A, X_i) \cong Ext_R^n(A, \varinjlim X_i)$ for any n-presented right R-module A and direct system $\{X_i\}_{i\in I}$ of right R-modules

(3) $Tor_n^R(A, X^+) \cong Ext_R^n(A, X)^+$ for any n-presented right R-module A and each right R-module X

(4) $Tor_n^R(A, E^+) = 0$ for any n-presented right R-module A and each injective right R-module E

(5) If X is a weakly n-FP-injective module, then X is n-FP-injective

(6) Any direct limit of n-FP-injective modules is n-FP-injective

(7) Any direct limit of FP-injective modules is n-FP-injective

(8) Any direct limit of injective modules is n-FP-injective

(9) A right R-module X is n-FP-injective if and only if X^+ is n-flat

(10) If Y is a pure submodule of an n-FP-injective right R-module X, then X/Y is n-FP-injective

We note that the equivalences of (1), (2), (6), and (9) in Corollary 2 appeared in Theorem 3.1 in [2].

Corollary 3. *Let τ be a hereditary torsion theory of mod-R and n a positive integer. If R is right τ-n-coherent, then a τ-torsion-free module X is n-FP-injective if and only if X^{++} is n-FP-injective.*

Proof. $\Rightarrow$ Let X be a τ-torsion-free n-FP-injective module. Since R is right τ-n-coherent, by Theorem 2 (9), X^+ is n-flat, and so X^{++} is n-FP-injective by Proposition 2.3 in [2].

$\Leftarrow$ Since X^{++} is n-FP-injective and X is a pure submodule of X^{++}, so, by Proposition 2.6 in [2], X is n-FP-injective. □

Let $n = 1$; then, by Theorem 2, we can obtained several results on right τ-coherent rings.

Corollary 4. *Let τ be a hereditary torsion theory of mod-R. Then, the following statements are equivalent for the ring R:*

(1) R is right τ-coherent

(2) $\varinjlim Ext_R^1(A, X_i) \cong Ext_R^1(A, \varinjlim X_i)$ for any finitely presented right R-module A and direct system $\{X_i\}_{i\in I}$ of τ-torsion-free modules

(3) $Tor_1^R(A, X^+) \cong Ext_R^n(A, X)^+$ for any finitely presented right R-module A and each τ-torsion-free module X

(4) $Tor_1^R(A, E^+) = 0$ for any finitely presented right R-module A and each τ-torsion-free injective module E

(5) If X is a τ-1-FP-injective module, then X is FP-injective

(6) Any direct limit of τ-torsion-free FP-injective modules is FP-injective

(7) Any direct limit of τ-torsion-free injective modules is FP-injective.

(8) A τ-torsion-free module X is FP-injective if and only if X^+ is flat

(9) If Y is a pure submodule of a τ-torsion-free FP-injective module X, then X/Y is FP-injective

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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