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Research Article

On $M_{\lambda_{m,n}}$-Statistical Convergence

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In this paper, we introduce a new type of statistical convergence method for double sequences by using the $(M, \lambda_{m,n})$-method of summability which was defined by Natarajan. We also obtain some inclusion relations between statistical convergence and $M_{\lambda m,n}$-statistical convergence for double sequences.

1. Introduction

The subject of statistical convergence has been studied by many researchers since the emergence of the idea of statistical convergence in 1935. Statistical convergence was introduced by Fast [2] and Steinhaus [3] independently in the same year 1951 as a generalization of ordinary convergence and was later reintroduced by Schoenberg [4]. Quite a few researchers have generalized or extended this concept and applied different fields of mathematics such as Erdös and Tenenbaum [5], Miller [6], Zygmund [7], Freedmanet al. [8], Connor [9], Salat [10], Duman and Orhan [11], Et et al. [12], Çakallı [13, 14], Çakallı and Savaş [15], Edely et al. [16], Mursaleen et al. [17, 18], Natarajan [19], Tok and Başarır [20], Aral and Küçükaslan [21–23], and Taylan [24].

Let $x = (x_{m,n})$ be a double sequence. Then, $x = (x_{m,n})_{m,n=0}^{\infty}$ is said to be convergent to L in the Pringsheim sense if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$, whenever $m, n > N$. In this case, we write $P - \lim x = L$ [25].

Also, a double sequence $x = (x_{m,n})_{m,n=0}^{\infty}$ is said to be bounded if there exists a positive number M such that $|x_{m,n}| < M$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$. By ℓ_{∞}^{2}, we denote the set of all bounded double sequences.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K(m, n) = \{(m, n)\colon k \le m, j \le n\}$. Then, the double natural density of K is given by

$$\delta(K) \coloneqq P - \lim_{m,n \longrightarrow \infty} \frac{1}{mn} \sum_{k,j=1}^{m,n} \chi_{K(m,n)}(k, j) \tag{1}$$

if the limit exists. A double sequence $x = (x_{m,n})$ is said to be statistically convergent to L provided that, for every $\varepsilon > 0$, the set

$$\left\{(m, n)\colon k \le m \text{ and } j \le n\colon \left|x_{m,n} - L\right| \ge \varepsilon\right\} \tag{2}$$

has double natural density zero [26]. In this case, we write $st_2-\lim x = L$. By st_2, we denote the set of all statistically convergent double sequences. Later, a lot of works have been done on the statistical convergence of double sequences (see [27–33]).

The following definition which is required for our study is given by Natarajan.

Definition 1 (see [1]). Let $\lambda = \{\lambda_{m,n}\}$ be a double sequence such that $\sum_{m,n=0}^{\infty,\infty}|\lambda_{m,n}| < \infty$. Then, the $(M, \lambda_{m,n})$-method is defined by the 4-dimensional infinite matrix $(a_{m,n,k,j})$, where

$$a_{m,n,k,j} \coloneqq \begin{cases} \lambda_{m-k,n-j}, & \text{if } k \le m,\ j \le n, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

It is well known from Theorem 3.4 of [1] that the $(M, \lambda_{m,n})$-method is regular if and only if $\sum_{m,n=0}^{\infty,\infty}\lambda_{m,n} = 1$.

The purpose of this paper is to give a new statistical convergence definition using the above definition for double sequences and some relations between statistical convergence and $M_{\lambda_{m,n}}$-statistical convergence. In addition, we have used it to prove a Korovkin-type approximation theorem as an application of our method.

We acknowledge that the definition of $M_{\lambda m,n}$-statistical convergence for double sequences was presented at the International Conference on Multidisciplinary, Engineering, Science, Education and Technology in 2017 [34].

Let $x = (x_{m,n})$ be a given real-valued sequence. A sequence $(t_{m,n})$ of $M_{\lambda_{m,n}}$-mean of $(x_{m,n})$ is defined by

$$t_{m,n} = \sum_{k,j=0}^{m,n} \lambda_{m-k,n-j} x_{k,j}, \tag{4}$$

for all $m, n \in \mathbb{N}$.

Definition 2. The sequence $x = (x_{m,n})$ is said to be $M_{\lambda_{m,n}}$-summable to $L \in \mathbb{R}$ if $t_{m,n} \longrightarrow L$ and is denoted by

$$x_{m,n} \longrightarrow L\left(M_{\lambda_{m,n}}\right). \tag{5}$$

Definition 3. A double sequence $x = (x_{m,n})$ is said to be strongly $M_{\lambda_{m,n}}$-summable to $L \in \mathbb{R}$ if $(|x_{m,n} - L|)$ is $M_{\lambda_{m,n}}$-summable to zero. In this case, we write $x_{m,n} \longrightarrow L([M_{\lambda_{m,n}}])$. The set of all strongly $M_{\lambda_{m,n}}$-summable sequences is denoted by $[M_{\lambda_{m,n}}]$ as

$$\left[M_{\lambda_{m,n}}\right] = \left\{ x = \left(x_{k,j}\right): \lim_{m,n\longrightarrow\infty,\infty} \sum_{k,j=0}^{m,n} \lambda_{m-k,n-j} \left|x_{m,n} - L\right| = 0 \right\}. \tag{6}$$

By considering the matrix $M_{\lambda_{m,n}}$ in (3) for any $\lambda = (\lambda_{m,n})$, natural density and statistical convergence can be defined as follows.

Definition 4 ($M_{\lambda_{m,n}}$-density). Let K be a subset of $\mathbb{N} \times \mathbb{N}$. Then, $M_{\lambda_{m,n}}$-density of K is denoted by $\delta_{\lambda_{m,n}}(K)$ and defined by

$$\lim_{m,n\longrightarrow\infty} \sum_{k,j=0}^{m,n} \lambda_{m-k,n-j} \chi_K(k,j) \tag{7}$$

if the limit exists.

Definition 5 ($M_{\lambda_{m,n}}$-statistical convergence). A double sequence $x = (x_{m,n})$ is said to be $M_{\lambda_{m,n}}$-statistically convergent to L if for every $\varepsilon > 0$, $M_{\lambda_{m,n}}$-density of the set $K(\varepsilon) \coloneqq \left\{(m,n): |x_{m,n} - L| \geq \varepsilon\right\}$ is zero, i.e.,

$$\lim_{m,n\longrightarrow\infty} \sum_{k,j=0}^{m,n} \lambda_{m-k,n-j} \chi_{K(\varepsilon)}(k,j) = 0. \tag{8}$$

It is denoted by $x_{m,n} \longrightarrow L(M_{\lambda_{m,n}} - \text{st})$. The set of all $M_{\lambda_{m,n}}$-statistically convergent sequences is denoted by $M^{\text{st}}_{\lambda_{m,n}}$, i.e.,

$$M^{\text{st}}_{\lambda_{m,n}} \coloneqq \left\{x = \left(x_{m,n}\right): \exists L \in \mathbb{R} \text{ such that } x_{m,n} \longrightarrow L\left(M_{\lambda_{m,n}} - \text{st}\right)\right\}. \tag{9}$$

Let $p = (p_n)$ and $q = (q_n)$ be sequences of positive natural numbers and $P_m \coloneqq \sum_{k=0}^{m} p_k \longrightarrow \infty$ and $Q_n \coloneqq \sum_{k=0}^{n} q_k \longrightarrow \infty$. Take $\lambda = (\lambda_{m,n})$, where $(\lambda_{m,n}) = (\lambda_{m,n,k,j})$ with

$$\lambda_{m,n,k,j} \coloneqq \begin{cases} \dfrac{p_j q_k}{P_m Q_n}, & \text{if } 0 \leq j \leq m, 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

If we consider $\lambda = (\lambda_{m,n})$ as in (10), then it is clear that $M_{\lambda_{m,n}}$-statistical convergence coincides weighted statistical convergence (or $S_{\overline{N_2}}$) which was defined and studied by Çınar and Et in [29].

2. Main Results and Proofs

In this section, we will give some properties of $M_{\lambda_{m,n}}$-statistical convergence and comparison with strong $M_{\lambda_{m,n}}$-summability. Moreover, inclusion results for $M^{\text{st}}_{\lambda_{m,n}}$ are given.

Theorem 1. *Let $x = (x_{m,n})$ and $y = (y_{m,n})$ be two double sequences. Then,*

(i) If $x_{m,n} \longrightarrow L_1(M_{\lambda_{m,n}} - st)$ and $y_{m,n} \longrightarrow L_2(M_{\lambda_{m,n}} - st)$, then $(x_{m,n} + y_{m,n}) \longrightarrow (L_1 + L_2)(M_{\lambda_{m,n}} - st)$

(ii) If $x_{m,n} \longrightarrow L_1(M_{\lambda_{m,n}} - st)$ and $c \in \mathbb{C}$, then $cx_{m,n} \longrightarrow cL_1(M_{\lambda_{m,n}} - st)$

Proof. Omitted.

We define each of the following sets:

$$\begin{aligned} \Lambda &\coloneqq \left\{\lambda = \left(\lambda_{m,n}\right): \sum_{m,n=0}^{\infty,\infty} \left|\lambda_{m,n}\right| < \infty\right\}, \\ \Lambda_0 &\coloneqq \left\{\lambda \in \Lambda: \sum_{m,n=0}^{\infty,\infty} \lambda_{m,n} = 1\right\}. \end{aligned} \tag{11}$$

Theorem 2. *Let $\lambda = (\lambda_{m,n}) \in \Lambda_0$. If $x_{m,n} \longrightarrow L_1(M_{\lambda_{m,n}} - st)$ and $x_{m,n} \longrightarrow L_2(M_{\lambda_{m,n}} - st)$, then $L_1 = L_2$.*

Proof. Assume that $x_{m,n} \longrightarrow L_1(M_{\lambda_{m,n}} - \text{st})$, $x_{m,n} \longrightarrow L_2(M_{\lambda_{m,n}} - st)$, and $L_1 \neq L_2$. Take any $\varepsilon < (1/2)|L_1 - L_2|$, and denote the sets

$$\begin{aligned} A_\varepsilon(L_1) &= \left\{(m,n): \left|x_{m,n} - L_1\right| \geq \varepsilon\right\}, \\ A_\varepsilon(L_2) &= \left\{(m,n): \left|x_{m,n} - L_2\right| \geq \varepsilon\right\}. \end{aligned} \tag{12}$$

Since $x_{m,n} \longrightarrow L_1(M_{\lambda_{m,n}} - \text{st})$ and $x_{m,n} \longrightarrow L_2(M_{\lambda_{m,n}} - \text{st})$, we can write $\delta_{\lambda_{m,n}}(A_\varepsilon(L_1)) = 0$ and $\delta_{\lambda_{m,n}}(A_\varepsilon(L_2)) = 0$. It is clear that inclusion

$$\left\{(m,n): \left|x_{m,n} - L_2\right| < \varepsilon\right\} \subset A_\varepsilon(L_1) \tag{13}$$

holds. Therefore, the inequality

$$\delta_{\lambda_{m,n}}\left(\left\{(m,n): \left|x_{m,n} - L_2\right| < \varepsilon\right\}\right) \leq \delta_{\lambda_{m,n}}\left(A_\varepsilon(L_1)\right) = 0 \tag{14}$$

holds. Also, the sets $\left\{(m,n): |x_{m,n} - L_2| < \varepsilon\right\}$ and $A_\varepsilon(L_2)$ are disjoint and

$$\mathbb{N} \times \mathbb{N} = \left\{(m,n): \left|x_{m,n} - L_2\right| < \varepsilon\right\} \cup A_\varepsilon(L_2). \tag{15}$$

From (15), we obtain

$$\delta_{\lambda_{m,n}}(\mathbb{N}\times\mathbb{N}) = \delta_{\lambda_{m,n}}\left(\left\{(m,n)\colon \left|x_{m,n}-L_2\right|<\varepsilon\right\}\right) + \delta_{\lambda_{m,n}}\left(A_\varepsilon(L_2)\right). \tag{16}$$

The last equality gives that $\delta_{\lambda_{m,n}}(\mathbb{N}\times\mathbb{N}) = 0$, but this is a contradiction to $\delta_{\lambda_{m,n}}(\mathbb{N}\times\mathbb{N}) = 1$. So, $M_{\lambda_{m,n}}$-statistical limit of $x=(x_{m,n})$ is unique.

Theorem 3. *Let $\lambda=(\lambda_{m,n})$ be a sequence from Λ_0. Then, the following statements hold true:*

(i) If $x_{m,n}\longrightarrow L([M_{\lambda_{m,n}}])$, then $x_{m,n}\longrightarrow L(M_{\lambda_{m,n}}-st)$ and $(M_{\lambda_{m,n}})\subsetneq M^{st}_{\lambda_{m,n}}$

(ii) If $x=(x_{m,n})\in\ell^2_\infty$ and $x_{m,n}\longrightarrow L(M_{\lambda_{m,n}}-st)$, then $x_{m,n}\longrightarrow L(M_{\lambda_{m,n}})$

Proof

(i) Let $\varepsilon>0$ and $x_{m,n}\longrightarrow L(M_{\lambda_{m,n}})$. In this case, we obtain

$$\begin{aligned}\sum_{k,j=0}^{m,n}\lambda_{m-k,n-j}\left|x_{k,j}-L\right| &= \sum_{k,j=0,(k,j)\in K(\varepsilon)}^{m,n}\lambda_{m-k,n-j}\left|x_{k,j}-L\right| \\ &\quad+\sum_{k,j=0,(k,j)\notin K(\varepsilon)}^{m,n}\lambda_{m-k,n-j}\left|x_{k,j}-L\right| \\ &\geq \sum_{k,j=0,(k,j)\in K(\varepsilon)}^{m,n}\lambda_{m-k,n-j}\left|x_{k,j}-L\right| \\ &\geq \varepsilon\sum_{k,j=0}^{m,n}\lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j).\end{aligned} \tag{17}$$

(ii) After taking the limit when $m,n\longrightarrow\infty$ in the above inequality, we obtain $x_{m,n}\longrightarrow L(M_{\lambda_{m,n}}-\mathrm{st})$.

(iii) Assume that $x_{m,n}\longrightarrow L(M_{\lambda_{m,n}}-\mathrm{st})$ and $x\in\ell^2_\infty$. Then, there exists a positive constant $H>0$ such that

$$\left|x_{m,n}-L\right|\leq H, \tag{18}$$

for all $m,n\in\mathbb{N}$.

For a given $\varepsilon>0$, the following inequality holds:

$$\begin{aligned}\sum_{k,j=0}^{m,n}\lambda_{m-k,n-j}\left|x_{kj}-L\right| &= \sum_{k,j=0,(k,j)\in K(\varepsilon)}^{m,n}\lambda_{m-k,n-j}\left|x_{k,j}-L\right| \\ &\quad+\sum_{k,j=0,(k,j)\notin K(\varepsilon)}^{m,n}\lambda_{m-k,n-j}\left|x_{k,j}-L\right| \\ &\leq H\sum_{k,j=0}^{m,n}\lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \\ &\quad+\varepsilon\sum_{k,j=0,(k,j)\notin K(\varepsilon)}^{m,n}\lambda_{m-k,n-j}.\end{aligned} \tag{19}$$

If we take the limit as $m,n\longrightarrow\infty$, then we obtain that $x_{k,j}\longrightarrow L([M_{\lambda_{m,n}}])$.

Remark 1. The converse of (i) in Theorem 3 does not hold.

Let us consider $\lambda=(\lambda_{m,n})=(1/(mn+1)^2)$ and $x=(x_{m,n})$ as follows:

$$x_{m,n}=\begin{cases} s^3r^3, & m=r^2, n=s^2,\\ 0, & \text{otherwise.}\end{cases} \tag{20}$$

Therefore, we have

$$\begin{aligned}&\lim_{m,n\longrightarrow\infty}\sum_{k,j=0}^{m,n}\frac{1}{((m-k)(n-j)+1)^2}\cdot\chi_{K(\varepsilon)}(k,j) \\ &=\lim_{m,n\longrightarrow\infty}\sum_{r,s=0}^{[|\sqrt{m}|],[|\sqrt{n}|]}\frac{1}{((m-r)(n-s)+1)^2}\leq 0,\end{aligned} \tag{21}$$

but the following inequality holds:

$$\begin{aligned}&\lim_{m,n\longrightarrow\infty}\sum_{k,j=0}^{m,n}\frac{1}{((m-k)(n-j)+1)^2}\cdot x_{kj} \\ &=\lim_{m,n\longrightarrow\infty}\sum_{r,s=0}^{[|\sqrt{m}|],[|\sqrt{n}|]}\frac{1}{((m-r)(n-s)+1)^2}s^3r^3 \\ &\geq\lim_{m,n\longrightarrow\infty}\frac{1}{(mn+1)^2}\cdot\sum_{r,s=0}^{[|\sqrt{m}|],[|\sqrt{n}|]}s^3r^3=\infty.\end{aligned} \tag{22}$$

This gives that $M_{\lambda_{m,n}}\subset M^{\mathrm{st}}_{\lambda m,n}$ is strict.

Corollary 1. *If $x_{m,n}\longrightarrow L(m,n\longrightarrow\infty)$, then $x_{m,n}\longrightarrow L(M_{\lambda_{m,n}}-st)$ for any $\lambda\in\Lambda_0$.*

Proof. Since $x_{m,n}\longrightarrow L(m,n\longrightarrow\infty)$ and $M_{\lambda_{m,n}}$ is regular for any $\lambda\in\Lambda_0$, then $x_{m,n}\longrightarrow L([M_{\lambda_{m,n}}])$. Therefore, Theorem 3 (i) gives that $x_{m,n}\longrightarrow L(M_{\lambda_{m,n}}-\mathrm{st})$.

For $\lambda,\mu\in\Lambda_0$, let us consider the series

$$\begin{aligned}\lambda(x,y)&=\sum_{m,n=0}^{\infty,\infty}\lambda_{m,n}x^my^n,\\ \mu(x,y)&=\sum_{m,n=0}^{\infty,\infty}\mu_{m,n}x^my^n,\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\mu(x,y)}{\lambda(x,y)}&=k(x,y)=\sum_{m,n=0}^{\infty,\infty}k_{m,n}x^my^n,\\ \frac{\lambda(x,y)}{\mu(x,y)}&=l(x,y)=\sum_{m,n=0}^{\infty,\infty}l_{m,n}x^my^n.\end{aligned} \tag{24}$$

The series $\lambda(x,y)$ and $\mu(x,y)$ are convergent for all $|x|<1, |y|<1$.

Theorem 4. *For any $\lambda,\mu\in\Lambda$, there exists a double sequence $\gamma=(\gamma_{m,n})\in\Lambda$ such that $M^{st}_{\lambda_{m,n}}\subset M^{st}_{\gamma_{m,n}}$ and $M^{st}_{\mu_{m,n}}\subset M^{st}_{\gamma_{m,n}}$.*

Proof. For $\lambda = (\lambda_{m,n}) \in \Lambda_0$ and $\mu = (\mu_{m,n}) \in \Lambda_0$, let us consider the sequence $\gamma = (\gamma_{m,n})$ as

$$\gamma_{m,n} := \lambda_{m,n}\mu_{0,0} + \lambda_{m-1,n-1}\mu_{1,1} + \cdots + \lambda_{0,0}\mu_{m,n}, \tag{25}$$

for $m, n \in \mathbb{N}$. Let us show that $\gamma \in \Lambda$. Since $\lambda, \mu \in \Lambda$, we have

$$\begin{aligned}\sum_{m,n=0}^{\infty,\infty} |\gamma_{m,n}| &= \left|\sum_{m,n=0}^{\infty,\infty} \left(\lambda_{m,n}\mu_{00} + \lambda_{m-1,n-1}\mu_{11} + \cdots + \lambda_{00}\mu_{m,n}\right)\right| \\ &\le \sum_{m,n=0}^{\infty,\infty} |\lambda_{m,n}||\mu_{0,0}| + |\lambda_{m-1,n-1}||\mu_{1,1}| + \cdots \\ &\quad + |\lambda_{0,0}||\mu_{m,n}| = \left(\sum_{m,n=0}^{\infty,\infty} |\lambda_{m,n}|\right)\left(\sum_{m,n=0}^{\infty,\infty} |\mu_{m,n}|\right) < \infty.\end{aligned} \tag{26}$$

Let $(t^{\lambda}_{m,n})$ and $(t^{\mu}_{m,n})$ be the $M_{\lambda_{m,n}}-\text{st}$ and $M_{\mu_{m,n}}-\text{st}$ transformations of $x = (x_{m,n})$, respectively.

$(t^{\gamma}_{m,n})$ is also the $M_{\gamma_{m,n}}$-st transformation of $(x_{m,n})$, where

$$\begin{aligned}t^{\gamma}_{m,n} := \sum_{k,j=0}^{m,n} \gamma_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) &= \gamma_{m,n}\chi_{K(\varepsilon)}(0,0) + \cdots \\ &\quad + \gamma_{0,0}\chi_{K(\varepsilon)}(m,n).\end{aligned} \tag{27}$$

From (25), we have the following equality:

$$\begin{aligned}t^{\gamma}_{m,n} &= \left(\lambda_{m,n}\mu_{0,0} + \lambda_{m-1,n-1}\mu_{1,1} + \cdots + \lambda_{0,0}\mu_{m,n}\right)\chi_{K(\varepsilon)}(0,0) \\ &\quad + \cdots + \lambda_{0,0}\mu_{0,0}\chi_{K(\varepsilon)}(m,n) \\ &= \lambda_{0,0}\mu_{0,0}\chi_{K(\varepsilon)}(m,n) \\ &\quad + \cdots + \left(\lambda_{m,n}\mu_{0,0} + \cdots + \lambda_{0,0}\mu_{m,n}\right)\chi_{K(\varepsilon)}(0,0) \\ &= \lambda_{1,1}\left(\mu_{0,0}\chi_{K(\varepsilon)}(m,n) + \cdots + \mu_{m,n}\chi_{K(\varepsilon)}(0,0)\right) \\ &\quad + \cdots + \lambda_{m,n}\left(\mu_{m,n}\chi_{K(\varepsilon)}(0,0)\right) \\ &= \lambda_{1,1}t^{\mu}_{m,n} + \lambda_{2,2}t^{\mu}_{m-1,n-1} + \cdots + \lambda_{m,n}t^{\mu}_{1,1}.\end{aligned} \tag{28}$$

Since $t^{\mu}_{m,n} \longrightarrow 0$, as $m, n \longrightarrow \infty$, we obtain $\lim_{m,n\longrightarrow\infty} t^{\gamma}_{m,n} = 0$. This implies that $x_{m,n} \longrightarrow L(M_{\gamma_{m,n}} - \text{st})$.

By using the same argument, we can get $M^{\text{st}}_{\lambda_{m,n}} \subset M^{\text{st}}_{\gamma_{m,n}}$.

Remark 2. Theorem 4 is valid for any $\lambda, \mu \in \Lambda_0$.

Theorem 5. *For any two $\lambda, \mu \in \Lambda$, the methods $M^{st}_{\lambda_{m,n}}$ and $M^{st}_{\mu_{m,n}}$ are consistent.*

Proof. Let $(x_{m,n})$ be a sequence such that $x_{m,n} \longrightarrow s(M_{\lambda_{m,n}} - \text{st})$ and $x_{m,n} \longrightarrow t(M_{\mu_{m,n}} - \text{st})$. If we consider the sequence $\gamma = (\gamma_{m,n}) \in \Lambda$ as (25) in Theorem 4, we obtain $x_{m,n} \longrightarrow s(M_{\gamma_{m,n}} - \text{st})$ and $x_{m,n} \longrightarrow t(M_{\gamma_{m,n}} - \text{st})$. From the uniqueness of the limit, we obtain that $s = t$.

Theorem 6. *Let $\lambda, \mu \in \Lambda_0$. Then, $M^{st}_{\lambda_{m,n}} \subset M^{st}_{\mu_{m,n}}$ if and only if $\sum_{m,n=0}^{\infty} |k_{m,n}| < \infty$ and $\sum_{m,n=0}^{\infty,\infty} k_{m,n} = 1$.*

Proof. Let $(t^{\lambda}_{m,n})$ and $(t^{\mu}_{m,n})$ be the $M^{\text{st}}_{\lambda_{m,n}}$- and $M^{\text{st}}_{\mu_{m,n}}$-transforms of the sequence $x = (x_{m,n})$ and $y = (y_{m,n})$, respectively.

If $|x| < 1$ and $|y| < 1$, we obtain

$$\begin{aligned}\sum_{m,n=0}^{\infty,\infty} t^{\mu}_{m,n}x^{m,n}y^{m,n} &= \sum_{m,n=0}^{\infty} \left(\mu_{m,n}\chi_{K(\varepsilon)}(0,0) + \cdots + \mu_{0,0}\chi_{K(\varepsilon)}(m,n)\right)x^{mn}y^{mn} \\ &= \left(\sum_{m,n=0}^{\infty,\infty} \mu_{m,n}x^{mn}y^{mn}\right)\left(\sum_{m,n=0}^{\infty,\infty} \chi_{K(\varepsilon)}(m,n)x^{mn}y^{mn}\right).\end{aligned} \tag{29}$$

Similarly, if $|x| < 1$ and $|y| < 1$,

$$\begin{aligned}&\sum_{m,n=0}^{\infty,\infty} t^{\lambda}_{m,n}x^{mn}y^{mn} \\ &= \left(\sum_{m,n=0}^{\infty,\infty} \lambda_{m,n}x^{mn}y^{mn}\right)\left(\sum_{m,n=0}^{\infty,\infty} \chi_{K(\varepsilon)}(m,n)x^{mn}y^{mn}\right).\end{aligned} \tag{30}$$

Then, it is clear from (24) that

$$\mu(x,y) = \lambda(x,y)k(x,y), \tag{31}$$

for all $|x| < 1$ and $|y| < 1$.

It follows from the above equality and (23) and (24) that

$$\begin{aligned}k(x,y)\lambda(x,y)\sum_{m,n=0}^{\infty,\infty} \chi_{K(\varepsilon)}(m,n)x^{mn}y^{mn} \\ = \mu(x,y)\sum_{m,n=0}^{\infty,\infty} \chi_{K(\varepsilon)}(m,n)x^{mn}y^{mn}.\end{aligned} \tag{32}$$

So, we obtain

$$k(x,y)\sum_{m,n=0}^{\infty,\infty} t^{\lambda}_{m,n}x^{m,n}y^{m,n} = \sum_{m,n=0}^{\infty,\infty} t^{\mu}_{m,n}x^{m,n}y^{m,n}. \tag{33}$$

Then,

$$t^{\mu}_{m,n} := k_{0,0}t^{\lambda}_{m,n} + k_{1,1}t^{\lambda}_{m-1,n-1} + \cdots + k_{m,n}t^{\lambda}_{0,0} = \sum_{i,j=0}^{\infty,\infty} a_{m,n,i,j}t^{\lambda}_{i,j}, \tag{34}$$

where

$$a_{m,n,i,j} = \begin{cases} k_{m-i,n-j}, & \text{if } i \le m,\ j \le n, \\ 0, & \text{otherwise.} \end{cases} \tag{35}$$

Then, for the last discussion, we have $M_{\lambda}^{st} \subset M_{\mu}^{st}$ if and only if the matrix $(a_{m,n,i,j})$ is regular. So, we obtain

$$\sup_{m,n\ge 0} \sum_{i,j=0}^{\infty,\infty} \left|a_{m,n,i,j}\right| = \sup_{m,n\ge 0} \sum_{i,j=0}^{\infty,\infty} \left|k_{m-i,n-j}\right| < \infty. \tag{36}$$

That is, $\sum_{m,n=0}^{\infty,\infty} |k_{m,n}| < \infty$. Also, we have

$$\lim_{m,n\longrightarrow\infty} \sum_{i,j=0}^{\infty,\infty} a_{m,n,i,j} = \lim_{m,n\longrightarrow\infty} \sum_{i,j=0}^{m,n} k_{m-i,n-j} = 1, \tag{37}$$

i.e., $\sum_{m,n=0}^{\infty,\infty} k_{m,n} = 1$.

The proof is completed.

Corollary 2. *Let $\lambda, \mu \in \Lambda_0$. Then, $M_{\lambda_{m,n}}^{st} = M_{\mu_{m,n}}^{st}$ if and only if $\sum_{m,n=0}^{\infty,\infty} |k_{m,n}| < \infty$, $\sum_{m,n=0}^{\infty,\infty} |h_{m,n}| < \infty$, and $\sum_{m,n=0}^{\infty,\infty} k_{m,n} = \sum_{m,n=0}^{\infty,\infty} h_{m,n} = 1$.*

Definition 6 (see [25]). Two sequences $\lambda = (\lambda_{m,n})$ and $\mu = (\mu_{m,n})$ in Λ are said to be equivalent if

$$\lim_{m,n\longrightarrow\infty} \frac{\lambda_{m,n}}{\mu_{m,n}} = 1, \tag{38}$$

and it is denoted by $\lambda \sim \mu$.

Theorem 7. *Let λ and μ be sequences in Λ such that $\lambda \sim \mu$. Then, $M_{\lambda}^{st} = M_{\mu}^{st}$.*

Proof. Let $x = (x_{m,n}) \in M_{\lambda_{m,n}}^{\mathrm{st}}$ be an arbitrary sequence such that

$$\lim_{m,n\longrightarrow\infty} \sum_{k,j=0}^{m,n} \lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) = 0, \tag{39}$$

for any $\varepsilon > 0$. Therefore,

$$\begin{aligned} &\lim_{m,n\longrightarrow\infty} \sum_{k,j=0}^{m,n} \mu_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \\ &\quad = \lim_{m,n\longrightarrow\infty} \sum_{k,j=0}^{m,n} \frac{\mu_{m-k,n-j}}{\lambda_{m-k,n-j}} \lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j). \end{aligned} \tag{40}$$

Since $\lambda \sim \mu$, for any fixed $\varepsilon_0 > 0$, there exists $n_0 \equiv n_0(\varepsilon_0) \in \mathbb{N}$ and $m_0 \equiv m_0(\varepsilon_0) \in \mathbb{N}$ such that

$$1 - \varepsilon_0 < \frac{\mu_{m-k,n-j}}{\lambda_{m-k,n-j}} < 1 + \varepsilon_0 \tag{41}$$

holds for all $n \ge n_0$ and $n \ge m_0$. Hence, the following inequality holds:

$$\begin{aligned} &\sum_{k,j=0}^{m,n} \mu_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \\ &\quad = \sum_{k,j=0}^{m_0(\varepsilon_0),n_0(\varepsilon_0)} \frac{\mu_{m-k,n-j}}{\lambda_{m-k,n-j}} \lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \\ &\qquad + \sum_{m_0(\varepsilon_0),n_0(\varepsilon_0)}^{m,n} \frac{\mu_{m-k,n-j}}{\lambda_{m-k,n-j}} \lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \\ &\quad \le \sum_{k,j=0}^{m_0(\varepsilon_0),n_0(\varepsilon_0)} \frac{\mu_{m-k,n-j}}{\lambda_{m-k,n-j}} \lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \\ &\qquad + (1+\varepsilon_0) \sum_{m_0(\varepsilon_0),n_0(\varepsilon_0)}^{m,n} \lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \\ &\quad \le c \sum_{k,j=0}^{m,n} \lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \end{aligned} \tag{42}$$

when

$$c = \max\left\{1 + \varepsilon, \frac{\mu_{mn}}{\lambda_{mn}}, \frac{\mu_{m-1,n-1}}{\lambda_{m-1,n-1}}, \ldots, \frac{\mu_{m-m_0,n-n_0}}{\lambda_{m-m_0,n-n_0}}\right\}. \tag{43}$$

Now, by taking the limit as $m, n \longrightarrow \infty$, we obtain that $x_{m,n} \longrightarrow L(M_{\mu_{m,n}} - \mathrm{st})$. We conclude that $M_{\lambda_{m,n}}^{\mathrm{st}} \subset M_{\mu_{m,n}}^{\mathrm{st}}$. Converse of this inclusion result can be obtained by the same way. Hence, the proof is completed.

Corollary 3. *Let $E = \{\lambda_{m,n}\colon m, n \in \mathbb{N}\}$ and $F = \{\mu_{m,n}\colon m, n \in \mathbb{N}\}$ be associated sets of $\lambda = (\lambda_{m,n}), \mu = (\mu_{m,n}) \in \Lambda$. If $E\backslash F$ (or $F\backslash E$) is finite, then $M_{\lambda_{m,n}}^{st} = M_{\mu_{m,n}}^{st}$.*

Theorem 8. *Let $x = (x_{m,n})$ be a sequence and $\lambda \in \Lambda$. Then, $x_{m,n} \longrightarrow L(st)$ implies $x_{m,n} \longrightarrow L(M_{\lambda_{m,n}} - st)$ if and only if $mn\lambda_{m-k,n-j} = O(1)$ for $0 \le k \le n$ and $0 \le j \le m$.*

Proof. Let $K(\varepsilon) = \{(m,n)\colon |x_{m,n} - L| \ge \varepsilon\}$ for any $\varepsilon > 0$. Since $mn\lambda_{m-k,n-j} = O(1)$, we have

$$\begin{aligned} \sum_{k,j=0}^{m,n} \lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) &= \frac{1}{mn} \sum_{k,j=0}^{m,n} mn\lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \\ &\le O(1) \frac{1}{mn} \sum_{k,j=0}^{m,n} \chi_{K(\varepsilon)}(k,j). \end{aligned} \tag{44}$$

If we take the limit as $m, n \longrightarrow \infty$, the desired result is obtained if and only if $mn\lambda_{m-k,n-j} = O(1)$.

Theorem 9. *Let $x = (x_{m,n})$ be a sequence and $\lambda \in \Lambda$. Then, $x_{m,n} \longrightarrow L(M_{\lambda} - st)$ implies $x_{m,n} \longrightarrow L(st)$ if and only if the sequence $(1/mn\lambda_{m-k,n-j}) = O(1)$ for $k \le m$ and $j \le n$.*

Proof. For any $\varepsilon > 0$, let $K(\varepsilon) = \{(m,n)\colon |x_{m,n} - L| \ge \varepsilon\}$. Then, the inequality

$$\lim_{m,n\longrightarrow\infty}\frac{1}{mn}\sum_{k,j=0}^{m,n}\chi_{K(\varepsilon)}(k,j)$$
$$=\lim_{m,n\longrightarrow\infty}\frac{1}{mn}\sum_{k,j=0}^{m,n}\frac{\lambda_{m-k,n-j}}{\lambda_{m-k,n-j}}\chi_{K(\varepsilon)}(k,j)$$
$$=\lim_{m,n\longrightarrow\infty}\sum_{k,j=0}^{m,n}\frac{1}{mn\lambda_{m-k,n-j}}\lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j) \tag{45}$$
$$\leq O(1)\lim_{m,n\longrightarrow\infty}\sum_{k,j=0}^{m,n}\lambda_{m-k,n-j}\chi_{K(\varepsilon)}(k,j)$$

holds. This gives the proof.

On the contrary, let us recall that $C(D)$ is the space of all continuous real-valued functions on any compact subset of the real two-dimensional space. We know that $C(D)$ is a Banach space with norm

$$\|f\|_\infty \coloneqq \sup_{(x,y)\in D}|f(x,y)|, \quad (f\in C(D)). \tag{46}$$

Suppose that L is a linear operator from $C(D)$ into $C(D)$. It is clear that if $f\geq 0$ implies $\mathrm{Lf}\geq 0$, then the linear operator L is positive on $C(D)$. Also, we denote the value of Lf at a point (x,y) by $L(f(u,v);x,y)$ or only $L(f;x,y)$. The classical Korovkin approximation theorem (see [35]) was extended from single sequences to double sequences [36].

Now, we give the following theorem to prove the approximation theorem.

Theorem 10. *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators from $C(D)$ into $C(D)$. Then,*

$$\lim_{m,n}\|L_{m,n}(f)-f\|_\infty \longrightarrow 0\left(M_{\lambda_{m,n}}-\mathrm{st}\right) \tag{47}$$

for all $f\in C(D)$ if and only if

$$\lim_{m,n}\|L_{m,n}(f_i)-f_i\|_\infty \longrightarrow 0, \left(M_{\lambda_{m,n}}-\mathrm{st}\right), \tag{48}$$

where $f_0(x,y)=1$, $f_1(x,y)=x$, $f_2(x,y)=y$, $f_3(x,y)=x^2+y^2$, and $i=0,1,2,3$.

Proof. Since each $f_i\in C(D)(i=0,1,2,3)$, assertion (48) follows immediately from assertion (47). Assume that (48) holds. Since $f\in C(D)$, we have $|f(x,y)|\leq M$, where $M=\|f\|_\infty$. Using the continuity of f on D for every $\varepsilon>0$, there is $\delta>0$ such that $|f(v,z)-f(x,y)|<\varepsilon$ for all $(v,z)\in D$ satisfying $|v-x|<\delta$ and $|z-y|<\delta$. Then, we get

$$|f(v,z)-f(x,y)|<\varepsilon+\frac{2M}{\delta_2}\left\{(v-x)^2+(z-y)^2\right\}. \tag{49}$$

Also, by the linearity and positivity of the operators $L_{m,n}$ and from (49), we obtain

$$\begin{aligned}\left|L_{m,n}(f;x,y)-f(x,y)\right| &= \left|L_{m,n}(f(v,z)-f(x,y);x,y)-f(x,y)\left(L_{m,n}(f_0;x,y)-f_0(x,y)\right)\right|\\ &\leq L_{m,n}(|f(v,z)-f(x,y)|;x,y)+M\left|L_{m,n}(f_0;x,y)-f_0(x,y)\right|\\ &\leq \left|L_{m,n}\left(\varepsilon+\frac{2M}{\delta^2}\left\{(v-x)^2+(z-y)^2;x,y\right\}\right)\right|+M\left|L_{m,n}(f_0;x,y)-f_0(x,y)\right|\\ &\leq \varepsilon+M+\frac{2M}{\delta^2}\left(A^2+B^2\right)\left|L_{m,n}(f_0;x,y)-f_0(x,y)\right|+\frac{4M}{\delta^2}A\left|L_{m,n}(f_1;x,y)-f_1(x,y)\right|\\ &\quad+\frac{4M}{\delta^2}B\left|L_{m,n}(f_2;x,y)-f_2(x,y)\right|+\frac{2M}{\delta^2}\left|L_{m,n}(f_3;x,y)-f_3(x,y)\right|+\varepsilon,\end{aligned} \tag{50}$$

where $A\coloneqq \max|x|$ and $B\coloneqq\max|y|$. Then, taking $\sup_{(x,y)\in D}$, we obtain

$$\begin{aligned}\|L_{m,n}f-f\|_\infty &\leq R\|L_{m,n}f_0-f_0\|_\infty+\|L_{m,n}f_1-f_1\|_\infty\\ &\quad+\|L_{m,n}f_2-f_2\|_\infty+\|L_{m,n}f_3-f_3\|_\infty+\varepsilon,\end{aligned} \tag{51}$$

where

$$R=\max\left\{\varepsilon+M+\frac{2M}{\delta^2}\left(E^2+F^2\right),\frac{4M}{\delta^2}E,\frac{4M}{\delta^2}F,\frac{2M}{\delta^2}\right\}. \tag{52}$$

Similarly, we obtain

$$\begin{aligned}&\left\|L_{m,n}(f;x,y)\lambda_{m-k,n-j}-f(x,y)\right\|_\infty\\ &\quad\leq\varepsilon+R\sum_{i=0}^{3}\left\|L_{m,n}(f_i;x,y)\lambda_{m-k,n-j}-f_i(x,y)\right\|_\infty.\end{aligned} \tag{53}$$

We now replace $L_{m,n}(.;x,y)\lambda_{m-k,n-j}$ by

$$T_{m,n}(.;x,y)=\sum_{(k,l)}L_{m,n}(.;x,y)\lambda_{m-k,n-j}. \tag{54}$$

We choose $\varepsilon'>0$ such that $\varepsilon'<r$ for a given $r>0$. Then, define the sets

$$\begin{aligned}S&\coloneqq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}\colon\ \|T_{m,n}f-f\|_{C(K)}\geq r\right\},\\ S_i&=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}\colon\ \|T_{m,n}f_i-f_i\|_{C(K)}\geq\frac{r-\varepsilon'}{4R}\right\},\end{aligned} \tag{55}$$

$i = 0, 1, 2, 3$. It is clear that $S \subset \cup_{i=0}^{3} S_i$, and so, $\delta(S) \le \delta(S_0) + \delta(S_1) + \delta(S_2) + \delta(S_3)$. Therefore, using condition (53), we obtain

$$\lim_{m,n} \left\| L_{m,n} f - f \right\|_\infty \longrightarrow 0\left(M_{\lambda m,n} - \mathrm{st}\right). \quad (56)$$

This completes the proof.

Corollary 4. *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators from $C(D)$ into $C(D)$. Then,*

$$P - \lim_{m,n} \left\| L_{m,n}(f) - f \right\|_\infty = 0 \quad (57)$$

for all $f \in C(D)$ if and only if

$$P - \lim_{m,n} \left\| L_{m,n}(f_i) - f_i \right\|_\infty = 0, \quad (58)$$

where $f_0(x, y) = 1$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = x^2 + y^2$, and $i = 0, 1, 2, 3$.

Remark 3. We now show in the following an example of a sequence of positive linear operators of two variables satisfying the conditions of Theorem 10 but does not satisfy the conditions of the Korovkin theorem. Consider the following Bernstein operators:

$$B_{m,n}(f; x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} f\left(\frac{k}{m}, \frac{j}{n}\right) C_m^k x^k C_n^j y^j (1 - y)^{n-j}, \quad (59)$$

where $(x, y) \in [0, 1] \times [0, 1]$.

Let $L_{m,n}(f_0; x, y) = 1, L_{m,n}(f_1; x, y) = x$, $L_{m,n}(f_2; x, y) = y$, and $L_{m,n}(f_3; x, y) = x^2 + y^2 + (x - x^2/m) + (y - y^2/n)$. Then, by Corollary 4, we obtain

$$P - \lim_{m,n} \left\| L_{m,n} f - f \right\|_\infty = 0, \quad (60)$$

for all $f \in C(D)$. $L_{m,n}$: $C(D) \longrightarrow C(D)$ with $L_{m,n}(f; x, y) = (1 + s_{m,n}) B_{m,n}(f; x, y)$, where

$$s_{m,n} = \begin{cases} 1, & \text{if } m \text{ and } n \text{ are even,} \\ 0, & \text{otherwise.} \end{cases} \quad (61)$$

Let $\lambda_{m,n} = 1$. The double sequence $(s_{m,n})$ is neither P-convergent nor statistically convergent, but $(s_{m,n})$ is statistically summable $M_{\lambda_{m,n}}$ to zero. $B_{m,n}(1; x, y) = 1$, $B_{m,n}(x; x, y) = x$, $B_{m,n}(y; y, x) = y$, and $B_{m,n}(x^2 + y^2; x, y) = x^2 + y^2 + (x - x^2/m) + (y - y^2/n)$, and the double sequence $L_{m,n}$ satisfies condition (48) for $i = 0, 1, 2, 3$. Hence, we get

$$M_{\lambda_{m,n}} \mathrm{st} - \lim_{m,n} \left\| L_{m,n} f - f \right\|_\infty = 0. \quad (62)$$

We have $L_{m,n}(f, 0, 0) = (1 + s_{m,n}) B_{m,n}(f; 0, 0)$ since $B_{m,n}(f; 0, 0) = f(0, 0)$, and hence,

$$\begin{aligned} \left\| L_{m,n}(f; x, y) - f(x, y) \right\|_\infty &\ge \left| L_{m,n}(f; x, y) - f(x, y) \right| \\ &\ge s_{m,n} |f(0, 0)|. \end{aligned} \quad (63)$$

It is easy to see that $(L_{m,n})$ does not satisfy the conditions of the classical Korovkin-type theorem since $\lim s_{m,n}$ and $\mathrm{st}^2 - \lim s_{m,n}$ do not exist; this proves the claim.

3. Conclusion

In this paper, we introduce $M_{\lambda m,n}$-statistical convergence for double sequences and give the inclusion results for different $\lambda_{m,n}$'s. These new results can be viewed as a generalization of previously known results. The new concept can be applied to the approximation theory, Fourier analysis, topology, and so on. The $M_{\lambda m,n}$-statistically Cauchy sequence can be described, and its properties can be studied. Also, ideal convergent sequence spaces can be given.

Data Availability

All the data in the manuscript are available upon request to the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors of the manuscript read and agreed to its content and were accountable for all aspects of the accuracy and integrity of the manuscript.

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