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Research Article

The Unique Periodic Solution of Abel's Differential Equation

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In this paper, the existence of a periodic solution for Abel's differential equation is obtained first by using the fixed-point theorem. Then, by constructing the Lyapunov function, the uniqueness and stability of the periodic solution of the equation are obtained.

1. Introduction

The nonlinear Abel-type first-order differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x^3 + b(t)x^2 + c(t)x + d(t), \tag{1}$$

plays an important role in many physical and technical applications [1, 2]. The mathematical properties of equation (1) have been intensively investigated in the mathematical and physical literature [3–15].

Recently, Ni [16] supposed that $\gamma = \gamma(t)$ is a periodic particular solution of Abel's differential equation under some conditions, then by means of the transformation method and the fixed-point theory, the author presented an alternative method of generating the other periodic solutions of Abel's differential equation. Ni and Tian [17] got the sufficient conditions which guarantee the existence of three periodic solutions for Abel's differential equation.

All the time, the existence, uniqueness, and stability of the periodic solution of differential equations have always been an important research hotspot in the field of differential equations. So, in this paper, equation (1) is considered and a new criterion is given to judge the existence, uniqueness, and stability of the periodic solution of equation (1) by using the fixed-point theorem and constructing the Lyapunov function, and some new results are obtained.

The rest of the paper is arranged as follows: in Section 2, some lemmas and abbreviations are introduced to be used later; in Section 3, the existence and stability of the unique periodic solution of equation (1) are obtained; in Section 4, two examples are given to verify the main results of the paper; the paper ends with a short conclusion.

2. Preliminaries

In this section, some definitions, lemmas, and abbreviations are given, which will be used to prove the main results.

Definition 1 (see [18]). Suppose $f(t)$ is an ω-periodic continuous function on R, then

$$a(f,\lambda) = \int_0^{\omega} f(t)e^{-i\lambda t}\mathrm{d}t, \tag{2}$$

must exist, where $a(f,\lambda)$ is called the Fourier coefficient of $f(t)$ and λ, such that $a(f,\lambda) \neq 0$, is called the Fourier index of $f(t)$; there is a countable set Λ_f, when $\lambda \in \Lambda_f$ and $a(f,\lambda) \neq 0$, and as long as $\lambda \notin \Lambda_f$, there must be $a(f,\lambda) = 0$, where Λ_f is called the exponential set of $f(t)$.

Definition 2. (see [18]). A set of real numbers composed of linear combinations of integer coefficients of elements in Λ_f is called a module or a frequency module of $f(t)$, which is denoted as $\mathrm{mod}(f)$, that is,

$$\mathrm{mod}(f) = \left\{ \mu \mid \mu = \sum_{j=1}^{N} n_j\lambda_j, \quad n_j, N \in Z^+, N \geq 1, \lambda_j \in \Lambda_f \right\}. \tag{3}$$

Lemma 1 (see [19]). *Consider the following equation:*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x + b(t), \tag{4}$$

where $a(t)$ and $b(t)$ are the ω-periodic continuous functions, and if $\int_0^\omega a(t)dt \neq 0$, then equation (4) has a unique ω-periodic continuous solution $\eta(t), \mathrm{mod}(\eta) \subseteq \mathrm{mod}(a(t), b(t))$ and $\eta(t)$ can be written as follows:

$$\eta(t) = \begin{cases} \int_{-\infty}^{t} e^{\int_s^t a(t)\mathrm{d}\tau} b(s)\mathrm{d}s, & \int_0^\omega a(t)\mathrm{d}t < 0, \\ -\int_t^{+\infty} e^{\int_s^t a(t)\mathrm{d}\tau} b(s)\mathrm{d}s, & \int_0^\omega a(t)\mathrm{d}t > 0. \end{cases} \tag{5}$$

Lemma 2 (see [20]). *Suppose that an ω-periodic function sequence $\{f_n(t)\}$ is convergent uniformly on any compact set of R, $f(t)$ is an ω-periodic function, and $\mathrm{mod}(f_n) \subseteq \mathrm{mod}(f)$ $(n = 1, 2, \cdots, n)$, then $\{f_n(t)\}$ is convergent uniformly on R.*

Lemma 3 (see [21]). *Suppose V is a metric space, C is a convex closed set of V, and its boundary is ∂C, if $T\colon V \longrightarrow V$ is a continuous compact mapping, such that $T(\partial C) \subseteq C$, then T has at least a fixed point on C.*

For the sake of convenience, suppose that $f(t)$ is an ω-periodic continuous function on R, then it is denoted that

$$\begin{aligned} f_M &= \sup_{t \in [0,\omega]} f(t), \\ f_L &= \inf_{t \in [0,\omega]} f(t). \end{aligned} \tag{6}$$

3. Main Results

In this section, the existence, uniqueness, and stability of the periodic solution of equation (1) are studied.

Theorem 1. *Consider equation (1), and $a(t), b(t), c(t)$, and $d(t)$ are all ω-periodic continuous functions on R, suppose that the following conditions hold:*

$$\begin{aligned} &(H_1) a(t) < 0, \\ &(H_2) b^2(t) - 3a(t)c(t) < 0, \end{aligned} \tag{7}$$

then, equation (1) has a unique ω-periodic continuous solution which is uniformly asymptotic stable.

Proof. Let

$$f(t,x) = a(t)x^3 + b(t)x^2 + c(t)x + d(t). \tag{8}$$

By (H_1) and (8), it follows that

$$\begin{aligned} f(t,x) &\longrightarrow +\infty (x \longrightarrow -\infty), \\ f(t,x) &\longrightarrow -\infty (x \longrightarrow +\infty). \end{aligned} \tag{9}$$

By zero point theorem, there is at least a zero point $\gamma(t) \in (-\infty, +\infty)$ such that

$$f[t, \gamma(t)] = a(t)\gamma^3(t) + b(t)\gamma^2(t) + c(t)\gamma(t) + d(t) = 0. \tag{10}$$

By (H_1) and (H_2), it follows that

$$f'_x(t,x) = 3a(t)x^2 + 2b(t)x + c(t) < 0, \tag{11}$$

thus, $f(t,x)$ has a unique zero point $\gamma(t)(\gamma(t) \in (-\infty, +\infty))$ such that (10) holds. Since $a(t), b(t), c(t)$, and $d(t)$ are all continuous functions on R, and from (10) and (11), it follows that $\gamma(t)$ is a continuous function on R.

By (10), it follows that

$$\begin{aligned} f[t+\omega, \gamma(t+\omega)] = {}& a(t+\omega)\gamma^3(t+\omega) + b(t+\omega)\gamma^2(t+\omega) \\ &+ c(t+\omega)\gamma(t+\omega) + d(t+\omega) = 0. \end{aligned} \tag{12}$$

$f(t,x)$ is ω-periodic on t, that is,

$$\begin{aligned} f[t, \gamma(t+\omega)] = {}& a(t)\gamma^3(t+\omega) + b(t)\gamma^2(t+\omega) \\ &+ c(t)\gamma(t+\omega) + d(t) = 0. \end{aligned} \tag{13}$$

If $\gamma(t)$ is not an ω-periodic function, then

$$\gamma(t+\omega) \equiv \gamma(t) (t \in R), \tag{14}$$

thus, $f(t,x)$ has two zero points, $\gamma(t)$ and $\gamma(t+\omega)$, and it is a contradiction to the fact that $f(t,x)$ has a unique zero point $\gamma(t)$ on $(-\infty, +\infty)$. Therefore,

$$\gamma(t+\omega) \equiv \gamma(t) (t \in R), \tag{15}$$

that is to say, $\gamma(t)$ is an ω-periodic continuous function on R.

In addition,

$$\begin{aligned} f(t,x) &= f(t,x) - f[t,\gamma(t)] \\ &= f'_x[t, \gamma(t) + \theta(x - \gamma(t))](x - \gamma(t)) (0 < \theta = \theta(x) < 1) \\ &= (x - \gamma(t))\big[3a(t)(\gamma(t) + \theta(x - \gamma(t)))^2 \\ &\quad + 2b(t)(\gamma(t) + \theta(x - \gamma(t))) + c(t)\big]. \end{aligned} \tag{16}$$

Comparing the coefficients of the third power of x of the left and right hands of (16), the following is obtained:

$$\theta = \frac{\sqrt{3}}{3}. \tag{17}$$

By (16), let $x = 0$, and then $\gamma(t)$ satisfies

$$[2\sqrt{3} - 4]a(t)\gamma^3(t) + 2\left(\frac{\sqrt{3}}{3} - 1\right)b(t)\gamma^2(t) - c(t)\gamma(t) = d(t), \tag{18}$$

thus, equation (1) becomes

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= (x-\gamma(t))\left[3a(t)\left(\gamma(t)+\frac{\sqrt{3}}{3}(x-\gamma(t))\right)^2+2b(t)\left(\gamma(t)+\frac{\sqrt{3}}{3}(x-\gamma(t))\right)+c(t)\right]\\ &= (x-\gamma(t))\left\{a(t)x^2+\left[2\sqrt{3}\,a(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t)+\frac{2\sqrt{3}\,b(t)}{3}\right]x+3a(t)\left(1-\frac{\sqrt{3}}{3}\right)^2\gamma^2(t)+2b(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t)+c(t)\right\}.\end{aligned}\tag{19}$$

By simple computation and (H_2), it is obtained that the discriminant of the quadratic polynomial of one variable satisfies the following:

$$\begin{aligned}\Delta &= \left[2\sqrt{3}\,a(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t)+\frac{2\sqrt{3}\,b(t)}{3}\right]^2-4a(t)\left\{3a(t)\left(1-\frac{\sqrt{3}}{3}\right)^2\gamma^2(t)+2b(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t)+c(t)\right\}\\ &= \frac{4\left(b^2(t)-3a(t)c(t)\right)}{3a^2(t)}<0.\end{aligned}\tag{20}$$

Next, the existence of the unique periodic solution $\phi(t)$ of equation (1) is proven by using the fixed-point theorem.

Suppose

$$S=\{\varphi(t)\in C(R,R)\mid \varphi(t+\omega)=\varphi(t)\}.\tag{21}$$

Given any $\varphi(t),\psi(t)\in S$, the distance is defined as follows:

$$\rho(\varphi,\psi)=\sup_{t\in[0,\omega]}|\varphi(t)-\psi(t)|,\tag{22}$$

thus, (S,ρ) is a complete metric space. A convex closed set B of S is taken as follows:

$$B=\{\varphi(t)\in S\mid \varphi(t+\omega)=\varphi(t),\gamma_L\le\varphi(t)\le\gamma_M,\operatorname{mod}(\varphi)\subseteq\operatorname{mod}(a,b,c,d)\}.\tag{23}$$

Given any $\varphi(t)\in B$, consider the following equation:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= (x-\gamma(t))\left\{a(t)\varphi^2(t)+\left[2\sqrt{3}\,a(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t)+\frac{2\sqrt{3}\,b(t)}{3}\right]\varphi(t)+3a(t)\left(1-\frac{\sqrt{3}}{3}\right)^2\gamma^2(t)+2b(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t)+c(t)\right\}\\ &= f(t)(x-\gamma(t))\\ &= f(t)x-f(t)\gamma,\end{aligned}\tag{24}$$

where

$$a(t)\varphi^2(t)+\left[2\sqrt{3}\,a(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t)+\frac{2\sqrt{3}\,b(t)}{3}\right]\varphi(t)+3a(t)\left(1-\frac{\sqrt{3}}{3}\right)^2\gamma^2(t)+2b(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t)+c(t)\triangleq f(t).\tag{25}$$

It is easy to see that $f(t)$ is an ω-periodic continuous function, and by (H_1) and (20),

$$f(t)<0,\tag{26}$$

and $f(t)$ is bounded above and below by some negative constants $-\mu$ and $-\lambda$ for all $t\in[0,\omega]$ and any $\varphi(t)\in B$, that is,

$$-\mu\leq f(t)\leq-\lambda. \tag{27}$$

Thus,

$$\int_0^\omega f(t)\mathrm{d}t<0. \tag{28}$$

By Lemma 1, equation (24) has a unique ω-periodic continuous solution as follows:

$$\begin{aligned}&\eta(t)=-\int_{-\infty}^{t}e^{\int_s^t f(\tau)\mathrm{d}\tau}f(s)\gamma(s)\mathrm{d}s,\\&\operatorname{mod}(\eta)\subseteq\operatorname{mod}(f(t),f(t)\gamma(t)).\end{aligned} \tag{29}$$

By (18), it follows that

$$\operatorname{mod}(\gamma(t))\subseteq\operatorname{mod}(a,b,c,d). \tag{30}$$

By (25), it follows that

$$\begin{aligned}&\operatorname{mod}(f(t))\subseteq\operatorname{mod}(a,b,c,\gamma),\\&\operatorname{mod}(f(t)\gamma(t))\subseteq\operatorname{mod}(a,b,c,\gamma),\end{aligned} \tag{31}$$

thus, it follows that

$$\begin{aligned}&\operatorname{mod}(f(t))\subseteq\operatorname{mod}(a,b,c,d),\\&\operatorname{mod}(f(t)\gamma(t))\subseteq\operatorname{mod}(a,b,c,d).\end{aligned} \tag{32}$$

Hence,

$$\operatorname{mod}(\eta)\subseteq\operatorname{mod}(a,b,c,d). \tag{33}$$

By (23), (28), and (29), the following is obtained:

$$\begin{aligned}\eta(t)&=-\int_{-\infty}^{t}e^{\int_s^t f(\tau)\mathrm{d}\tau}f(s)\gamma(s)\mathrm{d}s\geq-\gamma_L\int_{-\infty}^{t}e^{\int_s^t f(\tau)\mathrm{d}\tau}f(s)\mathrm{d}s\\&=\gamma_L\int_{-\infty}^{t}e^{\int_s^t f(\tau)\mathrm{d}\tau}\mathrm{d}\left(\int_s^t f(\tau)\mathrm{d}\tau\right)\\&=\gamma_L\left[e^{\int_s^t f(\tau)\mathrm{d}\tau}\right]_{-\infty}^{t}\\&=\gamma_L\left[1-e^{\int_{-\infty}^t f(\tau)\mathrm{d}\tau}\right]\\&=\gamma_L,\\\eta(t)&=-\int_{-\infty}^{t}e^{\int_s^t f(\tau)\mathrm{d}\tau}f(s)\gamma(s)\mathrm{d}s\leq-\gamma_M\int_{-\infty}^{t}e^{\int_s^t f(\tau)\mathrm{d}\tau}f(s)\mathrm{d}s\\&=\gamma_M\int_{-\infty}^{t}e^{\int_s^t f(\tau)\mathrm{d}\tau}d\left(\int_s^t f(\tau)\mathrm{d}\tau\right)\\&=\gamma_M\left[e^{\int_s^t f(\tau)\mathrm{d}\tau}\right]_{-\infty}^{t}\\&=\gamma_M\left[1-e^{\int_{-\infty}^t f(\tau)\mathrm{d}\tau}\right]\\&=\gamma_M,\end{aligned} \tag{34}$$

thus $\eta(t)\in B$.

Define a mapping as follows:

$$(T\varphi)(t)=\eta(t)=-\int_{-\infty}^{t}e^{\int_s^t f(\tau)\mathrm{d}\tau}f(s)\gamma(s)\mathrm{d}s, \tag{35}$$

thus, if given any $\varphi(t)\in B$, then $(T\varphi)(t)\in B$; hence $T\colon B\longrightarrow B$.

Now, it is proven that the mapping T is a compact mapping.

Consider any sequence $\{\varphi_n(t)\}\subseteq B\,(n=1,2,\cdots)$, then it follows that

$$\gamma_L \leq \varphi_n(t) \leq \gamma_M, \operatorname{mod}(\varphi_n) \subseteq \operatorname{mod}(a,b,c,d)\,(n=1,2,\cdots). \tag{36}$$

On the other hand, $(T\varphi_n)(t) = x_{\varphi_n}(t)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}x_{\varphi_n}(t)}{\mathrm{d}t} &= \left(x_{\varphi_n}(t)-\gamma(t)\right)\left\{a(t)\varphi_n^2(t) + \left[2\sqrt{3}\,a(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t) + \frac{2\sqrt{3}\,b(t)}{3}\right]\varphi_n(t) + 3a(t)\left(1-\frac{\sqrt{3}}{3}\right)^2\gamma^2(t)\right.\\ &\quad \left. + 2b(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t) + c(t)\right\}\\ &= f_n(t)\left(x_{\varphi_n}(t)-\gamma(t)\right)\\ &= f_n(t)x_{\varphi_n}(t) - f_n(t)\gamma(t),\end{aligned} \tag{37}$$

where

$$a(t)\varphi_n^2(t) + \left[2\sqrt{3}\,a(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t) + \frac{2\sqrt{3}\,b(t)}{3}\right]\varphi_n(t) + 3a(t)\left(1-\frac{\sqrt{3}}{3}\right)^2\gamma^2(t) + 2b(t)\left(1-\frac{\sqrt{3}}{3}\right)\gamma(t) + c(t) \triangleq f_n(t). \tag{38}$$

It is easy to see that $f_n(t)$ is an ω-periodic continuous function, and by (23), (25), (27), (36), and (38), the following is obtained:

$$-\mu \leq f_n(t) \leq -\lambda, \tag{39}$$

thus,

$$\left|\frac{\mathrm{d}x_{\varphi_n}(t)}{\mathrm{d}t}\right| \leq 2\mu|\gamma|_M, \operatorname{mod}\left(x_{\varphi_n}(t)\right) \subseteq \operatorname{mod}(a,b,c,d), \tag{40}$$

hence, $\left\{\mathrm{d}x_{\varphi_n}(t)/\mathrm{d}t\right\}$ is uniformly bounded; therefore, $\left\{x_{\varphi_n}(t)\right\}$ is uniformly bounded and equicontinuous on R. By the theorem of Ascoli–Arzela, for any sequence $\left\{x_{\varphi_n}(t)\right\} \subseteq B$, there exists a subsequence (also denoted by $\left\{x_{\varphi_n}(t)\right\}$) such that $\left\{x_{\varphi_n}(t)\right\}$ is convergent uniformly on any compact set of R. By (40), combined with Lemma 2, $\left\{x_{\varphi_n}(t)\right\}$ is convergent uniformly on R, that is to say, T is relatively compact on B.

Next, it is proven that T is a continuous mapping. Suppose $\{\varphi_n(t)\} \subseteq B, \varphi(t) \in B$ and

$$\varphi_n(t) \longrightarrow \varphi(t)\,(n \longrightarrow \infty). \tag{41}$$

By (35),

$$\begin{aligned}\left|(T\varphi_n)(t) - (T\varphi)(t)\right| &= \left|\int_{-\infty}^t e^{\int_s^t f_n(\tau)\mathrm{d}\tau} f_n(s)\gamma(s)\mathrm{d}s - \int_{-\infty}^t e^{\int_s^t f(\tau)\mathrm{d}\tau} f(s)\gamma(s)\mathrm{d}s\right|\\ &= \left|\int_{-\infty}^t e^{\int_s^t f_n(\tau)\mathrm{d}\tau}[f_n(s)-f(s)]\gamma(s)\mathrm{d}s + \int_{-\infty}^t \left[e^{\int_s^t f_n(\tau)\mathrm{d}\tau} - e^{\int_s^t f(\tau)\mathrm{d}\tau}\right] f(s)\gamma(s)\mathrm{d}s\right|\\ &= \left|\int_{-\infty}^t e^{\int_s^t f_n(\tau)\mathrm{d}\tau}[f_n(s)-f(s)]\gamma(s)\mathrm{d}s + \int_{-\infty}^t \left(e^{\xi}\int_s^t [f_n(\tau)-f(\tau)]\mathrm{d}\tau\right) f(s)\gamma(s)\mathrm{d}s\right|\\ &\leq \left(\int_{-\infty}^t e^{\int_s^t f_n(\tau)\mathrm{d}\tau}|\gamma(s)|\mathrm{d}s + \int_{-\infty}^t \left(e^{\xi}\int_s^t \mathrm{d}\tau\right)|f(s)||\gamma(s)|\mathrm{d}s\right)\rho(f_n, f),\end{aligned} \tag{42}$$

where ξ is between $\int_s^t f_n(\tau)\mathrm{d}\tau$ and $\int_s^t f(\tau)\mathrm{d}\tau$, thus $\xi \leq -\lambda(t-s)$; hence,

$$\begin{aligned} \left|(T\varphi_n)(t) - T\varphi(t)\right| &\leq \left(\int_{-\infty}^t e^{-\lambda(t-s)}|\gamma(s)|\mathrm{d}s + \int_{-\infty}^t \left(e^{-\lambda(t-s)}(t-s)\right)|f(s)\||\gamma(s)|\mathrm{d}s\right)\rho(f_n, f) \\ &\leq \left(\int_{-\infty}^t e^{-\lambda(t-s)}|\gamma|_M\mathrm{d}s + \int_{-\infty}^t \left(e^{-\lambda(t-s)}(t-s)\right)\mu|\gamma|_M\mathrm{d}s\right)\rho(f_n, f) \\ &= \left(\frac{|\gamma|_M}{\lambda} + \frac{\mu|\gamma|_M}{\lambda^2}\right)\rho(f_n, f) \\ &= \frac{(\lambda+\mu)|\gamma|_M}{\lambda^2}\rho(f_n, f). \end{aligned} \tag{43}$$

By (41) and the above inequality, it follows that

$$(T\varphi_n)(t) \longrightarrow (T\varphi)(t)(n \longrightarrow \infty), \tag{44}$$

therefore, T is continuous. By (35), $T(\partial B)\subseteq B$. According to Lemma 3, T has at least a fixed point on B and the fixed point is the periodic continuous solution $\phi(t)$ of equation (1), and

$$\gamma_L \leq \phi(t) \leq \gamma_M. \tag{45}$$

Define a Lyapunov function as follows:

$$V(t, x(t)) = (x(t) - \phi(t))^2, \tag{46}$$

where $x(t)$ is the unique solution with an initial value $x(t_0) = x_0$ of equation (1), and differentiating both sides of (46) along the solution of equation (1), the following is obtained:

$$\begin{aligned} \frac{\mathrm{d}V(t, x(t))}{\mathrm{d}t} &= 2(x(t) - \phi(t))\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \frac{\mathrm{d}\phi(t)}{\mathrm{d}t}\right) \\ &= 2(x(t) - \phi(t))\left(a(t)\left(x^3(t) - \phi^3(t)\right) + b(t)\left(x^2(t) - \phi^2(t)\right) + c(t)(x(t) - \phi(t))\right) \\ &= 2(x(t) - \phi(t))^2\left(a(t)\left(x^2(t) + x(t)\phi(t) + \phi^2(t)\right) + b(t)(x(t) + \phi(t)) + c(t)\right) \\ &= 2(x(t) - \phi(t))^2\left(a(t)x^2(t) + (a(t)\phi(t) + b(t))x(t) + a(t)\phi^2(t) + b(t)\phi(t) + c(t)\right) \\ &\leq 2(x(t) - \phi(t))^2\left(\frac{4a(t)\left(a(t)\phi^2(t) + b(t)\phi(t) + c(t)\right) - (a(t)\phi(t) + b(t))^2}{4a(t)}\right) \\ &= 2(x(t) - \phi(t))^2\left(\frac{3a^2(t)\phi^2(t) + 2a(t)b(t)\phi(t) + 4a(t)c(t) - b^2(t)}{4a(t)}\right) \\ &\leq \frac{(x(t) - \phi(t))^2}{2a(t)}\left(\frac{12a^2(t)\left(4a(t)c(t) - b^2(t)\right) - 4a^2(t)b^2(t)}{12a^2(t)}\right) \\ &= \frac{2\left(3a(t)c(t) - b^2(t)\right)}{3a(t)}(x(t) - \phi(t))^2. \end{aligned} \tag{47}$$

By (H_2), there is a positive number δ such that $3a(t)c(t) - b^2(t) \geq \delta > 0$; hence,

$$\frac{\mathrm{d}V(t, x(t))}{\mathrm{d}t} \leq \frac{2\delta}{3a_L}(x(t) - \phi(t))^2, \tag{48}$$

therefore, the periodic solution $\phi(t)$ of equation (1) is uniformly asymptotic stable; thus,

$$|x(t) - \phi(t)| \longrightarrow 0(t \longrightarrow +\infty), \tag{49}$$

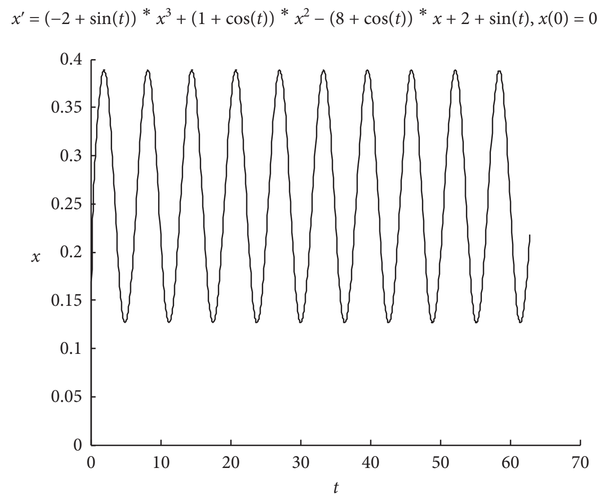

Figure 1: The solution curve of equation (51) with an initial value $x(0) = 0$.

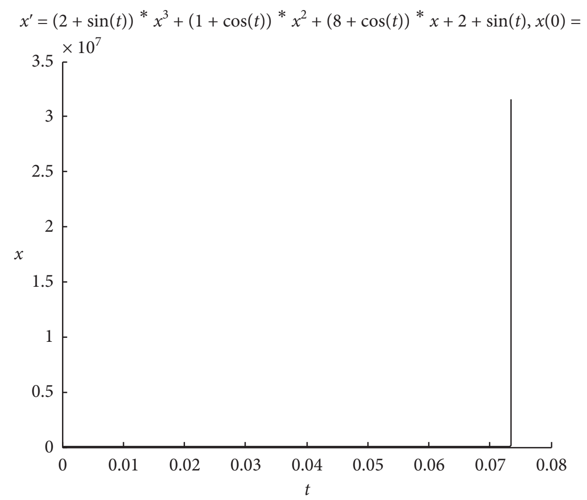

Figure 2: The solution curve of equation (53) with an initial value $x(0) = 1$.

so $x(t)$ cannot be a periodic solution of equation (1), and equation (1) has a unique periodic solution $\phi(t)$ which is uniformly asymptotic stable.

This is the end of the proof of Theorem 1.

Similarly, the following theorem is obtained. □

Theorem 2. *Consider equation (1), and $a(t), b(t), c(t)$, and $d(t)$ are all ω-periodic continuous functions on R, suppose that the following conditions hold:*

$$\begin{gathered}(H_1) a(t) > 0,\\ (H_2) b^2(t) - 3a(t)c(t) < 0,\end{gathered} \tag{50}$$

then, equation (1) has a unique ω-periodic continuous solution which is unstable.

4. Two Examples

In this section, two examples are given to verify the main results of the paper.

Example 1. Consider the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (-2 + \sin t)x^3 + (1 + \cos t)x^2 - (8 + \cos t)x + 2 + \sin t, \tag{51}$$

here $a(t) = -2 + \sin t,\ b(t) = 1 + \cos t,\ c(t) = -(8 + \cos t)$, and $d(t) = 2 + \sin t$.

By simple computation, it is obtained that

$$a(t) < 0, b^2(t) - 3a(t)c(t) < 0. \tag{52}$$

Clearly, equation (51) satisfies all the conditions of Theorem 1. It follows from Theorem 1 that equation (51) has a unique 2π-periodic solution $\phi(t)$, which is uniformly asymptotic stable.

Clearly, by the graph of the solution curve, given any initial value $x(0) = x_0$(e.g., $x(0) = 0$), the solution curve of equation (51) tends to the curve of the periodic solution $\gamma(t)$ (see Figure 1).

Example 2. Consider the following equation:

$$\frac{dx}{dt} = (2 + \sin t)x^3 + (1 + \cos t)x^2 + (8 + \cos t)x + 2 + \sin t, \tag{53}$$

here $a(t) = 2 + \sin t$, $b(t) = 1 + \cos t$, $c(t) = 8 + \cos t$, and $d(t) = 2 + \sin t$.

By simple computation, it is obtained that

$$a(t) > 0, b^2(t) - 3a(t)c(t) < 0. \tag{54}$$

Clearly, equation (53) satisfies all the conditions of Theorem 2. It follows from Theorem 2 that equation (53) has a unique 2π-periodic solution $\phi(t)$, which is unstable.

Clearly, by the graph of the solution curve, given any initial value $x(0) = x_0$(e.g., $x(0) = 1$), the solution curve of equation (53) tends to ∞ at some time t (see Figure 2).

5. Conclusion

In this paper, Abel's differential equation (1) is considered and a simple criterion to judge the existence and stability of the unique periodic solution of the equation is obtained.

In fact, condition (H_2) of Theorems 1 or 2 implies that condition $a(t) \neq 0$; thus, the following is obtained.

Corollary 1. *Consider equation (1), and $a(t), b(t), c(t)$, and $d(t)$ are all ω-periodic continuous functions on R, suppose that the following condition holds:*

$$(H_1) b^2(t) - 3a(t)c(t) < 0, \tag{55}$$

then, equation (1) has a unique ω-periodic continuous solution.

It is easy to see that when $b(t) \equiv 0, a(t)c(t) > 0$, Theorems 1 and 2 are also true; thus, the following corollary is obtained.

Corollary 2. *Consider the following Abel's type equation:*

$$\frac{dx}{dt} = a(t)x^3 + c(t)x + d(t), \tag{56}$$

where $a(t), c(t)$, and $d(t)$ are all the ω-periodic continuous functions on R, suppose that the following condition holds:

$$(H_1) a(t)c(t) > 0, \tag{57}$$

then, equation (56) has a unique ω-periodic continuous solution which is uniformly asymptotic stable if $a(t) < 0$ (unstable if $a(t) < 0$).

When $d(t) \equiv 0$, the following is obtained.

Corollary 3. *Consider the following Abel's type equation:*

$$\frac{dx}{dt} = a(t)x^3 + b(t)x^2 + c(t)x, \tag{58}$$

where $a(t), b(t)$, and $c(t)$ are all the ω-periodic continuous functions on R, suppose that the following condition holds:

$$(H_1) b^2(t) - 3a(t)c(t) < 0, \tag{59}$$

then, equation (58) has no nonzero periodic continuous solution, and the zero solution is uniformly asymptotic stable if $a(t) < 0$ (unstable if $a(t) > 0$).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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