Research Article

Some New Tempered Fractional Pólya-Szegő and Chebyshev-Type Inequalities with Respect to Another Function

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In this present article, we establish certain new Pólya-Szegő-type tempered fractional integral inequalities by considering the generalized tempered fractional integral concerning another function Ψ in the kernel. We then prove certain new Chebyshev-type tempered fractional integral inequalities for the said operator with the help of newly established Pólya-Szegő-type tempered fractional integral inequalities. Also, some new particular cases in the sense of classical tempered fractional integrals are discussed. Additionally, examples of constructing bounded functions are considered. Furthermore, one can easily form new inequalities for Katugampola fractional integrals, generalized Riemann-Liouville fractional integral concerning another function Ψ in the kernel, and generalized fractional conformable integral by applying different conditions.

1. Introduction

The well-known Chebyshev functional [1] is defined by

$$\mathcal{T}(f_1, f_2) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f_1(\theta) f_2(\theta) d\theta - \frac{1}{x_2 - x_1} \left(\int_{x_1}^{x_2} f_1(\theta) d\theta \right) \frac{1}{x_2 - x_1} \left(\int_{x_1}^{x_2} f_2(\theta) d\theta \right), \quad (1)$$

where the functions f_1 and f_2 are integrable on $[x_1, x_2]$. If the functions f_1 and f_2 are synchronous on $[x_1, x_2]$, i.e.,

$$(f_1(\vartheta) - f_1(\zeta))(f_2(\vartheta) - f_2(\zeta)) \geq 0, \quad (2)$$

for any $\vartheta, \zeta \in [x_1, x_2]$, then $\mathcal{T}(f_1, f_2) \geq 0$. Functional (1) has gained more recognition due to its diverse applications in the fields of transform theory, numerical quadrature, probability, and statistical problems. Additionally, the researchers have established a large number of integral

inequalities by utilizing functional (1). The interesting readers may consult [2–5]. In [6], Tassaddiq et al. recently established certain inequalities via fractional conformable integrals by considering functional (1).

In [7], Grüss introduced the following inequality:

$$|\mathcal{T}(f_1, f_2)| \leq \frac{(M_1 - m_1)(N_1 - n_1)}{4}, \quad (3)$$

where the functions f_1 and f_2 are integrable on $[x_1, x_2]$ such that f_1 and f_2 satisfy the inequalities $m_1 \leq f_1(\vartheta) \leq M_1$ and $n_1 \leq f_2(\zeta) \leq N_1$, for all $\vartheta, \zeta \in [x_1, x_2]$ and for some constant $m_1, n_1, M_1, N_1 \in \mathbb{R}$.

In [8], Pólya–Szegő presented the following inequality:

$$\frac{\int_{x_1}^{x_2} f_1^2(\theta) d\theta \int_{x_1}^{x_2} f_2^2(\theta) d\theta}{\left(\int_{x_1}^{x_2} f_1(\theta) f_2(\theta) d\theta\right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{M_1 N_1}{m_1 n_1}} + \sqrt{\frac{m_1 n_1}{M_1 N_1}} \right)^2. \quad (4)$$

In [9], Dragomir and Diamond presented the following inequality with the help of Pólya–Szegő inequality:

$$|\mathcal{T}(f_1, f_2)| \leq \frac{(M_1 - m_1)(N_1 - n_1)}{4(x_2 - x_1)^2 \sqrt{m_1 M_1 n_1 N_1}} \int_{x_1}^{x_2} f_1(\theta) d\theta \cdot \int_{x_1}^{x_2} f_2(\theta) d\theta, \quad (5)$$

where the functions f_1 and f_2 are positive and integrable on $[x_1, x_2]$ such that f_1 and f_2 satisfy the inequalities $m_1 \leq f_1(\theta) \leq M_1$ and $n_1 \leq f_2(\zeta) \leq N_1$, for all $\theta, \zeta \in [x_1, x_2]$ and for some constant $m_1, n_1, M_1, N_1 \in \mathbb{R}$.

In the last few decades, the researchers have considered that fractional integral inequalities are the most powerful tools for the development of both applied and pure mathematics. In [10], the authors presented some Grüss-type integral inequalities by considering fractional integrals. Some new integral inequalities in sense of Riemann–Liouville fractional integrals can be found in the work of Dahmani [11].

In [12], Sarikaya et al. gave the idea of generalized (k, s) -fractional integrals with applications. Set et al. [13]

investigated some Grüss-type inequalities by considering generalized k -fractional integrals.

Very recently, the idea of fractional conformable and proportional fractional integral operators was proposed by Jarad et al. [14, 15]. Later on, Huang et al. [16] presented generalized Hermite–Hadamard-type inequalities by considering generalized fractional conformable integrals. In [17], Qi et al. established Chebyshev-type inequalities for generalized fractional conformable integrals.

In [18], Ntouyas et al. investigated some new Pólya–Szegő- and Chebyshev-type inequalities by considering Riemann–Liouville fractional integrals. The tempered fractional integral was first studied by Buschman [19], but Li et al. [20] and Meerschaert et al. [21] have described the associated tempered fractional calculus more explicitly. Fernandez and Ustaoglu [22] investigated several analytic properties of the tempered fractional integral. In [23], Fahad et al. proposed the general form of the generalized tempered fractional integral concerning another function. In this paper, we investigate the said inequalities for the so-called tempered fractional integrals containing another function in the kernel.

The structure of the paper as follows.

In Section 2, some basic definitions are presented. Some new Pólya–Szegő-type for the so-called generalized tempered fractional integral in the sense of another function is presented in Section 3. In Section 4, we present some new generalized Chebyshev-type tempered fractional integral inequalities. In Section 5, certain new particular cases in terms of classical tempered fractional integrals are discussed. An example of constructing bounding functions is considered in Section 6. Finally, the concluding remarks are discussed in Section 7.

2. Preliminaries

In this section, we consider some well-known definitions and mathematical preliminaries.

Definition 1 (see [7]). Suppose that the functions $f_1, f_2: [x_1, x_1] \rightarrow \mathbb{R}$ are positive with $\mathcal{A} \leq f_1(\vartheta) \leq \mathcal{B}$ and $\mathcal{C} \leq f_2(\vartheta) \leq \mathcal{D}$, for all $\vartheta \in [x_1, x_1]$, then the following inequality holds:

$$\left| \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f_1(\vartheta) f_2(\vartheta) d\vartheta - \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f_1(\vartheta) d\vartheta \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f_2(\vartheta) d\vartheta \right| \leq \frac{1}{4} (\mathcal{B} - \mathcal{A})(\mathcal{D} - \mathcal{C}), \quad (6)$$

where the constants $\mathcal{B}, \mathcal{A}, \mathcal{C}, \mathcal{D} \in \mathbb{R}$ and $1/4$ is the sharp of inequality (6).

Definition 2 (see [24, 25]). The function f_1 will be in the space $L_{p,r}[0, \infty[$ if

$$L_{p,r}[0, \infty[= \left\{ f_1: \|f_1\|_{L_{p,r}[0, \infty[} = \left(\int_r^s |f_1(\vartheta)|^p \vartheta^r d\vartheta \right)^{1/p} < \infty, 1 \leq p < \infty, r \geq 0 \right\}. \quad (7)$$

If we apply $r = 0$, then (7) gives

$$L_p[0, \infty[= \left\{ f_1 : \|f_1\|_{L_p[0, \infty[} = \left(\int_0^\infty |f_1(\vartheta)|^p d\vartheta \right)^{1/p} < \infty, 1 \leq p < \infty \right\}. \quad (8)$$

Definition 3 (see [26]). Suppose that the function $f_1 \in L_1[0, \infty[$ and assume that the function Ψ is positive, monotone, and increasing on $[0, \infty[$ and having continuous derivative Ψ' on $[0, \infty[$ with $\Psi(0) = 0$. Then, the Lebesgue real-valued measurable function f_1 defined on $[0, \infty[$ is said to be in the space $X_\Psi^p(0, \infty)$, ($1 \leq p < \infty$) if

$$\|f_1\|_{X_\Psi^p} = \left(\int_0^\infty |f_1(\vartheta)|^p \Psi'(\vartheta) d\vartheta \right)^{1/p} < \infty, 1 \leq p < \infty. \quad (9)$$

When $p = \infty$, then

$$\|f_1\|_{X_\Psi^\infty} = \text{ess sup}_{0 \leq \vartheta < \infty} [\Psi'(\vartheta) f_1(\vartheta)]. \quad (10)$$

Note that, the space $X_\Psi^p(0, \infty)$ coincides with the space $L_p[0, \infty[$ if $\Psi(\vartheta) = \vartheta$ for $1 \leq p < \infty$ and similarly with the space $L_{p,r}[1, \infty[$ if $\Psi(\vartheta) = \ln \vartheta$ for $1 \leq p < \infty$.

Definition 4 (see [20–22]). The left-sided tempered fractional integral of order $\kappa > 0$ and $\tau \geq 0$ with $\Re(\kappa) > 0$ and $\Re(\tau) \geq 0$ is defined by

$$({}_{x_1} \mathcal{R}^{\kappa, \tau} f_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^\theta e^{-\tau(\theta-\vartheta)} (\theta - \vartheta)^{\kappa-1} f_1(\vartheta) d\vartheta, x_1 < \theta. \quad (11)$$

Remark 1. By setting $\tau = 0$ in (11) yields the following Riemann–Liouville fractional integral, which is defined by

$$({}_{x_1}^\Psi \mathcal{R}^{\kappa, \tau} f_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^\theta e^{-\tau(\Psi(\theta)-\Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) d\vartheta, x_1 < \theta, \quad (15)$$

where $\tau \geq 0$, $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$, and $\Gamma(\cdot)$ is the well-known gamma function.

Remark 2. The following results can be obtained:

(i) Applying Definition 6 for $\Psi(\theta) = \theta$, we get (11).

$$({}_{x_1} \mathcal{R}^\kappa f_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^\theta (\theta - \vartheta)^{\kappa-1} f_1(\vartheta) d\vartheta, x_1 < \theta. \quad (12)$$

The following results for (11) hold:

$${}_{x_1} \mathcal{R}^{\kappa, \tau} {}_{x_1} \mathcal{R}^{\lambda, \tau} f_1(\theta) = {}_{x_1} \mathcal{R}^{\kappa+\lambda, \tau} f_1(\theta). \quad (13)$$

We define the following one-sided tempered fractional integral.

Definition 5. The one-sided tempered fractional integral of order $\kappa > 0$, $\tau \geq 0$ is defined by

$$(\mathcal{R}_0^{\kappa, \tau} f_1)(\theta) = (\mathcal{R}^{\kappa, \tau} f_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\theta-\vartheta)} (\theta - \vartheta)^{\kappa-1} f_1(\vartheta) d\vartheta. \quad (14)$$

Definition 6 (see [23]). Let the function f_1 be an integrable in the space $X_\Psi^p(0, \infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0, \infty[$, and its derivative Ψ' is continuous on $[0, \infty[$ with $\Psi(0) = 0$. Then, the left-sided generalized tempered fractional integral of the function f_1 concerning another function Ψ in the kernel is defined by

(ii) Applying Definition 6 for $\tau = 0$, then it will reduce to the left-sided generalized Riemann–Liouville fractional integral operator [27].

(iii) Applying Definition 6 for $\Psi(\theta) = \ln \theta$, then it will reduce to the following left-sided Hadamard tempered integral defined by [23]

$$({}_{x_1}^\Psi \mathcal{R}^{\kappa, \tau} f_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^\theta \exp[-\tau(\ln \theta - \ln \vartheta)] (\ln \theta - \ln \vartheta)^{\kappa-1} \frac{f_1(\vartheta)}{\vartheta} d\vartheta, x_1 < \theta. \quad (16)$$

- (iv) Applying Definition 6 for $\Psi(\theta) = \theta^\eta/\eta$, $\eta > 0$ and $\tau = 0$, then it will reduce to the left-sided Katugampola [24] fractional integral.
- (v) Applying Definition 6 for $\Psi(\theta) = \theta$ and $\tau = 0$, the left Riemann–Liouville fractional integral (12) will be obtained
- (vi) Applying Definition 6 for $\Psi(\theta) = \theta^{\alpha+s}/\alpha + s$ and $\tau = 0$ (where $\alpha \in (0, 1]$, $s \in \mathbb{R}$ and $\alpha + s \neq 0$), then it reduces to the left-sided generalized fractional conformable integral given by [28]

- (vii) Applying Definition 6 for $\Psi(\theta) = (\theta - x_1)^\alpha/\alpha$, $\alpha > 0$, then it reduces to the fractional conformable integral defined by Jarad et al. [14].

In this article, we consider the following one-sided GTF-integral.

Definition 7. Let the function f_1 be integrable in the space $X_\Psi^p(0, \infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0, \infty[$, and its derivative Ψ' is continuous on $[0, \infty[$ with $\Psi(0) = 0$. Then, the one-sided generalized tempered fractional integral of the function f_1 concerning another function Ψ in the kernel is defined by

$$({}^\Psi \mathcal{R}_0^{\kappa, \tau} f_1)(\theta) = \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) d\vartheta. \quad (17)$$

Definition 8. For $0 = x_0 < x_1 < \dots < x_p < x_{p+1} = X$, we define the following subintegrals for the generalized tempered integral

$$({}^\Psi \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau} f_1)(X) = \frac{1}{\Gamma(\kappa)} \int_{x_i}^{x_{i+1}} e^{-\tau(\Psi(X) - \Psi(\vartheta))} (\Psi(X) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) d\vartheta. \quad (18)$$

Note that

$$\begin{aligned} ({}^\Psi \mathcal{R}_0^{\kappa, \tau} f_1)(X) &= \sum_{i=0}^p {}^\Psi \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau} (f_1)(X) = \frac{1}{\Gamma(\kappa)} \int_0^{x_1} e^{-\tau(\Psi(X) - \Psi(\vartheta))} (\Psi(X) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) d\vartheta \\ &\quad + \frac{1}{\Gamma(\kappa)} \int_{x_1}^{x_2} e^{-\tau(\Psi(X) - \Psi(\vartheta))} (\Psi(X) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) d\vartheta \\ &\quad + \dots + \frac{1}{\Gamma(\kappa)} \int_{x_p}^X e^{-\tau(\Psi(X) - \Psi(\vartheta))} (\Psi(X) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1(\vartheta) d\vartheta. \end{aligned} \quad (19)$$

Remark 3. If we set $\Psi(X) = X$ and $\tau = 0$, then (18) will reduce to the subintegrals of Riemann–Liouville fractional integral defined by [18].

3. Pólya–Szegő-Type Tempered Fractional Integral Inequalities

In this section, we provide some new Pólya–Szegő-type tempered fractional integral inequalities for positive and integrable functions via tempered fractional integral (17) containing another function Ψ in the kernel.

Lemma 1. Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$ and assume that the function Ψ is positive,

monotone, and increasing on $[0, \infty[$, and its derivative Ψ' is continuous on $[0, \infty[$ with $\Psi(0) = 0$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$ and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ such that

$$(H_1) 0 < \mathcal{U}_1(\vartheta) \leq f_1(\vartheta) \leq \mathcal{U}_2(\vartheta), 0 < \mathcal{V}_1(\vartheta) \leq f_2(\vartheta) \leq \mathcal{V}_2(\vartheta), \vartheta \in [0, \theta], \theta > 0. \quad (20)$$

Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\frac{{}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{V}_1 \mathcal{V}_2 f_1^2)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{U}_1 \mathcal{U}_2 f_2^2)(\theta)}{({}^\Psi \mathcal{R}_0^{\kappa, \tau} \{(\mathcal{U}_1 \mathcal{V}_1 + \mathcal{U}_2 \mathcal{V}_2) f_1 f_2\}(\theta))^2} \leq \frac{1}{4}. \quad (21)$$

Proof. From the given hypothesis, we have

$$\left(\frac{\mathcal{U}_2(\vartheta)}{\mathcal{V}_1(\vartheta)} - \frac{f_1(\vartheta)}{f_2(\vartheta)} \right) \geq 0. \quad (22)$$

Similarly, we have

$$\left(\frac{f_1(\vartheta)}{f_2(\vartheta)} - \frac{\mathcal{U}_1(\vartheta)}{\mathcal{V}_2(\vartheta)} \right) \geq 0. \quad (23)$$

Taking product of (22) and (23), we get

$$\left(\frac{\mathcal{U}_2(\vartheta)}{\mathcal{V}_1(\vartheta)} - \frac{f_1(\vartheta)}{f_2(\vartheta)} \right) \left(\frac{f_1(\vartheta)}{f_2(\vartheta)} - \frac{\mathcal{U}_1(\vartheta)}{\mathcal{V}_2(\vartheta)} \right) \geq 0. \quad (24)$$

From (24), it can be written as

$$(\mathcal{U}_1(\vartheta)\mathcal{V}_1(\vartheta) + \mathcal{U}_2(\vartheta)\mathcal{V}_2(\vartheta))f_1(\vartheta)f_2(\vartheta) \geq \mathcal{V}_1(\vartheta)\mathcal{V}_2(\vartheta)f_1^2(\vartheta) + \mathcal{U}_1(\vartheta)\mathcal{U}_2(\vartheta)f_2^2(\vartheta). \quad (25)$$

Now, taking product of (25) with $e^{-\tau(\Psi(\theta) - \Psi(\vartheta))}(\Psi(\theta) - \Psi(\vartheta))^{\kappa-1}\Psi'(\vartheta)/\Gamma(\kappa)$ and integrating the resultant identity with respect to ϑ over $(0, \theta)$, we have

$$\begin{aligned} & \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) (\mathcal{U}_1(\vartheta)\mathcal{V}_1(\vartheta) + \mathcal{U}_2(\vartheta)\mathcal{V}_2(\vartheta)) f_1(\vartheta)f_2(\vartheta) d\vartheta \\ & \geq \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) \mathcal{V}_1(\vartheta)\mathcal{V}_2(\vartheta) f_1^2(\vartheta) d\vartheta \\ & \quad + \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) \mathcal{U}_1(\vartheta)\mathcal{U}_2(\vartheta) f_2^2(\vartheta) d\vartheta. \end{aligned} \quad (26)$$

With the aid of Definition 8, we can write

$${}^\Psi \mathcal{R}_0^{\kappa, \tau} [(\mathcal{U}_1\mathcal{V}_1 + \mathcal{U}_2\mathcal{V}_2)f_1f_2](\theta) \geq {}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{V}_1\mathcal{V}_2f_1^2)(\theta) + {}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{U}_1\mathcal{U}_2f_2^2)(\theta). \quad (27)$$

By applying AM-GM inequality, i.e., $x_1 + x_2 \geq 2\sqrt{x_1x_2}$, $x_1, x_2 \in \mathbb{R}^+$, we get

$${}^\Psi \mathcal{R}_0^{\kappa, \tau} [(\mathcal{U}_1\mathcal{V}_1 + \mathcal{U}_2\mathcal{V}_2)f_1f_2](\theta) \geq 2\sqrt{{}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{V}_1\mathcal{V}_2f_1^2)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{U}_1\mathcal{U}_2f_2^2)(\theta)}. \quad (28)$$

It follows that

$$\begin{aligned} & {}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{V}_1\mathcal{V}_2f_1^2)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{U}_1\mathcal{U}_2f_2^2)(\theta) \\ & \leq \frac{1}{4} \left({}^\Psi \mathcal{R}_0^{\kappa, \tau} [(\mathcal{U}_1\mathcal{V}_1 + \mathcal{U}_2\mathcal{V}_2)f_1f_2](\theta) \right)^2, \end{aligned} \quad (29)$$

which gives the desired assertion (21). $\square$

Corollary 1. Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0, \infty[$, and its derivative Ψ' is continuous on $[0, \infty[$ with $\Psi(0) = 0$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$, and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ such that

$$\begin{aligned} (H_2) & 0 < m_1 \leq f_1(\vartheta) \leq M_1 < \infty, 0 < n_1 \leq f_2(\vartheta) \leq N_1 < \infty, \\ & \vartheta \in [0, \theta], \theta > 0. \end{aligned} \quad (30)$$

Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\begin{aligned} & \frac{{}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{V}_1\mathcal{V}_2f_1^2)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau} (\mathcal{U}_1\mathcal{U}_2f_2^2)(\theta)}{({}^\Psi \mathcal{R}_0^{\kappa, \tau} \{f_1f_2\}(\theta))^2} \\ & \leq \frac{1}{4} \left(\sqrt{\frac{m_1n_1}{M_1N_1}} + \sqrt{\frac{M_1N_1}{m_1n_1}} \right)^2. \end{aligned} \quad (31)$$

Lemma 2. Let all the conditions of Lemma 1 hold. Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\frac{{}^\Psi \mathcal{R}_0^{\kappa, \tau}(f_1^2)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(f_2^2)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{V}_1 \mathcal{V}_2)(\theta)}{({}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_1 f_1)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{V}_1 f_2)(\theta) + {}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_2 f_1)(\theta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{V}_2 f_2)(\theta))^2} \leq \frac{1}{4} \quad (32)$$

Proof. From the hypothesis (H_1) defined by (20), we have

$$\left(\frac{\mathcal{U}_2(\vartheta)}{\mathcal{V}_1(\zeta)} - \frac{f_1(\vartheta)}{f_2(\zeta)} \right) \geq 0, \quad (33)$$

$$\left(\frac{f_1(\vartheta)}{f_2(\zeta)} - \frac{\mathcal{U}_1(\vartheta)}{\mathcal{V}_2(\zeta)} \right) \geq 0, \quad (34)$$

which follows that

$$\left(\frac{\mathcal{U}_1(\vartheta)}{\mathcal{V}_2(\zeta)} + \frac{\mathcal{U}_2(\vartheta)}{\mathcal{V}_1(\zeta)} \right) \frac{f_1(\vartheta)}{f_2(\zeta)} \geq \frac{f_1^2(\vartheta)}{f_2^2(\zeta)} + \frac{\mathcal{U}_1(\vartheta) \mathcal{U}_2(\vartheta)}{\mathcal{V}_1(\zeta) \mathcal{V}_2(\zeta)}. \quad (35)$$

Taking product on both sides of (18) by $\mathcal{V}_1(\zeta) \mathcal{V}_2(\zeta) f_2^2(\zeta)$, we obtain

$$\begin{aligned} & \mathcal{U}_1(\vartheta) f_1(\vartheta) \mathcal{V}_1(\zeta) f_2(\zeta) + \mathcal{U}_2(\vartheta) f_1(\vartheta) \mathcal{V}_2(\zeta) f_2(\zeta) \\ & \geq \mathcal{V}_1(\zeta) \mathcal{V}_2(\zeta) f_1^2(\vartheta) + \mathcal{U}_1(\vartheta) \mathcal{U}_2(\vartheta) f_2^2(\zeta). \end{aligned} \quad (36)$$

Taking product of both sides of (36) with $e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) / \Gamma(\kappa)$ and integrating the resultant identity with respect to ϑ over $(0, \theta)$, we have

$$\begin{aligned} & \mathcal{V}_1(\zeta) f_2(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) \mathcal{U}_1(\vartheta) f_1(\vartheta) d\vartheta \\ & + \mathcal{V}_2(\zeta) f_2(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) \mathcal{U}_2(\vartheta) f_1(\vartheta) d\vartheta \\ & \geq \mathcal{V}_1(\zeta) \mathcal{V}_2(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1^2(\vartheta) d\vartheta \\ & + f_2^2(\zeta) \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) \mathcal{U}_1(\vartheta) \mathcal{U}_2(\vartheta) d\vartheta, \end{aligned} \quad (37)$$

which by applying (8) becomes

$$\begin{aligned} & \mathcal{V}_1(\zeta) f_2(\zeta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_1 f_1)(\theta) + \mathcal{V}_2(\zeta) f_2(\zeta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_2 f_1)(\theta) \\ & \geq \mathcal{V}_1(\zeta) \mathcal{V}_2(\zeta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(f_1^2)(\theta) + f_2^2(\zeta) {}^\Psi \mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta). \end{aligned} \quad (38)$$

Again, taking product of both sides of (38) with $e^{-\tau(\Psi(\theta) - \Psi(\zeta))} (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\zeta) / \Gamma(\lambda)$ and integrating the resultant identity with respect to ζ over $(0, \theta)$ and then applying (17), we get

$$\begin{aligned} & {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_1 f_2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{U}_1 f_1)(\theta) + {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_2 f_2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{U}_2 f_1)(\theta) \\ & \geq {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_1 \mathcal{V}_2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_1^2)(\theta) + {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_2^2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta). \end{aligned} \quad (39)$$

By using AM-GM inequality, we get

$$\begin{aligned} & {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_1 f_2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{U}_1 f_1)(\theta) + {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_2 f_2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{U}_2 f_1)(\theta) \\ & \geq 2 \sqrt{{}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_1 \mathcal{V}_2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_1^2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_2^2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta)}. \end{aligned} \quad (40)$$

It follows that

$$\begin{aligned} & {}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(\mathcal{V}_1\mathcal{V}_2)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_1^2)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_2^2)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(\mathcal{U}_1\mathcal{U}_2)(\theta) \\ & \leq \frac{1}{4}\left({}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(\mathcal{V}_1f_2)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(\mathcal{U}_1f_1)(\theta) + {}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(\mathcal{V}_2f_2)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(\mathcal{U}_2f_1)(\theta)\right)^2, \end{aligned} \quad (41)$$

which completes the desired assertion (17). $\square$

Corollary 2. Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$ satisfying the hypothesis (H_2) defined by (16) and assume that the function Ψ is positive, monotone, and increasing on $[0, \infty[$, and its derivative Ψ' is continuous on $[0, \infty[$ with $\Psi(0) = 0$. Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\begin{aligned} & \frac{\gamma(\kappa, \tau\Psi(\theta))\gamma(\kappa, \tau\Psi(\theta))}{\tau^{\kappa+\lambda}\Gamma(\kappa)\Gamma(\lambda)} \frac{{}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1^2)(\theta){}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_2^2)(\theta)}{\left({}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta)\right)^2} \\ & \leq \frac{1}{4} \left(\sqrt{\frac{m_1n_1}{M_1N_1}} + \sqrt{\frac{M_1N_1}{m_1n_1}} \right)^2, \end{aligned} \quad (42)$$

where

$$\begin{aligned} {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}[1] &= \frac{1}{\Gamma(\kappa)} \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) d\vartheta \\ &= \frac{1}{\Gamma(\kappa)} \int_0^{\Psi(\theta)} e^{-\tau z} z^{\kappa-1} dz, \quad (\Psi(0) = 0) \\ &= \frac{1}{\tau^\kappa \Gamma(\kappa)} \int_0^{\tau\Psi(\theta)} e^{-u} u^{\kappa-1} du \\ &= \frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^\kappa \Gamma(\kappa)}, \end{aligned} \quad (43)$$

and $\gamma(\kappa, t) = \int_0^t e^{-u} u^{\kappa-1} du$ is the well-known incomplete gamma function (see [22]).

Lemma 3. Suppose that all the conditions of Lemma 1 hold and assume that the function Ψ is positive, monotone, and

increasing on $[0, \infty[$, and its derivative Ψ' is continuous on $[0, \infty[$ with $\Psi(0) = 0$. Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$${}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1^2)(\theta){}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_2^2)(\theta) \leq {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}\left(\frac{\mathcal{U}_2f_1f_2}{\mathcal{V}_1}\right)(\theta){}^{\Psi}\mathcal{R}_0^{\kappa,\tau}\left(\frac{\mathcal{V}_2f_1f_2}{\mathcal{U}_1}\right)(\theta). \quad (44)$$

Proof. From the hypothesis (H_1) defined by (20), we have

$$\begin{aligned} & \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) f_1^2(\vartheta) d\vartheta \\ & \leq \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\vartheta))} (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \Psi'(\vartheta) \frac{\mathcal{U}_2(\vartheta)}{\mathcal{V}_1(\vartheta)} f_1(\vartheta) f_2(\vartheta) d\vartheta, \end{aligned} \quad (45)$$

which in view of (17) yields

$${}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1^2)(\theta) \leq {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}\left(\frac{\mathcal{U}_2 f_1 f_2}{\mathcal{V}_1}\right)(\theta). \quad (46)$$

Similarly, one can obtain

$${}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_2^2)(\theta) \leq {}^{\Psi}\mathcal{R}_0^{\lambda,\tau}\left(\frac{\mathcal{V}_2 f_1 f_2}{\mathcal{U}_1}\right)(\theta). \quad (47)$$

Hence, the product of (46) and (47) yields the desired assertion (44). $\square$

Corollary 3. Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$ satisfying the hypothesis (H_2) defined by (30) and assume that the function Ψ is positive, monotone, and increasing on $[0, \infty[$, and its derivative Ψ' is continuous on $[0, \infty[$ with $\Psi(0) = 0$. Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\frac{{}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1^2)(\theta){}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_2^2)(\theta)}{{}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1 f_2)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_1 f_2)(\theta)} \leq \frac{M_1 N_1}{m_1 n_1}. \quad (48)$$

$$\begin{aligned} & \left| \frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)} {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1 f_2)(\theta) + \frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^{\lambda}\Gamma(\lambda)} {}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_1 f_2)(\theta) - {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta) - {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_1)(\theta) \right| \\ & \leq |F_1(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) + F_2(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta)|^{1/2} \times |F_1(f_2, \mathcal{V}_1, \mathcal{V}_2)(\theta) + F_2(f_1, \mathcal{V}_1, \mathcal{V}_2)(\theta)|^{1/2}, \end{aligned} \quad (49)$$

where

$$F_1(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) = \frac{\gamma(\lambda, \tau\Psi(\theta))}{4\tau^{\lambda}\Gamma(\lambda)} \frac{({}^{\Psi}\mathcal{R}_0^{\kappa,\tau}\{(\mathcal{U}_1 + \mathcal{U}_2)f_1\})^2}{{}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta)} - {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta), \quad (50)$$

$$F_2(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) = \frac{\gamma(\kappa, \tau\Psi(\theta))}{4\tau^{\kappa}\Gamma(\kappa)} \frac{({}^{\Psi}\mathcal{R}_0^{\lambda,\tau}\{(\mathcal{U}_1 + \mathcal{U}_2)f_1\})^2}{{}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta)} - {}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_1)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_1)(\theta). \quad (51)$$

Proof. By the given hypothesis, both the functions f_1 and f_2 are positive and integrable functions on $[0, \infty)$. Therefore, for $\vartheta, \zeta \in (0, \theta)$ with $\theta > 0$, we define $\mathcal{A}(\vartheta, \zeta)$ by

$$\begin{aligned} \mathcal{A}(\vartheta, \zeta) &= (f_1(\vartheta) - f_1(\zeta))(f_2(\vartheta) - f_2(\zeta)) \\ &= f_1(\vartheta)f_2(\vartheta) + f_1(\zeta)f_2(\zeta) - f_1(\vartheta)f_2(\zeta) - f_1(\zeta)f_2(\vartheta). \end{aligned} \quad (52)$$

Multiplying (52) by $(1/\Gamma(\kappa)\Gamma(\lambda))\exp[-\tau(\Psi(\theta) - \Psi(\vartheta))]\exp[-\tau(\Psi(\theta) - \Psi(\zeta))](\Psi(\theta) - \Psi(\vartheta))^{\kappa-1}(\Psi(\theta) - \Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)$ and double integrating the resultant identity with respect to ϑ and ζ over $(0, \theta)$ and then using (17), we obtain

4. Chebyshev-Type Tempered Fractional Integral Inequalities

In this section, certain Chebyshev-type inequalities via tempered fractional integral (20) are presented with the help of Pólya–Szegő integral inequality given by Lemma 1.

Theorem 1. Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$ and assume that the function Ψ is positive, monotone, and increasing on $[0, \infty[$, and its derivative Ψ' is continuous on $[0, \infty[$ with $\Psi(0) = 0$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$, and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ satisfying the hypothesis (H_1) defined by (10). Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\begin{aligned} & \frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))]\exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \\ & \quad \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1}(\Psi(\theta) - \Psi(\zeta))^{\lambda-1}\Psi'(\vartheta)\Psi'(\zeta)\mathcal{A}(\vartheta, \zeta)d\vartheta d\zeta \\ &= \frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^{\kappa}\Gamma(\kappa)} {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1 f_2)(\theta) + \frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^{\lambda}\Gamma(\lambda)} {}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_1 f_2)(\theta) \\ & \quad - {}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta){}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_2)(\theta) - {}^{\Psi}\mathcal{R}_0^{\lambda,\tau}(f_1)(\theta){}^{\Psi}\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta). \end{aligned} \quad (53)$$

By applying Cauchy–Schwartz inequality for double integrals, we have

$$\begin{aligned}
& \left| \frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] \exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \right. \\
& \quad \cdot (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\vartheta) \Psi'(\zeta) \mathcal{A}(\vartheta, \zeta) d\vartheta d\zeta \Big| \\
& \leq \left[\frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] \exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \right. \\
& \quad \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\vartheta) \Psi'(\zeta) f_1^2(\vartheta) d\vartheta d\zeta \\
& \quad + \frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] \exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \\
& \quad \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\vartheta) \Psi'(\zeta) f_1^2(\zeta) d\vartheta d\zeta \\
& \quad - 2 \frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] \exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \\
& \quad \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\vartheta) \Psi'(\zeta) f_1(\vartheta) f_1(\zeta) d\vartheta d\zeta \Big]^{1/2} \\
& \quad \times \left[\frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] \exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \right. \\
& \quad \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\vartheta) \Psi'(\zeta) f_2^2(\vartheta) d\vartheta d\zeta \\
& \quad + \frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] \exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \\
& \quad \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\vartheta) \Psi'(\zeta) f_2^2(\zeta) d\vartheta d\zeta \\
& \quad - 2 \frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] \exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \\
& \quad \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\vartheta) \Psi'(\zeta) f_2(\vartheta) f_2(\zeta) d\vartheta d\zeta \Big]^{1/2}.
\end{aligned} \tag{54}$$

In view of (17) and (43), we get

$$\begin{aligned}
& \left| \frac{1}{\Gamma(\kappa)\Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau(\Psi(\theta) - \Psi(\vartheta))] \exp[-\tau(\Psi(\theta) - \Psi(\zeta))] \times (\Psi(\theta) - \Psi(\vartheta))^{\kappa-1} \right. \\
& \quad \cdot (\Psi(\theta) - \Psi(\zeta))^{\lambda-1} \Psi'(\vartheta) \Psi'(\zeta) \mathcal{A}(\vartheta, \zeta) d\vartheta d\zeta \Big| \\
& \leq \left[\frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^\kappa \Gamma(\kappa)} {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_1^2)(\theta) + \frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^\lambda \Gamma(\lambda)} {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_1^2)(\theta) - 2 {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_1)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_1)(\theta) \right]^{1/2} \\
& \quad \times \left[\frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^\kappa \Gamma(\kappa)} {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_2^2)(\theta) + \frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^\lambda \Gamma(\lambda)} {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_2^2)(\theta) - 2 {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_2)(\theta) {}^\Psi \mathcal{R}_0^{\lambda, \tau}(f_2)(\theta) \right]^{1/2}.
\end{aligned} \tag{55}$$

Applying Lemma 1 for $\mathcal{V}_1(\theta) = \mathcal{V}_2(\theta) = f_2(\theta) = 1$, we get

$$\frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^\lambda\Gamma(\lambda)} {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1^2)(\theta) \leq \frac{\gamma(\lambda, \tau\Psi(\theta))}{4\tau^\lambda\Gamma(\lambda)} \frac{({}^\Psi\mathcal{R}_0^{\kappa,\tau}\{(\mathcal{U}_1 + \mathcal{U}_2)f_1\})^2}{{}^\Psi\mathcal{R}_0^{\kappa,\tau}(\mathcal{U}_1\mathcal{U}_2)(\theta)}. \quad (56)$$

It follows that

$$\begin{aligned} & \frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^\lambda\Gamma(\lambda)} {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1^2)(\theta) - {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta) {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta) \\ & \leq \frac{\gamma(\lambda, \tau\Psi(\theta))}{4\tau^\lambda\Gamma(\lambda)} \frac{({}^\Psi\mathcal{R}_0^{\kappa,\tau}\{(\mathcal{U}_1 + \mathcal{U}_2)f_1\})^2}{{}^\Psi\mathcal{R}_0^{\kappa,\tau}(\mathcal{U}_1\mathcal{U}_2)(\theta)} \\ & \quad - {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta) {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta) \\ & = F_1(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta). \end{aligned} \quad (57)$$

Similarly, one can get

$$\begin{aligned} & \frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^\kappa\Gamma(\kappa)} {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_1^2)(\theta) - {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_1)(\theta) {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_1)(\theta) \\ & \leq \frac{\gamma(\kappa, \tau\Psi(\theta))}{4\tau^\kappa\Gamma(\kappa)} \frac{({}^\Psi\mathcal{R}_0^{\lambda,\tau}\{(\mathcal{U}_1 + \mathcal{U}_2)f_1\})^2}{{}^\Psi\mathcal{R}_0^{\lambda,\tau}(\mathcal{U}_1\mathcal{U}_2)(\theta)} \\ & \quad - {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_1)(\theta) {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_1)(\theta) \\ & = F_2(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta). \end{aligned} \quad (58)$$

Again applying Lemma 1 for $\mathcal{U}_1(\theta) = \mathcal{U}_2(\theta) = f_1(\theta) = 1$, we get

$$\begin{aligned} & \frac{\gamma(\lambda, \tau\Psi(\theta))}{\tau^\lambda\Gamma(\lambda)} {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2^2)(\theta) - {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta) {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta) \\ & \leq \frac{\gamma(\lambda, \tau\Psi(\theta))}{4\tau^\lambda\Gamma(\lambda)} \frac{({}^\Psi\mathcal{R}_0^{\kappa,\tau}\{(\mathcal{V}_1 + \mathcal{V}_2)f_2\})^2}{{}^\Psi\mathcal{R}_0^{\kappa,\tau}(\mathcal{V}_1\mathcal{V}_2)(\theta)} \\ & \quad - {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta) {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta) \\ & = F_1(f_2, \mathcal{U}_1, \mathcal{U}_2)(\theta), \end{aligned} \quad (59)$$

$$\begin{aligned} & \frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^\kappa\Gamma(\kappa)} {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_2^2)(\theta) - {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_2)(\theta) {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_2)(\theta) \\ & \leq \frac{\gamma(\kappa, \tau\Psi(\theta))}{4\tau^\kappa\Gamma(\kappa)} \frac{({}^\Psi\mathcal{R}_0^{\lambda,\tau}\{(\mathcal{V}_1 + \mathcal{V}_2)f_2\})^2}{{}^\Psi\mathcal{R}_0^{\lambda,\tau}(\mathcal{V}_1\mathcal{V}_2)(\theta)} \\ & \quad - {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_2)(\theta) {}^\Psi\mathcal{R}_0^{\lambda,\tau}(f_2)(\theta) \\ & = F_2(f_2, \mathcal{U}_1, \mathcal{U}_2)(\theta). \end{aligned} \quad (60)$$

Thus, by considering (53) to (60), we arrive at the desired assertion (49). This is the desired proof of Theorem 1. $\square$

Theorem 2. Suppose that all the conditions of Theorem 1 are satisfied. Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\begin{aligned} & \left| \frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^\kappa\Gamma(\kappa)} {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1f_2)(\theta) - {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta) {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta) \right| \\ & \leq |F(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta)F(f_2, \mathcal{V}_1, \mathcal{V}_2)(\theta)|^{1/2}, \end{aligned} \quad (61)$$

where

$$\begin{aligned} F(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) &= \frac{\gamma(\kappa, \tau\Psi(\theta))}{4\tau^\kappa\Gamma(\kappa)} \frac{({}^\Psi\mathcal{R}_0^{\kappa,\tau}\{(\mathcal{U}_1 + \mathcal{U}_2)f_1\})^2}{{}^\Psi\mathcal{R}_0^{\kappa,\tau}(\mathcal{U}_1\mathcal{U}_2)(\theta)} \\ & \quad - ({}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta))^2. \end{aligned} \quad (62)$$

Proof. Applying Theorem 1 for $\kappa = \lambda$, we get the desired result in (61). $\square$

Remark 4. If we consider $\mathcal{U}_1 = m_1$, $\mathcal{U}_2 = M_1$, $\mathcal{V}_1 = n_1$, and $\mathcal{V}_2 = N_1$, then we have

$$F(f_1, m_1, M_1)(\theta) = \frac{(M_1 - m_1)^2}{4M_1m_1} ({}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta))^2, \quad (63)$$

$$F(f_2, m_1, M_1)(\theta) = \frac{(N_1 - n_1)^2}{4N_1n_1} ({}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta))^2. \quad (64)$$

Corollary 4. Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$ and satisfying the hypothesis (H_2) given by (30). Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\begin{aligned} & \left| \frac{\gamma(\kappa, \tau\Psi(\theta))}{\tau^\kappa\Gamma(\kappa)} {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1f_2)(\theta) - {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta) {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta) \right| \\ & \leq \frac{(M_1 - m_1)(N_1 - n_1)}{4\sqrt{m_1M_1n_1N_1}} {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_1)(\theta) {}^\Psi\mathcal{R}_0^{\kappa,\tau}(f_2)(\theta). \end{aligned} \quad (65)$$

5. Particular Cases

The following new Pólya–Szegő- and Chebyshev-type inequalities for classical tempered fractional integral (14) can be easily established.

Lemma 4. Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$, and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ satisfying the hypothesis (H_1) defined by (20). Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\frac{\mathcal{R}_0^{\kappa, \tau}(f_1^2)(\theta) \mathcal{R}_0^{\lambda, \tau}(f_2^2)(\theta) \mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta) \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_1 \mathcal{V}_2)(\theta)}{(\mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_1 f_1)(\theta) \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_1 f_2)(\theta) + \mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_2 f_1)(\theta) \mathcal{R}_0^{\lambda, \tau}(\mathcal{V}_2 f_2)(\theta))^2} \leq \frac{1}{4}. \quad (67)$$

Proof. Applying Lemma 2 for $\Psi(\theta) = \theta$, we get Lemma 5. $\square$

Similarly, we can derive the particular case of Lemma 3. The following theorem represents the particular case of Theorem 1 in terms of classical tempered fractional integral.

Proof. Applying Lemma 1 for $\Psi(\theta) = \theta$, we get Lemma 4. $\square$

Lemma 5. Let all the conditions of Lemma 4 are satisfied. Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

Theorem 3. Let the functions f_1 and f_2 be positive and integrable on $[0, \infty)$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$, and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ satisfying the hypothesis (H_1) defined by (20). Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\left| \frac{\gamma(\kappa, \tau\theta)}{\tau^\kappa \Gamma(\kappa)} \mathcal{R}_0^{\kappa, \tau}(f_1 f_2)(\theta) + \frac{\gamma(\lambda, \tau\theta)}{\tau^\lambda \Gamma(\lambda)} \mathcal{R}_0^{\lambda, \tau}(f_1 f_2)(\theta) - \mathcal{R}_0^{\kappa, \tau}(f_1)(\theta) \mathcal{R}_0^{\lambda, \tau}(f_2)(\theta) - \mathcal{R}_0^{\kappa, \tau}(f_2)(\theta) \mathcal{R}_0^{\lambda, \tau}(f_1)(\theta) \right| \leq |F_1(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) + F_2(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta)|^{1/2} \times |F_1(f_2, \mathcal{V}_1, \mathcal{V}_2)(\theta) + F_2(f_2, \mathcal{V}_1, \mathcal{V}_2)(\theta)|^{1/2}, \quad (68)$$

where

$$F_1(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) = \frac{\gamma(\lambda, \tau\theta)}{4\tau^\lambda \Gamma(\lambda)} \frac{(\mathcal{R}_0^{\kappa, \tau}\{(\mathcal{U}_1 + \mathcal{U}_2)f_1\})^2}{\mathcal{R}_0^{\kappa, \tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta)} - \mathcal{R}_0^{\kappa, \tau}(f_1)(\theta) \mathcal{R}_0^{\lambda, \tau}(f_1)(\theta), \quad (69)$$

$$F_2(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) = \frac{\gamma(\kappa, \tau\theta)}{4\tau^\kappa \Gamma(\kappa)} \frac{(\mathcal{R}_0^{\lambda, \tau}\{(\mathcal{U}_1 + \mathcal{U}_2)f_1\})^2}{\mathcal{R}_0^{\lambda, \tau}(\mathcal{U}_1 \mathcal{U}_2)(\theta)} - \mathcal{R}_0^{\kappa, \tau}(f_1)(\theta) \mathcal{R}_0^{\lambda, \tau}(f_1)(\theta). \quad (70)$$

Proof. Applying Theorem 1 for $\Psi(\theta) = \theta$, then we get the desired Theorem 3. $\square$

Similarly, we can derive particular result of Theorem 2.

6. Applications

In this section, we define a way for constructing four bounded functions and then utilize them to present certain estimates of Chebyshev-type tempered fractional integral inequalities of two unknown functions.

Let the unit function $h(\theta)$ be defined by

$$h(\theta) = \begin{cases} 1, & \theta > 0, \\ 0, & \theta \leq 0, \end{cases} \quad (71)$$

and let the Heaviside unit step function $h_a(\theta)$ be defined by

$$h_a(\theta) = \begin{cases} 1, & \theta > a, \\ 0, & \theta \leq a. \end{cases} \quad (72)$$

Suppose that the function $\mathcal{U}_1$ is piecewise continuous function on $[0, X]$ defined by

$$\begin{aligned}
\mathcal{U}_1(x) &= m_{1_1}(\hbar_0(x) - \hbar_{x_1}(x)) + m_{1_2}(\hbar_{x_1}(x) - \hbar_{x_2}(x)) + \cdots + m_{1_{p+1}}\hbar_{x_p}(x) \\
&= m_{1_1}\hbar_0(x) + (m_{1_2} - m_{1_1})\hbar_{x_1}(x) + (m_{1_3} - m_{1_2})\hbar_{x_2}(x) + \cdots + (m_{1_{p+1}} - m_{1_p})\hbar_{x_p}(x) \\
&= \sum_{i=0}^p (m_{1_{i+1}} - m_{1_i})\hbar_{x_i}(x),
\end{aligned} \tag{73}$$

where $m_{1_0} = 0$ and $0 = x_0 < x_1 < x_2 < \cdots < x_p < x_{p+1} = X$. Similarly, we define

$$\mathcal{U}_2(x) = \sum_{i=0}^p (M_{1_{i+1}} - M_{1_i})\hbar_{x_i}(x), \tag{74}$$

$$\mathcal{V}_1(x) = \sum_{i=0}^p (n_{1_{i+1}} - n_{1_i})\hbar_{x_i}(x), \tag{75}$$

$$\mathcal{V}_2(x) = \sum_{i=0}^p (N_{1_{i+1}} - N_{1_i})\hbar_{x_i}(x), \tag{76}$$

where the constants $n_{1_0} = N_{1_0} = M_{1_0} = 0$. If there exists an integrable function f_1 on $[0, X]$ satisfying the hypothesis (H_1) , then we have $m_{1_{i+1}} \leq f_1(x) \leq M_{1_{i+1}}$ for each $x \in (x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, p$.

Proposition 1. Let the functions f_1 and f_2 be two positive and integrable on $[0, X]$. Assume that the functions $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}_1$, and $\mathcal{V}_2$ are defined by (73)–(76), respectively, and satisfying the hypothesis (H_1) defined by (30). Then, for $\kappa > 0$, the following inequality for tempered fractional integral holds:

$$\begin{aligned}
&\left(\sum_{i=0}^p n_{1_{i+1}} N_{1_{i+1}} {}^\Psi \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_1^2)(X) \right)^2 \left(\sum_{i=0}^p m_{1_{i+1}} M_{1_{i+1}} {}^\Psi \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_2^2)(X) \right)^2 \\
&\leq \frac{1}{4} \sum_{i=0}^p (n_{1_{i+1}} N_{1_{i+1}} + m_{1_{i+1}} M_{1_{i+1}}) {}^\Psi \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_1 f_2)(X).
\end{aligned} \tag{77}$$

Proof. By applying the Definition 8, we have

$${}^\Psi \mathcal{R}_{0, X}^{\kappa, \tau}(\mathcal{V}_1 \mathcal{V}_2 f_1^2)(X) = \sum_{i=0}^p n_{1_{i+1}} N_{1_{i+1}} {}^\Psi \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_1^2)(X), \tag{78}$$

$${}^\Psi \mathcal{R}_{0, X}^{\kappa, \tau}(\mathcal{U}_1 \mathcal{U}_2 f_2^2)(X) = \sum_{i=0}^p m_{1_{i+1}} M_{1_{i+1}} {}^\Psi \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_2^2)(X), \tag{79}$$

$${}^\Psi \mathcal{R}_{0, X}^{\kappa, \tau}\{(\mathcal{U}_1 \mathcal{V}_1 + \mathcal{U}_2 \mathcal{V}_2) f_1 f_2\}(X) = \sum_{i=0}^p (m_{1_{i+1}} n_{1_{i+1}} + M_{1_{i+1}} N_{1_{i+1}}) {}^\Psi \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_1 f_2)(X). \tag{80}$$

Hence, by applying Lemma 1, we get the desired assertion (77). $\square$

Proposition 2. By setting $\Psi(\theta) = \theta$ in Proposition 1, then we arrive to the following result in terms of classical tempered fractional integral:

$$\begin{aligned}
&\left(\sum_{i=0}^p n_{1_{i+1}} N_{1_{i+1}} \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_1^2)(X) \right)^2 \left(\sum_{i=0}^p m_{1_{i+1}} M_{1_{i+1}} \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_2^2)(X) \right)^2 \\
&\leq \frac{1}{4} \sum_{i=0}^p (n_{1_{i+1}} N_{1_{i+1}} + m_{1_{i+1}} M_{1_{i+1}}) \mathcal{R}_{x_i, x_{i+1}}^{\kappa, \tau}(f_1 f_2)(X).
\end{aligned} \tag{81}$$

Remark 5. Throughout in the paper, if we apply $\Psi(\theta) = \theta$ and $\tau = 0$, then all the newly presented inequalities will be reduced to the work derived earlier by Ntouyas et al. [18].

7. Concluding Remarks

Certain new Pólya–Szegő- and Chebyshev-type inequalities by utilizing tempered fractional integral are presented in this paper. These inequalities generalized the existing inequalities. We can easily get the said Pólya–Szegő- and Chebyshev-type inequalities for Katugampola, generalized Riemann–Liouville, classical Riemann–Liouville, generalized conformable, and conformable fractional integrals by applying different conditions on function Ψ given in Remark 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally, and they have read and approved the final manuscript for publication.

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