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*Research Article*

# On the Hybrid Power Mean Involving the Character Sums and Dedekind Sums

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The main purpose of this paper is to use the elementary and analytic methods, the properties of Gauss sums, and character sums to study the computational problem of a certain hybrid power mean involving the Dedekind sums and a character sum analogous to Kloosterman sum and give two interesting identities for them.

## 1. Introduction

We all know that the classical Dedekind sums $S(h,q)$ is defined (see [1]) as

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right), \tag{1}$$

where $q\geq 2$ is a positive integer, $h$ is any integer prime to $q$, and

$$((x))=\begin{cases} x-[x]-\dfrac{1}{2}, & \text{if } x \text{ is not an integer,}\\ 0, & \text{if } x \text{ is an integer.}\end{cases} \tag{2}$$

This sum describes the behaviour of the logarithm of the eta function under modular transformations, see [1, 2], for related references. Because of the importance of this sum in analytic number theory, many scholars have studied its various properties and obtained a series of important results. Perhaps, the most important property of $S(h,q)$ is its reciprocity theorem (see [3]). That is, for any positive integers $h$ and $q$ with $(h,q)=1$, one has the identity

$$S(h,q)+S(q,h)=\frac{h^2+k^2+1}{12hq}-\frac{1}{4}. \tag{3}$$

Some other papers related to Dedekind sums can be found in [4–6], and we do not want to list them all here.

On the contrary, we also introduce another character sums analogous to Kloosterman sums as follows. For any integer $q\geq 3$, let $\chi$ be a Dirichlet character $\bmod q$. For any positive integer $k$ and integer $h$, we define

$$G(k,h,\chi;q)=\sideset{}{'}\sum_{a_1=1}^{q}\sideset{}{'}\sum_{a_2=1}^{q}\ldots\sum_{a_k=1}^{q}\chi\left(a_1+a_2+\ldots+a_k+h\overline{a_1\cdot a_2\ldots a_k}\right), \tag{4}$$

where $\sum_{a=1}^{q}{}'$ denotes the summation over all $1 \le a \le q$ such that $(a, q) = 1$ and $\overline{a}$ denotes the inverse of $a$. That is, $a \cdot \overline{a} \equiv 1 \bmod q$.

About the properties of $G(k, h, \chi; q)$, some people had studied it and obtained some important results. For example, from the very special case of Weil's work [7] one can obtain the estimate

$$\left|\sum_{a=1}^{p-1} \chi(a + m\overline{a})\right| \le 2\sqrt{p}, \tag{5}$$

where $p$ is a prime and $m$ is any integer. Some related important works can also be found in [7–11].

In this paper, we consider the computational problem of the hybrid power mean involving the Dedekind sums $S(h, q)$ and $G(k, h, \chi; q)$. That is,

$$\sum_{h=1}^{q}{}' G(k, h, \chi; q) \cdot S(h, q). \tag{6}$$

However, for this hybrid power mean, it seems that none has studied it yet; at least, we have not seen any related results before. The problem is interesting because it is closely related to Dirichlet L-functions. In fact, for some special positive integers $k$, we can give an exact computational formula for (6). The main work of this paper is to reveal this point. That is, we shall use the elementary and analytic methods, and the properties of character sums to prove the following two conclusions.

**Theorem 1.** *Let $p$ be an odd prime with $p \equiv 3 \bmod 4$, and $\chi_2 = (*/p)$ denotes Legendre's symbol* mod *$p$. Then, for any positive integer $k$ with $(2k+1, p-1) = 1$, we have the identity*

$$\sum_{h=1}^{p-1} G(2k, h, \chi_2; p) \cdot S(h, p) = (-1)^k \cdot p^k \cdot h_p^2, \tag{7}$$

*where $h_p$ denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.*

**Theorem 2.** *Let $p$ be an odd prime with $p \equiv 1 \bmod 8$ and $k$ be any positive integer with $(k, p-1) = 1$. Then, we have the identity*

$$\sum_{h=1}^{p-1} G(2k-1, h, \chi_2; p) \cdot S(h, p) = 0. \tag{8}$$

If $p$ be an odd prime with $p \equiv 5 \bmod 8$, then we have the identity

$$\sum_{h=1}^{p-1} G(2k-1, h, \chi_2; p) \cdot S(h, p) = \frac{1}{\pi^2} \cdot p^{(k+1)/2} \cdot \left|L(1, \chi_4)\right|^2 \cdot \sum_{h=0}^{[k/2]} (-1)^h \cdot \frac{k}{k-h} \cdot \binom{k-h}{h} \cdot (2\alpha)^{k-2h} \cdot p^h, \tag{9}$$

*where $\chi_4$ denotes the fourth-order character* mod *$p$, $\alpha = \sum_{a=1}^{(p-1)/2} ((a + \overline{a})/p)$ is an integer, and $L(s, \chi)$ denotes the Dirichlet L-function corresponding to $\chi$.*

Taking $k = 1$ in Theorem 1 and Theorem 2, then we have the following.

**Corollary 1.** *Let $p$ be an odd prime with $p \equiv 3 \bmod 4$ and $(3, p-1) = 1$; then, we have the identity*

$$\sum_{h=1}^{p-1} G(2, h, \chi_2; p) \cdot S(h, p) = -p \cdot h_p^2. \tag{10}$$

**Corollary 2.** *Let $p$ be an odd prime with $p \equiv 5 \bmod 8$; then, we have*

$$\sum_{h=1}^{p-1} G(1, h, \chi_2; p) \cdot S(h, p) = \frac{2\alpha}{\pi^2} \cdot p \cdot \left|L(1, \chi_4)\right|^2. \tag{11}$$

Notes: Obviously, in a sense, Corollary 1 gives us efficient methods to compute the class number $h_p$ that can be done on a computer.

It is easy to prove that if $p \equiv 3 \bmod 4$, then, for any positive integer $k$, we have

$$\sum_{h=1}^{p-1} G(2k-1, h, \chi_2; p) \cdot S(h, p) = 0. \tag{12}$$

If $p \equiv 1 \bmod 4$, then, for any positive integer $k$, we also have

$$\sum_{h=1}^{p-1} G(2k, h, \chi_2; p) \cdot S(h, p) = 0. \tag{13}$$

For general composite number $q > 3$, whether there exists an exact computational formula for (6) will be our further research problem.

## 2. Several Lemmas

In this section, we shall give several simple lemmas, and they are necessary in the proofs of our theorems. First, we have the following.

**Lemma 1.** *Let $p > 3$ be a prime, and $\lambda$ and $\chi$ are two nonprincipal characters* mod *$p$ with $\chi(-1) = -1$. Then, for any positive integer $k$, we have the identity*

$$\sum_{m=1}^{p-1}\chi(m)G(k,m,\lambda;p)=\begin{cases}(p-1)\cdot\dfrac{\tau^{k+1}(\chi)}{\tau(\overline{\lambda})}, & \text{if } \lambda=\overline{\chi}^{k+1};\\ 0, & \text{otherwise,}\end{cases}\tag{14}$$

*where $\tau(\chi)$ denotes the classical Gauss sums.*

*Proof.* For any integer $n$ and nonprincipal character $\chi \bmod p$, from the properties of Gauss sums $\tau(\chi)$ (see Theorem 8.20 in [12]), we have

$$\chi(n)=\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{p-1}\overline{\chi}(a)e\left(\frac{na}{p}\right).\tag{15}$$

Using (15) and the properties of the reduced residue system $\bmod p$, we have

$$\begin{aligned}G(k,m,\lambda;p)&=\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\lambda\left(a_1+a_2+\cdots+a_k+m\overline{a_1a_2,\ldots,a_k}\right)\\&=\frac{1}{\tau(\overline{\lambda})}\sum_{b=1}^{p-1}\overline{\lambda}(b)\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}e\left(\frac{b(a_1+a_2+\cdots+a_k)+bm\overline{a_1,\ldots,a_k}}{p}\right)\\&=\frac{1}{\tau(\overline{\lambda})}\sum_{b=1}^{p-1}\overline{\lambda}(b)\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}e\left(\frac{a_1+\cdots+a_k+b^{k+1}m\overline{a_1,\ldots,a_k}}{p}\right).\end{aligned}\tag{16}$$

So, with the repeated use of (15) in (16), we have

$$\begin{aligned}&\sum_{m=1}^{p-1}\chi(m)G(k,m,\lambda;p)\\&=\frac{1}{\tau(\overline{\lambda})}\sum_{b=1}^{p-1}\overline{\lambda}(b)\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{m=1}^{p-1}\chi(m)e\left(\frac{a_1+\cdots+a_k+b^{k+1}m\overline{a_1\ldots a_k}}{p}\right)\\&=\frac{\tau(\chi)}{\tau(\overline{\lambda})}\sum_{b=1}^{p-1}\overline{\lambda}(b)\overline{\chi}^{k+1}(b)\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi(a_1\ldots a_k)e\left(\frac{a_1+\cdots+a_k}{p}\right)\\&=\frac{\tau^{k+1}(\chi)}{\tau(\overline{\lambda})}\sum_{b=1}^{p-1}\overline{\lambda}(b)\overline{\chi}^{k+1}(b)=\begin{cases}(p-1)\cdot\dfrac{\tau^{k+1}(\chi)}{\tau(\overline{\lambda})}, & \text{if } \lambda=\overline{\chi}^{k+1},\\ 0, & \text{otherwise.}\end{cases}\end{aligned}\tag{17}$$

This proves Lemma 1. □

**Lemma 2.** *Let $q>2$ be an integer; then, for any integer $a$ with $(a,q)=1$, we have the identity*

$$S(a,q)=\frac{1}{\pi^2 q}\sum_{d/q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2,\tag{18}$$

*where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to character $\chi \bmod d$.*

*Proof.* See Lemma 2 of [6]. □

**Lemma 3.** *If $p$ is a prime with $p\equiv 1 \bmod 4$ and $\psi$ is any fourth-order character $\bmod p$, then we have the identity*

$$\tau^2(\psi)+\tau^2(\overline{\psi})=2\sqrt{p}\cdot\alpha,\tag{19}$$

*where $\alpha=\sum_{a=1}^{(p-1)/2}((a+\overline{a})/p)$ is an integer.*

*Proof.* See Lemma 2.2 of [13] or Lemma 3 of [14]. □

**Lemma 4.** *If $p$ is a prime with $p\equiv 5 \bmod 8$ and $\psi$ is any fourth-order character $\bmod p$, then, for any positive integer $k$, we have the identity*

$$\tau^{2k}(\psi)+\tau^{2k}(\overline{\psi})=\sum_{h=0}^{[k/2]}(-1)^h\cdot\frac{k}{k-h}\cdot\binom{k-h}{h}\cdot(2\alpha)^{k-2h}\cdot p^{(1/2)(k+2h)}. \tag{20}$$

*Proof.* First, for all nonnegative integers $u$ and real numbers $X$ and $Y$, we have the identity

$$X^u+Y^u=\sum_{h=0}^{[u/2]}(-1)^h\cdot\frac{u}{u-h}\binom{u-h}{h}\cdot(X+Y)^{u-2h}\cdot(XY)^h, \tag{21}$$

where $[x]$ denotes the greatest integer $\le x$. This formula is obtained because of Waring [15]. It can also be found in [16].

Note that if $p\equiv 5 \bmod 8$, then, for any fourth-order character $\psi \bmod p$, we have $\tau(\overline{\psi})=\overline{\psi}(-1)\cdot\overline{\tau(\psi)}=-\overline{\tau(\psi)}$. So, $\tau(\overline{\psi})\cdot\tau(\psi)=-p$. Thus, taking $X=\tau^2(\psi)$ and $Y=\tau^2(\overline{\psi})$, from (4) and Lemma 3, we have

$$\begin{aligned}\tau^{2k}(\psi)+\tau^{2k}(\overline{\psi})&=\sum_{h=0}^{[k/2]}(-1)^h\cdot\frac{k}{k-h}\binom{k-h}{h}\cdot\left(\tau^2(\psi)+\tau^2(\overline{\psi})\right)^{k-2h}\cdot p^{2h}\\&=\sum_{h=0}^{[k/2]}(-1)^h\cdot\frac{k}{k-h}\binom{k-h}{h}\cdot(2\sqrt{p}\cdot\alpha)^{k-2h}\cdot p^{2h}\\&=\sum_{h=0}^{[k/2]}(-1)^h\cdot\frac{k}{k-h}\cdot\binom{k-h}{h}\cdot(2\alpha)^{k-2h}\cdot p^{1/2(k+2h)}.\end{aligned} \tag{22}$$

This proves Lemma 4. □

## 3. Proofs of the Theorems

In this section, we shall complete the proofs of our theorems. First, we prove Theorem 1. From Lemma 2, we have

$$S(a,p)=\frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2. \tag{23}$$

If $p$ is a prime with $p\equiv 3 \bmod 4$, then, for any positive integer $k$ with $(2k+1,p-1)=1$, let $\overline{2k+1}\cdot(2k+1)\equiv 1 \bmod (p-1)$; then, $\overline{2k+1}$ must be an odd number. If characters $\lambda$ and $\chi$ satisfy $\lambda=\overline{\chi}^{2k+1}$ with $\chi(-1)=-1$, then $\overline{\chi}=\lambda^{\overline{2k+1}}$. If $\lambda=(*/p)=\chi_2$ is Legendre's symbol mod $p$, then we have $\overline{\chi}=\chi_2^{\overline{2k+1}}=\chi_2$. Note that if $p\equiv 3 \bmod 4$, then $\chi_2(-1)=-1$, $\tau(\chi_2)=i\sqrt{p}$ and $L(1,\chi_2)=\pi h_p/\sqrt{p}$. So, from (23) and Lemma 1, we have

$$\begin{aligned}&\sum_{h=1}^{p-1}G(2k,h,\chi_2;p)\cdot S(h,p)\\&=\frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{h=1}^{p-1}\chi(h)\cdot G(2k,h,\chi_2;p)\cdot|L(1,\chi)|^2\\&=\frac{1}{\pi^2}\cdot p\cdot\sum_{\substack{\chi \bmod p\\ \chi=\chi_2}}\frac{\tau^{2k+1}(\chi)}{i\cdot\sqrt{p}}\cdot|L(1,\chi)|^2\\&=\frac{1}{\pi^2}\cdot p\cdot(i\cdot\sqrt{p})^{2k}\cdot\frac{\pi^2}{p}\cdot h_p^2=(-1)^k\cdot p^k\cdot h_p^2.\end{aligned} \tag{24}$$

This proves Theorem 1.

Now, we prove Theorem 2. Let $p$ be a prime with $p \equiv 5 \bmod 8$; then, $\tau(\chi_2) = \sqrt{p}$. For any four-order character $\chi \bmod p$, we have $\chi(-1) = -1$. So, for any positive integer $k$ with $(k, p-1) = 1$, note that $\overline{k}$ is an odd number; if $\chi_2 = \chi^{2k}$, then $\chi_2 = \chi^2$. In this time, $\chi = \chi_4$ must be a fourth-order character $\bmod p$ and $|L(1,\chi_4)| = |L(1,\overline{\chi}_4)|$; from (23), Lemma 1, and Lemma 4, we have

$$\begin{aligned}
&\sum_{h=1}^{p-1} G(2k-1,h,\chi_2;p)\cdot S(h,p)\\
&= \frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{h=1}^{p-1}\chi(h)\cdot G(2k-1,h,\chi_2;p)\cdot|L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot p\cdot\sum_{\substack{\chi \bmod p\\ \chi^2=\chi_2}}\frac{\tau^{2k}(\chi)}{\sqrt{p}}\cdot|L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot p\cdot\left(\frac{\tau^{2k}(\chi_4)}{\sqrt{p}}\cdot\left|L(1,\chi_4)\right|^2+\frac{\tau^{2k}(\overline{\chi}_4)}{\sqrt{p}}\cdot\left|L(1,\overline{\chi}_4)\right|^2\right)\\
&= \frac{1}{\pi^2}\cdot\sqrt{p}\cdot\left(\tau^{2k}(\chi_4)+\tau^{2k}(\overline{\chi}_4)\right)\cdot\left|L(1,\chi_4)\right|^2\\
&= \frac{1}{\pi^2}\cdot\sqrt{p}\cdot\left(\sum_{h=0}^{[k/2]}(-1)^h\cdot\frac{k}{k-h}\cdot\binom{k-h}{h}\cdot(2\alpha)^{k-2h}\cdot p^{(1/2)(k+2h)}\right)\cdot\left|L(1,\chi_4)\right|^2\\
&= \frac{1}{\pi^2}\cdot p^{(k+1)/2}\cdot\left(\sum_{h=0}^{[k/2]}(-1)^h\cdot\frac{k}{k-h}\cdot\binom{k-h}{h}\cdot(2\alpha)^{k-2h}\cdot p^{h}\right)\cdot\left|L(1,\chi_4)\right|^2.
\end{aligned} \tag{25}$$

If $p \equiv 1 \bmod 8$, then $\chi_4(-1) = 1$, so in this time, we have

$$\begin{aligned}
&\sum_{h=1}^{p-1} G(2k-1,h,\chi_2;p)\cdot S(h,p)\\
&= \frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{h=1}^{p-1}\chi(h)\cdot G(2k-1,h,\chi_2;p)\cdot|L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot p\cdot\sum_{\substack{\chi \bmod p\\ \chi^2=\chi_2}}\frac{\tau^{2k}(\chi)}{\sqrt{p}}\cdot|L(1,\chi)|^2 = 0.
\end{aligned} \tag{26}$$

Combining (25) and (26), we may immediately deduce Theorem 2.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

Xu Xiaoling has contributed to this work and read and approved the final manuscript.

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