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Research Article

Equidistribution Modulo 1

Wiem Gadri [1,2]

[1]Faculty of Science, Northern Border University, Arar, Saudi Arabia
[2]University of Gabes, Gabes, Tunisia

Correspondence should be addressed to Wiem Gadri; wiemgadri@yahoo.fr

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The generalisation of questions of the classic arithmetic has long been of interest. One line of questioning, introduced by Car in 1995, inspired by the equidistribution of the sequence $(n^{\alpha})_{n\in N}$ where $0<\alpha<1$, is the study of the sequence $(K^{(1/l)})$, where K is a polynomial having an l-th root in the field of formal power series. In this paper, we consider the sequence $(L'^{(1/l)})$, where $L\prime$ is a polynomial having an l-th root in the field of formal power series and satisfying $L\prime \equiv B \bmod C$. Our main result is to prove the uniform distribution in the Laurent series case. Particularly, we prove the case for irreducible polynomials.

1. Introduction

In 1952, Carlitz [1] introduced the definition of equidistribution modulo 1 in the formal power series case which reveals profitable; it uses Weyl's criterion [1], the generalisation of van der Corput inequality by Dijksma [2], and the theorem of Koksma by Mathan [3].

Car in [4], inspired by equidistribution modulo 1 of the sequence $(n^{\alpha})_{n\in\mathbb{N}}$ where $0<\alpha<1$, characterised equidistribution modulo 1 of the sequences $(L^{(1/l)})$ and $(P^{(1/l)})$, where L describes the sequence of polynomials in $\mathbb{F}_q[X]$ (resp. P describes the sequence of irreducible polynomials in $\mathbb{F}_q[X]$) with an l-th root $(L^{(1/l)})$ (resp. $(P^{(1/l)})$) in the field of formal power series.

In 2013, Mauduit and Car studied in [5] the Q−automaticity of the set of k-th power of polynomials in $\mathbb{F}_q[X]$. Moreover, they calculated the number of polynomials $K\in\mathbb{F}_q[X]$ with degree N such that the sum of digits of K^k in base Q is fixed. In the same subject, Madritsch and Thuswaldner in [6] called the maps $f\colon \mathbb{F}_q[X]\longrightarrow G$, where G is the group of Q-additives satisfying $f(AQ+B)=f(A)+f(B)$ for all polynomials $A,B\in\mathbb{F}_q[X]$ with $\deg(B)<\deg(Q)$. They proved the equidistribution of the sequence $h(W_i)$, where $h\in\mathbb{F}_q(X^{-1})[Y]$ is a polynomial with coefficients in the field of formal power series and (W_i) is an ordered sequence of polynomials in $\mathscr{C}(J)=\{A\in\mathbb{L}_n\colon f(A)\equiv J \bmod M\}$ if and only if one of the coefficients of $h(Y)-h(0)$ is irrational.

In this article, we are interested in the subsequences (L_n') of (L_n) and (P_n') of (P_n) of polynomials in arithmetic progression having an l-th root. We will prove that the sequences $(L_n^{\{'1/l\}})$ and $(P_n^{\{'1/l\}})$ are equidistributed modulo 1.

2. Preliminary

Let $\mathbb{F}_q$ be a finite field of characteristic p with q elements. We consider $\mathbb{F}_q[X]$, $\mathbb{F}_q(X)$, and $\mathbb{F}_q((X^{-1}))$ as analogues of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively.

An element $f\in\mathbb{F}_q((X^{-1}))$ is of the form $f=\sum_{i=n_0}^{+\infty} a_i X^{-i}$, with $a_i\in\mathbb{F}_q$, $n_0\in\mathbb{Z}$, and $a_{n_0}\neq 0$. We define $\nu(f)=\deg(f)=-n_0$ and $|f|=q^{\deg(f)}$. We note $[f]$ the polynomial part of f and $\{f\}$ its fractionary part. Let $\operatorname{Res}(f)=a_1$ if $f\neq 0$, and $\operatorname{sgn}(f)=a_{n_0}$. Let $\psi\colon \mathbb{F}_q\longrightarrow\mathbb{C}$ be a nontrivial additive character. For all $f\in\mathbb{F}_q((X^{-1}))$, we suppose that $E(f)=\psi(\operatorname{Res}(f))$.

Let l be a positive integer >2 which is not divisible by the characteristic p of the field $\mathbb{F}_q$. We introduce $\mathscr{L}=\{a_1,\ldots,a_r\}$ as the set of the r-th elements having an l-th root in $\mathbb{F}_q^*$, and we have

$$r=\frac{q-1}{(l,q-1)}. \tag{1}$$

Then, for f and $g\in\mathbb{F}_q^*((X^{-1}))$, g is called an l-th root of f; we note $f=g^l$ if and only if $\nu(f)\equiv 0 \bmod l$ and

$\operatorname{sgn}(f) \in \mathscr{L}$. In particular, a nonzero polynomial A has an l-th root in $\mathbb{F}_q^*((X^{-1}))$ if and only if $\deg(A) \equiv 0 \bmod l$ and $\operatorname{sgn}(A) \in \mathscr{L}$.

We denote by $\mathbb{L}$ the set of polynomials with an l-th root in $\mathbb{F}_q^*((X^{-1}))$:

$$\mathbb{L} = \left\{A \in \mathbb{F}_q[X] \setminus \{0\}\colon \deg(A) \equiv 0 \bmod l \text{ and } \operatorname{sgn}(A) \in \mathscr{L}\right\}, \tag{2}$$

and if $\mathbb{I}$ is the set of irreducible polynomials over $\mathbb{F}_q[X]$, we define $\mathbb{P} = \mathbb{L} \cap \mathbb{I}$.

If $n = \sum_{i=1}^{S} n_i q^i$, where $n_i \in \{0, \ldots, q-1\}$ for all $i \in \{0, \ldots, s\}$, is the representation in base q of the integer $n \geq 1$, then let

$$H_n = \chi_{n_0} + \cdots + \chi_{n_s} X^s, \tag{3}$$

where χ_{n_i} are given by the bijection $n_i \mapsto \chi_{n_i}$ from $\{0, \ldots, q-1\}$ to $\mathbb{F}_q$. For $n = 0$ and 1, it is convenient to suppose that $\chi_0 = 0$ and $\chi_1 = 1$. We define the order in $\mathbb{F}_q$ by $\chi_{n_i} < \chi_{n_i+1}$, for all $n_i \in \{0, \ldots, q-1\}$, and in $\mathbb{F}_q$ by

$$m < n \Longrightarrow \deg(H_m) \leq \deg(H_n). \tag{4}$$

Then, we order $\mathbb{F}_q$ by posing for all natural numbers n:

$$H_n < H_{n+1}. \tag{5}$$

This paper is devoted to the study of equidistribution modulo 1 of a certain sequence in the field of Laurent formal power series. In 1952, Carlitz introduced and characterised equidistribution modulo 1 in the field of Laurent formal power series and obtained the following result.

Lemma 1 (see [1]; Weyl's criterion). *ie sequence $\Theta = (\theta_n)$ with values in $\mathbb{F}_q((X^{-1}))$ is equidistributed modulo 1 if and only if*

$$\lim_{N \longrightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} E(H\theta_n) = 0, \tag{6}$$

for all $H \in \mathbb{F}_q[X]^*$.

Finally, we enounce a result which concerns a class of irreducible polynomials given by Artin in [7], which will be very useful later.

Theorem 1 (see [7]). *Let $C, B \in \mathbb{F}_q[X]$ be coprime polynomials. If $\pi(m\colon C, B)$ denotes the number of monic irreducible polynomials with degree m which are congruent to B modulo C, then*

$$\pi(m\colon C, B) = \frac{1}{\Phi(C)} \frac{q^m}{m} + O\left(\frac{q^{m\theta}}{m}\right), \tag{7}$$

where θ is a constant <1. This theorem is analogous to the theorem of prime numbers in arithmetic progression.

3. Results

Let $l \geq 2$ be an integer nondivisible by the characteristic p of the field $\mathbb{F}_q$; we order the set of the l-th powers of $\mathscr{L}$ under the increasing order of $\mathbb{F}_q$, and we fix a polynomial C with degree c. For all $B \in \mathbb{F}_q[X]$, we denote by $\mathbb{L}'$ the subset of $\mathbb{L}$ defined in (2),

$$\mathbb{L}' = \mathbb{L}_{C,B} = \{A \in \mathbb{L}\colon A \equiv B \bmod C\}, \tag{8}$$

and $\mathbb{P}'$ the subset of $\mathbb{P} = \mathbb{L} \cap \mathbb{I}$:

$$\mathbb{P}' = \mathbb{P}_{C,B} = \{A \in \mathbb{L} \cap \mathbb{I}\colon A \equiv B \bmod C \text{ with } (B, C) = 1\}. \tag{9}$$

We ordered the elements of $\mathbb{L}'$ and $\mathbb{P}'$ with the order relation defined in (2); hence,

$$\mathbb{L}' = \{L_1', \ldots, L_n', \ldots\} \text{ and } \mathbb{P}\prime = \{P_1', \ldots, P_n', \ldots\}. \tag{10}$$

The aim of this paper is to prove the following theorems.

Theorem 2. *Let (L_n') be the sequence of polynomials of $\mathbb{L}'$ indexed under the increasing order of $\mathbb{F}_q[X]$. Then, for $l \geq 2$, the sequence $(L_n^{\{'1/l\}})$ is equidistributed modulo 1.*

Theorem 3. *Let (P_n') be the sequence of polynomials of $\mathbb{P}'$ indexed under the increasing order of $\mathbb{F}_q[X]$. Then, the sequence $(P_n^{\{'1/l\}})$ is equidistributed modulo 1 for $l > (1/(1-\theta))$, and θ is a constant defined in Theorem 1. In particular, if $X^q - X$ does not divide C, then let $l \geq 3$.*

4. Proofs of Theorems 2 and 3

4.1. Tools. A generalisation of Theorem 1 was proved in 1965 by Hayes introducing the arithmetic progression.

Lemma 2 (see [8]). *Let $C \in \mathbb{F}_q[X]$ be a polynomial with degree c. Then, for all polynomials B, there exist exactly q^{m-c} monic polynomials with degree m which are congruent to B modulo C if $m \geq c$.*

Theorem 4 (see [8]). *Let $k \geq 1$ be a positive integer, $u = (u_1, \ldots, u_k)$ be a sequence of k elements in $\mathbb{F}_q$, and $C, B \in \mathbb{F}_q[X]$ be coprime polynomials. If, for $m \geq k$, $\pi(m; u, C, B)$ is the number of irreducible and monic polynomials P with degree m which are congruent to B modulo C such that* $\deg(P - X^m - u_1 X^{m-1} - \cdots - u_k X^{m-k} < m - k,)$ *then*

$$\pi(m\colon u, C, B) = \frac{q^{m-k}}{m\Phi(C)} + O\left(\frac{q^{m\theta}}{m}\right), \tag{11}$$

where θ is a constant <1.

Remark 1. In particular, if $X^q - X$ does not divide C, then (11) is verified for $\theta = (1/2)$.

The proofs of Theorems 2 and 3 are based on Corollary 1 whose proof needs the following lemmas.

Lemma 3 (see [9], Lemma II.1.1). *Let $k \in \mathbb{N}$, $H \in \mathbb{F}_q[X]^*$ with degree h, and $A \in \mathbb{L}$ with degree lk. Then, for all $Z \in \mathbb{F}_q[X]$ such that* $\deg(Z) = z < (l-1)k - h - 1$, *we have*

(1) $(A+Z)\in \mathbb{L}$

(2) $Res(HA^{(1/l)}) = Res(H(A+Z)^{(1/l)})$

Lemma 4 (see [9], Lemma II.1.2). *Let $H\in \mathbb{F}_q[X]^*$ with degree h and $k\in\mathbb{N}$ such that $(l-1)k\geq h$. Then, for all $a\in\mathscr{L}$, all $\xi\in\mathbb{F}_q$, and all $Y\in\mathbb{F}_q[X]$ with degree $<k+h$, there exists unique $\eta=\eta(a,Y)\in\mathbb{F}_q$ such that, for all $Z\in\mathbb{F}_q[X]$ with degree $<(l-1)k-h-1$, we obtain*

$$\xi = \mathrm{Res}\Big(H\big(aX^{lk}+YX^{(l-1)k-h}+\eta X^{(l-1)k-h-1}+Z\big)^{(1/l)}\Big). \tag{12}$$

Corollary 1. *Let $H\in\mathbb{F}_q[X]^*$ with degree h and $k\in\mathbb{N}$ such that $(l-1)k\geq h$. For all $a\in\mathscr{L}$, we have*

(i) $\sum_{A\in\mathscr{A}}\psi(Res(HA^{(1/l)}))=0$, *where* $\mathscr{A}=A\in\mathbb{L}'$, deg $(A)=lk$, *and* sgn $(A)=a$

(ii) $\sum_{A\in\mathscr{J}}\psi(Res(HA^{(1/l)}))=O(q^{lk\theta+k+h+1}/lk)$, *where* $\mathscr{J}=A\in\mathbb{P}'$, deg $(A)=lk$, *and* sgn $(A)=a$, *where* θ *is a constant* <1

Proof. For $\mathbb{A}=\mathscr{A}$ or $\mathscr{J}$, we note

$$\sigma(\mathbb{A}\colon k,a)=\sum_{A\in\mathbb{A}}\psi\Big(\mathrm{Res}\big(HA^{(1/l)}\big)\Big)=\sum_{\xi\in\mathbb{F}_q}\psi(\xi)\pi(\xi), \tag{13}$$

where $\pi(\xi)$ is the number of polynomials $A\in\mathbb{A}$ such that $\mathrm{Res}(HA^{(1/l)})=\xi$, but with Lemma 4, for $Y\in\mathbb{F}_q[X]$ with degree $<k+h$, there exists $\eta\in\mathbb{F}_q$ such that the polynomial

$$K=K(a,Y,\xi)=aX^{lk}+YX^{(l-1)k-h}+\eta X^{(l-1)k-h}, \tag{14}$$

satisfying

$$\mathrm{Res}\big(HK^{(1/l)}\big)=\xi. \tag{15}$$

Let $Z=A-K$; we denote by $\pi(lk\colon Y,\xi)$ the number of polynomials $A\in\mathbb{A}$ such that

$$\deg(Z)<(l-1)k-h-1. \tag{16}$$

We obtain

$$\sigma(\mathbb{A}\colon k,a)=\sum_{\xi\in\mathbb{F}_q}\psi(\xi)\sum_{\{Y\in\mathbb{F}_q[X]\ \backslash\backslash \deg(Y)<k+h\}}\pi(lk\colon Y,\xi). \tag{17}$$

(i) If $\mathbb{A}=\mathscr{A}$, then by Lemma 2, we have $\pi(lk\colon Y,\xi)=q^{(l-1)k-h-c-1}-1$. With the orthogonality criterion of ψ, it results in

$$\sigma(L'\colon k,a)=q^{k+h}\big(q^{(l-1)k-h-c-1}-1\big)\sum_{\xi\in\mathbb{F}_q}\psi(\xi)=0. \tag{18}$$

(ii) If $\mathbb{A}=\mathscr{J}$, then by Theorem 4, we have

$$\pi(lk\colon Y,\xi)=\frac{q^{(l-1)k-h-1}}{lk\Phi(C)}+O\bigg(\frac{q^{lk\theta}}{lk}\bigg),\quad \theta<1. \tag{19}$$

We deduce that

$$\sigma(\mathbb{P}'\colon k,a)=\frac{q^{lk-1}}{lk\Phi(C)}\sum_{\xi\in\mathbb{F}_q}\psi(\xi)+\sum_{\xi\in\mathbb{F}_q}O\bigg(\frac{q^{lk\theta+k+h}}{lk}\bigg). \tag{20}$$

Finally, with the orthogonality criterion, we obtain

$$\sigma(\mathbb{P}'\colon k,a)=O\bigg(\frac{q^{lk\theta+k+h+1}}{lk}\bigg),\quad \text{with } \theta<1. \tag{21}$$

□

4.2. Proof of Theorem 2. In $\mathbb{F}_q[X]$, there are q^{m-c} monic polynomials which are congruent to B modulo C with degree m, and let $c=\deg(C)$. We denote by a_m (resp. b_m) the number of polynomials in $\mathbb{L}\prime$ with degree lm (resp. $\leq lm$). It is sufficient to verify that

$$a_m = rq^{lm-c},$$
$$b_m = a_1+\cdots+a_m=\frac{r\big(q^{l(m+1)-c}-q^{l-c}\big)}{q^l-1}, \tag{22}$$

where r is defined in (1). Let $H\in\mathbb{F}_q[X]^*$ with degree h, and N is an integer such that

$$N>b_{[1+((h+1)/(l-1))]}, \tag{23}$$

where $[x]$ defines the least integer $\geq x$. The sequence (b_m) is strictly increasing, and there exists a unique integer t such that

$$b_{t-1}\leq N<b_t. \tag{24}$$

Moreover, there exists a unique integer $s\in 0,\ldots,r-1$, such that

$$b_{t-1}+1+sq^{lt}\leq N<b_{t-1}+(s+1)q^{lt}. \tag{25}$$

Let

$$W(N)=\sum_{n=1}^{N}E\Big(HL_n^{\prime 1/l}\Big). \tag{26}$$

To prove Theorem 2, we have to show that

$$\lim_{N\longrightarrow\infty}\frac{1}{N}|W(N)|=0. \tag{27}$$

Using relations (24) and (25), we rewrite the sum $W(N)$ to obtain

$$W(N)=W_1+W_2+W_3, \tag{28}$$

with

$$W_1 = \sum_{n=1}^{b_{t-1}} E\left(HL_n^{'(1/l)}\right),$$
$$W_2 = \sum_{j=0}^{s-1} \sum_{n=b_{t-1}+1+jq^{lt}}^{b_{t-1}+(j+1)q^{lt}} E\left(HL_n^{'(1/l)}\right), \quad (29)$$
$$W_3 = \sum_{n=b_{t-1}+1+sq^{lt}}^{N} E\left(HL_n^{'(1/l)}\right).$$

We start by giving an estimation of the sum W_1 which concerns the polynomials of $\mathbb{L}'$ with degree $\leq l(t-1)$. We have

$$W_1 = \sum_{k\leq t-1} \sum_{\substack{L'\in\mathbb{L}' \\ \deg(L')=lk \\ \operatorname{sgn}(L')\in\mathscr{L}}} E\left(HL_n^{'(1/l)}\right),$$
$$= \sum_{\substack{k\leq t-1 \\ (l-1)k<h+c}} \sum_{\substack{L'\in\mathbb{L}' \\ \deg(L')=lk \\ \operatorname{sgn}(L')\in\mathscr{L}}} E\left(HL_n^{'(1/l)}\right) + \sum_{\substack{k\leq t-1 \\ (l-1)k\geq h+c}} \sum_{\substack{L'\in\mathbb{L}' \\ \text{textdeg}(L')=lk \\ \text{textsgn}(L')\in\mathscr{L}}} E\left(HL_n^{'(1/l)}\right). \quad (30)$$

We have to just major the first part of the sum by the number of polynomials with degree $<((l(h+c))/(l-1))$, and we apply Corollary 1 on the second part to obtain

$$\left|W_1\right| \leq q^{((l(h+c))/(l-1))}. \quad (31)$$

We apply the same Corollary 1 on W_2 that represents the sum of polynomials with degree lt and and with signature $a_1, \ldots, a_{s-1}$, then

$$W_2 = \sum_{i=1}^{s-1} \sum_{\substack{L'\in\mathbb{L}' \\ \text{textdeg}(L')=lt \\ \text{textsgn}(L')\in a_i}} E\left(HL_n^{'(1/l)}\right) = 0. \quad (32)$$

The polynomials in W_3 can be written in the form

$$L_i' = a_s X^{lt} + H_{n_i}, \quad j \leq i \leq N, \quad (33)$$

where $j = 1 + b_{t-1} + sa_t$ and the sequence $(H_{n_i})_{j\leq i\leq N}$ is strictly increasing in $\mathbb{F}_q[X]$.

By the order relation on $\mathbb{F}_q[X]$ (4), if

$$n_N = c_0 + c_1 q + \cdots + c_m q^m \quad (34)$$

and is the presentation in base q of the integer n_N, we have

$$H_{n_N} = \chi_{c_0} + \chi_{c_1} X + \cdots + \chi_{c_m} X^m. \quad (35)$$

To estimate W_3, we will distinguish two cases: when the degree of H_{n_N} is up to the integer $(l-1)t-h-c-1$ and when it is not:

1st case: $m \leq (l-1)t-h-c-1$. Using (34) and the fact that the sequence $(n_i)_{j\leq i\leq N}$ is strictly increasing, we obtain

$$N - j \leq n_N - n_j \leq n_N \leq q^{m+1} - 1. \quad (36)$$

Thus,

$$\left|W_3\right| \leq N - j + 1 \leq q^{(l-1)t-h-c}. \quad (37)$$

2nd case: $m > (l-1)t-h-c-1$. The polynomials H_{n_i} defined in (33) are of the form

$$H_{n_i} = y_0 + y_1 X + \cdots + y_m X^m \leq H_{n_N} = \chi_{c_0} + \chi_{c_1} X + \cdots + \chi_{c_m} X^m. \quad (38)$$

Let $\mathscr{Y}$ be the set of polynomials of the form

$$Y = y_{(l-1)t-h-c} + y_{(l-1)t-h-c+1} X + \cdots + y_m X^{m-(l-1)t+h+c} \quad (39)$$

such that, for every polynomial Z with degree $<(l-1)t-h-c$, we have

$$YX^{(l-1)t-h-c} + Z \leq H_{n_N} \quad (40)$$

If k is the greatest index $i \in j, \ldots, N$ for which L_i' are written in the form

$$L_i' = a_s X^{lt} + YX^{(l-1)t-h-c} + Z, \quad (41)$$

with $Y \in \mathscr{Y}$ and Z being a polynomial with degree $<(l-1)t-h-c$, then we rewrite the sum W_3:

$$W_3 = W_4 + W_5, \quad (42)$$

with

$$W_4 = \sum_{i=j}^{k} E\left(HL_i^{'(1/l)}\right) \text{ and } W_5 = \sum_{i=k+1}^{N} E\left(HL_i^{'(1/l)}\right). \quad (43)$$

We have

$$W_4 = \sum_{\xi\in\mathbb{F}_q} \psi(\xi)\pi(\xi), \quad (44)$$

where $\pi(\xi)$ is the number of couples (Y, Z) such that $Y \in \mathscr{Y}$, $Z \in \mathbb{F}_q[X]$ with degree $<(l-1)t-h-c$, and $\operatorname{Res}(H(a_s X^{lt} + YX^{(l-1)t-h-c} + Z)^{(1/l)}) = \xi$. Moreover, we have

$$\pi(\xi) = \sum_{Y\in\mathscr{Y}} \pi(lt\colon Y, \xi), \quad (45)$$

where $\pi(lt\colon Y, \xi)$ denotes the number of polynomials $L' \in \mathbb{L}$ such that

$$\deg(L' - K) < (l-1)k - h - c - 1, \quad (46)$$

with

$$K = K(a, Y, \xi) = aX^{lk} + YX^{(l-1)k-h-c-1} + \eta X^{(l-1)k-h-c-1}, \quad (47)$$

which gives the same arguments presented in the proof of Corollary 1, and then we deduce that

$$W_4 = 0. \quad (48)$$

In (34), let v be the least index $> (l-1)t-h-c$ such that $c_v \neq 0$; then, we have

$$n_N = c_m q^m + \cdots + c_v q^v + c_{(l-1)t-h-c-1} q^{(l-1)t-h-c-1} + \cdots + c_0. \quad (49)$$

Since all polynomial L' of the form

$$L' = a_s X^{lt} + y_m X^m + \cdots + y_v X^v + y_{v-1} X^{v-1} + \cdots + y_0, \quad (50)$$

which coefficients satisfy the condition:

$$y_m \leq \chi_{c_m}, y_{m-1} \leq \chi_{c_{m-1}}, \ldots, y_{v-1} \leq \chi_{c_{v-1}}, y_v \leq \chi_{c_v}, \quad (51)$$

is less then L'_{n_N}, we obtain

$$\begin{aligned} n_k &\geq c_m q^m + \cdots + c_{v-1} q^{v-1} + (q-1) q^{v-1} + \cdots + (q-1), \\ n_k &\geq n_N - q^{(l-1)t-h-c}, \end{aligned} \quad (52)$$

which leads to

$$|W_5| \leq N - k \leq n_N - n_k \leq q^{(l-1)t-h-c}. \quad (53)$$

Then, with (48) and (53), it results in

$$|W_3| \leq q^{(l-1)t-h-c}. \quad (54)$$

With (31), (32), and (54), we obtain

$$\left| \frac{1}{N} \sum_{n=1}^{N} E\left(HL_n^{\prime(1/l)}\right) \right| \leq \frac{1}{N} \left(q^{((l(h+c))/(l-1))} + q^{(l-1)t-h-c} \right), \quad (55)$$

and finally, with (24), we obtain

$$\left| \frac{1}{N} \sum_{n=1}^{N} E\left(HL_n^{\prime(1/l)}\right) \right| \leq \frac{q^{((l(h+c))/(l-1))} + q^{(l-1)t-h-c}\left(q^l - 1\right)}{r\left(q^{l(t+1)} - q^{l-c}\right)} \ll q^{-t} \ll N^{-(1/l)}, \quad (56)$$

which ends the proof.

4.3. Proof of Theorem 3. The proof of Theorem 3 is treated as the proof of Theorem 2, and we will keep the same notations with the appropriate modifications. Let $\pi(m, C, B)$ be the number of monic irreducible polynomials with degree m in $\mathbb{F}_q[X]$, congruent to B modulo C, satisfying with [4] the following property:

$$\frac{q^m}{q-1} - 1 - 2q^{(m/2)} \leq m\pi(m\colon C, B) \leq \frac{q^m}{q-1}. \quad (57)$$

In $\mathbb{P}'$, we have $a_m = r\pi(lm\colon C, B)$, and there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \frac{q^{lm}}{m} \leq b_m \leq c_2 \frac{q^{lm}}{m}. \quad (58)$$

Let H be a nonzero polynomial with degree h and N be an integer satisfying (23). We suppose now that

$$W'(N) = \sum_{n=1}^{N} E\left(HP_n^{('1/l)}\right). \quad (59)$$

With relations (24) and (25), we obtain

$$W'(N) = W'_1 + W'_2 + W'_3. \quad (60)$$

By the same method used in the proof of Theorem 2, with Corollary 1, we obtain

$$|W'_1| = \left| \sum_{n=1}^{b_t - 1} E\left(HP_n^{('1/l)}\right) \right| \leq q^{((l(h+c))/(l-1))} + O\left(\frac{q^{(t-1)(l\theta+1)+h+1}}{l(h+c)} \right), \quad (61)$$

$$|W'_2| = O\left(\frac{q^{lt\theta+t+h+1}}{lt} \right), \quad (62)$$

and then

$$W'_2 = \sum_{j=0}^{s-1} \sum_{n=b_{t-1}+1+jq^{lt}}^{b_{t-1}+(j+1)q^{lt}} E\left(HP_n^{('1/l)}\right) = \sum_{i=1}^{r} \sum_{\substack{P_I \in \mathbb{P}_I \\ \text{textdeg}(P_I) = lt \\ \text{textsgn}(P_I) = a_i}} E\left(HP'^{(1/l)}\right).$$

$$|W'_4| = \left| \sum_{i=j}^{k} E\left(HP_i^{('1/l)}\right) \right| = O\left(\frac{q^{lt\theta+t+h}}{lt} \right), \quad (63)$$

where W'_4 is the sum defined in (42) concerning the polynomials in $\mathbb{P}'$. Finally, we treat the sum

$$W'_5 = \sum_{i=k+1}^{N} E\left(HP_i^{('1/l)}\right). \quad (64)$$

With Theorem 4, for all polynomials

$$Y = a_s X^{lt} + y_m X^m + \cdots + y_v X^v + \cdots + y_{(l-1)t-h-c} X^{(l-1)t-h-c}, \quad (65)$$

in which coefficients satisfy condition (51), there exists

$$\pi(lt\colon Y, C, B) = \frac{q^{(l-1)t-h-c}}{lt\Phi(C)} + O\left(\frac{q^{lt\theta}}{lt}\right), \quad \theta < 1. \tag{66}$$

Irreducible polynomials P' are congruent to B modulo C such that $\deg(P' - Y) < (l-1)t - h - c$. Such polynomials P' are in $\mathbb{P}'$, and we have

$$\begin{aligned} n_k &\geq c_m q^m + \cdots + c_{\nu-1} q^{\nu-1} + \cdots + (q-1) q^{(l-1)t-h-c} \\ & n_N - 2q^{(l-1)t-h-c} + 1. \end{aligned} \tag{67}$$

Then,

$$\left|W_5'\right| \leq N - k \leq 2q^{(l-1)t-h-c} - 1. \tag{68}$$

With (63and68, it results in

$$\left|W_3'\right| = \left|W_4' + W_5'\right| \leq 2q^{(l-1)t-h-c} + O\left(\frac{q^{lt\theta+t+h}}{lt}\right). \tag{69}$$

Finally, from (61), (62), and (69), we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} E\left(HP_n'^{\{1/l\}}\right)\right| \leq \left(q^{((l(h+c))/(l-1))} + O\left(\frac{q^{(t-1)(l\theta+1)+h+1}}{l(h+c)}\right) + O\left(\frac{q^{lt\theta+t+h+1}}{lt}\right) + 2q^{(l-1)t-h-c} + O\left(\frac{q^{lt\theta+t+h}}{lt}\right)\right). \tag{70}$$

Then,

$$\left|\frac{1}{N}\sum_{n=1}^{N} E\left(HP_n'^{\{1/l\}}\right)\right| \ll \frac{q^{(l\theta+1)t} + q^{(l-1)t}}{N} \ll N^{\theta+(1/l)}, \tag{71}$$

which gives the corresponding conclusion needed for $l > (1/(1-\theta))$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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