Research Article

Pricing Vulnerable Options in the Bifractional Brownian Environment with Jumps

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In this paper, we study the valuation of European vulnerable options where the underlying asset price and the firm value of the counterparty both follow the bifractional Brownian motion with jumps, respectively. We assume that default event occurs when the firm value of the counterparty is less than the default boundary. By using the actuarial approach, analytic formulae for pricing the European vulnerable options are derived. The proposed pricing model contains many existing models such as Black–Scholes model (1973), Merton jump-diffusion model (1976), Klein model (1996), and Tian et al. model (2014).

1. Introduction

Options are popular financial derivatives that play important roles in financial markets. An option is a financial contract that the option holder has the right to buy or sell a certain amount of underlying assets to the option issuer (the counterparty) at an agreed strike price, but the holder does not have the obligation to buy or sell the underlying assets. There is no free lunch in the world. Due to the asymmetry of right and obligation in an option trading, the holder must pay a certain amount that is called options price to the issuer in order to obtain the right given by the contract. Reasonable options price is the premise of the trading between the holder and the issuer, and it is not only a favorable guarantee for investors to hedge and make profit but also of great significance to improve the theoretical system of options pricing. Therefore, options pricing has always been received great attention by many researchers.

Black and Scholes [1] presented the famous Black–Scholes formulae based on Fama’s efficient market hypothesis [2]. The pricing model proposed by Black and Scholes marks the establishment of modern options theory. Since this pricing model is based on a pure diffusion process,

the process for the underlying stock prices may have the effect of understating the valuation of options in many business situations, using jumps to capture sudden shocks occurred in the underlying asset prices which are necessary and this can make options pricing more accurate. Merton [3] first incorporated jump risk into options pricing model and proposed a jump-diffusion model that effectively improved the Black–Scholes model. Since then, many researchers have further studied the pricing problem of options. For example, Jarrow and Rosenfeld [4] and Ball and Torous [5] observed that the underlying asset prices exhibited lognormal jumps. Kou [6] divided the stock price into continuous and discontinuous parts and proposed a double exponential jump-diffusion model with the assumption that the jump amplitude of discontinuous part obeyed double exponential distribution. Rodrigo [7] studied the pricing problem of barrier options where the underlying asset prices were governed by geometric Brownian motion with jumps and obtained explicit pricing formulae for barrier options. More studies about options pricing with jump-diffusion processes could be seen in reference [8–11] and the references therein.

All studies mentioned above mostly focused on the options pricing problem without credit risk. When options

trading activities have been moving from standardized products quoted on exchange-traded markets towards customized products traded on the over-the-counter (OTC) markets. OTC derivatives market cannot guarantee that the counterparty realizes the promise of payments to the option holder. Thus, the credit risk which is also called default risk exists in the OTC options trading, especially due to the frequent bankruptcy of domestic and foreign companies during the financial crisis, such as the sudden bankruptcy of Lehman Brothers after the harrowing experience of the subprime crisis in 2007. The issue of credit exposure has attracted much attention when pricing financial derivatives in the OTC market. Recently, OTC derivatives market has grown rapidly. In 2020, the notional amounts outstanding of the OTC options contract reached 107.4 trillion dollars based on the bank for International Settlements (in short, BIS) statistics on OTC derivatives outstanding. So, it is necessary to consider the potential credit risk caused by the counterparty when studying the options pricing in OTC market. The options with credit risk are defined as vulnerable options. As for the vulnerable options pricing, it is important to characterize the credit risk of the counterparty. Currently, there are two basic approaches in the existing literature that capture the credit risk; one is the structural approach. Merton [12] first proposed the structural model, which assumed that when the firm value of the counterparty was less than the default boundary, the counterparty would not realize the promise of payments to the option holder, that is, default occurred. The other is the reduced-form approach. Jarrow and Turnbull [13] introduced this model, which provided a new framework to model the default time and defined the first jump time as the default time. The default probability was controlled by a default intensity. Based on these two approaches, many researchers have studied the pricing problem of vulnerable options. Johnson and Stulz [14] first put forward the vulnerable options pricing model assuming that the options were the only liability of the options issuer and the underlying asset prices and the firm values of the counterparty were independent. Under the structural model, Klein [15] extended the model of Johnson and Stulz [14] by allowing the options issuer to have other liabilities and extended the results of Jarrow and Turnbull [13] by relaxing the independence assumption. Contrast to constant interest rate assumption in [15], Klein and Inglis [16] considered the valuation of vulnerable options with stochastic interest rate and derived an exact pricing formula of vulnerable option through partial differential equation techniques. In order to describe the phenomenon of volatility clustering and volatility smile, Lee et al. [17] incorporated Heston stochastic volatility model to capture the behavior of underlying asset prices and firm values of the counterparty and provided a closed-form analytic formula of vulnerable options price by using the Green function and Fourier transforms. Taking jump risk of financial assets into account, Tian et al. [18] investigated the pricing problem of vulnerable options where both of the underlying asset price and firm value of the counterparty followed jump-diffusion processes and jumps in these processes which were divided into idiosyncratic jumps and

common jumps besides diffusion parts and got explicit formulae of vulnerable option prices by using the theory of stochastic analysis. Wang [19] extended the framework of Tian et al. [18] to value vulnerable options when default barriers were stochastic. Since vulnerable options may face not only the jump risk but also the illiquidity risk, Yang et al. [20] constructed a pricing model for the European vulnerable options under incomplete information and discussed the impact of pricing parameters on option prices by numerical experiments. Niu et al. [21] studied the valuation of European vulnerable options where the underlying asset price was driven by a double exponential jump-diffusion process and the credit risk was described by the reduced-form approach and obtained the pricing formulae of vulnerable options via the equivalent martingale measure technique and two-dimensional Laplace transform. More valuation of vulnerable options could be found in references [22–30] and the references therein.

All the pricing models mentioned above were established under the efficient market hypothesis, which assumed that the logarithmic returns of the financial assets obeyed the standard Brownian motion. But this treatment is contrasted with a series of financial empirical studies that show financial asset prices have many characteristic properties such as high peaks, heavy tails, periods of constant values [31], and long-range dependence [32–34]. Therefore, many generalizations of the celebrated Black–Scholes model are studied. For example, in order to describe properly financial data exhibiting periods of constant values, Magdziarz [35] introduced the subdiffusive geometric Brownian motion (in short, SGBM) as a model of financial asset prices exhibiting subdiffusive dynamics. Then, the author obtained the fractional Fokker–Planck equation (in short, FFPE) governing the dynamics of the probability density function of the introduced process and the pricing formula for a European option in subdiffusive regime. This FFPE was first derived from the continuous-time random walks scheme with heavy-tailed waiting times (periods when the asset prices stay motionless) [36–40]. According to the distribution characteristics of waiting times, Magdziarz and Gajda [41] proposed a generalization of Black–Scholes model which was defined as standard GBM, subordinated by the infinitely divisible inverse subordinator. Since the waiting times could be chosen from the rich family of nonnegative infinitely divisible distributions, the proposed model was an extension of the model in [35]. The corresponding FFPE and formulae for fair prices of European options were obtained. Motivated by subdiffusive motion of the particle observed in biophysics, Jeon and Metzler [42] studied the stochastic properties of a non-Brownian particle whose motion is governed by either fractional Brownian motion (in short, FBM) or the fractional Langevin equation in combined geometries. FBM has been extensively used in the various fields, including but not limited to biology, physics, economics, and so on. Peter [43] put forward the fractal market hypothesis, analyzed different capital markets by using R/S method, and observed the existence of fractal structure and aperiodic cycle. That is to say that the financial asset prices depend not only on the current situation but also on the

historical moment. Thus, FBM that exhibits properties of self-similarity and long-range dependence has the ability to capture the behavior of financial asset prices. Although FBM had two important properties, Rogers [44] and Cheridito [45] discovered that there existed arbitrage opportunities when FBM was applied to financial markets. Björk and Hult [46] showed that the application of FBM based on Wick integral in financial market was limited.

To overcome these problems and take the long-range dependence of financial asset prices into consideration, many researchers began to use a bifractional Brownian motion to describe the dynamics of financial asset prices. The bifractional Brownian motion (in short, BFBM), known as a generalization of FBM, is a family of centered Gaussian processes that not only has the characteristics of self-similarity and long-range dependence but also has the property of nonstationary increment and can be a semimartingale under appropriate conditions, and then it can be applied to model the evolution of financial asset prices such as the underlying stock prices in options pricing. The BFBM was introduced by Houdre and Villa [47]. Russo and Tudor [48] studied several properties of the BFBM. Es-Sebaiy and Tudor [49] gave the stochastic integral of the BFBM and illustrated that the BFBM was a valuable stochastic process to capture dynamics of financial asset prices. Lifshits and Volkova [50] considered the existence of the BFBM for a given pair of parameters. Xu [51] proposed a pricing model for European options and compound options in the BFBM environment and derived the pricing formulae by the method of variable substitution and risk-neutral principle. More discussions of the BFBM could be seen in references [52–56] and the references therein.

In general, the valuation of options is derived by using some techniques such as equivalent martingale measure transform, partial differential equation, and Mellin transform. But, there are some assumptions for constructing the options pricing model before using these techniques, and the complexity of the calculation exists as appeared in these techniques. Therefore, an actuarial approach, which does not consider economic assumptions of financial markets and is not only valid in the arbitrage-free, equilibrium, and complete markets but also reliable in the arbitrage, non-equilibrium, and incomplete markets, is a very good method to solve the options pricing model. The actuarial approach proposed by Bladt and Rydberg [57] transformed the options pricing problem into an equivalent fair insurance premium problem. Yan et al. [58] obtained the pricing formulae of European options in the case where the underlying asset price followed Ornstein–Uhlenbeck process by actuarial approach. Falkowski [59] proposed a fractional Black–Scholes pricing model with time-dependent volatility for options and derived the analytic formulae via an actuarial method. Shokrollahi and Kılıçman [60] studied the strategy of fair insurance premium actuarial approach for pricing foreign currency option under the jump mixed fractional Brownian motion environment. A lot of the literature on options pricing using the actuarial approach could be found in [61–64].

Despite several literatures having considered vulnerable options pricing, applying bifractional Brownian motion with jumps to model vulnerable options pricing has not been studied. This paper is devoted to the valuation of European vulnerable options in the BFBM environment with jumps, and the structural approach is chosen to model the credit risk. Specifically, the dynamics of the underlying asset prices and firm values of the counterparty are both generated by a two-part stochastic process: one is the continuous price movements generated by a BFBM. The other one is discontinuous changes generated by a Poisson process due to sudden shocks. Following Tian et al. [18], the discontinuous changes in the underlying asset prices and firm values of the counterparty both consist of two parts: idiosyncratic shocks for each asset price and common ones that have an impact on all asset prices. Then, we show how to price European vulnerable options by actuarial approach in a jump bifractional Brownian environment. After obtaining the pricing formulae, the relationship between our pricing formulae and those by Black and Scholes [1], Merton [3], Klein [15], and Tian et al. [18] are discussed.

The rest of our paper is organized as follows. Section 2 provides some preliminaries about the BFBM and actuarial approach. In Section 3, a pricing model for vulnerable options in the jump BFBM environment based on the structural approach is proposed, and the pricing formulae are derived by actuarial approach. Finally, we give the concluding remarks in Section 4.

2. Preliminaries

In this section, we present some definitions and results that are needed for the rest of this paper.

2.1. Bifractional Brownian Motion

Definition 1. A bifractional Brownian motion $B^{H,K} = (B_t^{H,K})_{t \geq 0}$ with Hurst parameter $H \in (0, 1)$ and $K \in (0, 1]$ is a centered Gaussian process with the mean zero and covariance function

$$\begin{aligned} R_H(t, s) &:= E(B_t^{H,K} B_s^{H,K}) \\ &= \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t - s|^{2HK} \right), \quad s, t \geq 0, \end{aligned} \quad (1)$$

if $K = 1$, $B^{H,K}$ is the ordinary fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Moreover, when $H = (1/2)$, $B^{H,K}$ is the standard Brownian motion.

Now, recalling the results in references [48, 52, 53], some properties of the BFBM are summarized as follows:

- (i) For $t \geq 0$, $E(B_t^{H,K})^2 = t^{2HK}$.
- (ii) The BFBM is a self-similar process with Hurst parameter HK , that is,

$$\{B_{at}^{H,K}, t \geq 0\} \stackrel{d}{=} \{\alpha^{HK} B_t^{H,K}, t \geq 0\}, \quad \alpha > 0, \quad (2)$$

where $X \stackrel{d}{=} Y$ means the two processes X and Y have the same distributions.

- (iii) If $HK > (1/2)$, the BFBM exhibits long-range dependence in the case $\sum_{n=1}^{\infty} \rho(n) = +\infty$, where

$$\rho(n) = E[(B_1^{H,K})(B_{n+1}^{H,K} - B_n^{H,K})]. \quad (3)$$

- (iv) For every $s, t \geq 0$, we have

$$2^{-K}|t-s|^{2HK} \leq E(B_t^{H,K} - B_s^{H,K})^2 \leq 2^{1-K}|t-s|^{2HK}. \quad (4)$$

Note that $B^{H,K}$ does not have stationary increments, except in the case $K = 1$, and $B^{H,K}$ has $(HK - \delta)$ -Hölder continuous paths for any $\delta \in (0, HK)$, thanks to Kolmogorov's continuity criterion.

2.2. Actuarial Approach. The payoff of a European vulnerable call option and put option at maturity date T in this paper are expressed which are consistent with the reference [15] by the following equations, respectively,

$$\begin{aligned} C_T^d &= E\left[(S_T - X)^+ \left(I_{\{V_T \geq D^*\}} + \frac{1-\alpha}{D} V_T I_{\{V_T < D^*\}}\right)\right], \\ P_T^d &= E\left[(X - S_T)^+ \left(I_{\{V_T \geq D^*\}} + \frac{1-\alpha}{D} V_T I_{\{V_T < D^*\}}\right)\right], \end{aligned} \quad (5)$$

where S_T and V_T are the stock price and the firm value of the counterparty at maturity date T , respectively. X is the strike price of the options. D^* is a constant default boundary such

that no credit loss occurs if V_T is greater than D^* and the option holder receives full payments. If V_T is less than D^* , the option issuer cannot realize the promise of all payments to the holder, then the credit loss occurs. In the event of the credit loss, the option holder only obtains the proportion $((1-\alpha)/D)V_T$ of the nominal claim by the option issuer, where α denotes the deadweight costs associated with bankruptcy expressed as a percentage of the firm value of the counterparty and D is the value of total liabilities given by D^* and an additional liability.

Definition 2. Assume that $\beta_S(w)$ and $\beta_V(w)$ ($w \in [0, T]$) are the expected yield rates of the stock price process $\{S_t, t \geq 0\}$ and the firm value process of counterparty $\{V_t, t \geq 0\}$ on $t \in [0, T]$, respectively, which are defined as follows:

$$\frac{E(S_T)}{S_0} = \exp\left(\int_0^T \beta_S(w) dw\right), \quad (6)$$

$$\frac{E(V_T)}{V_0} = \exp\left(\int_0^T \beta_V(w) dw\right). \quad (7)$$

Definition 3. Suppose that C_0^d and P_0^d are the value of the European vulnerable call and put option at time 0 with the maturity date T , respectively, whose stock price is S_t , firm value of the counterparty is V_t , and the strike price is X . Then, the value of the European vulnerable options by actuarial approach is presented as follows, respectively:

$$C_0^d = E\left[\left(e^{-\int_0^T \beta_S(w) dw} S_T - e^{-rT} X\right) \left(I_{A_1, B_1} + \frac{1-\alpha}{e^{-rT} D} e^{-\int_0^T \beta_V(w) dw} V_T I_{A_1, B_2}\right)\right], \quad (8)$$

$$P_0^d = E\left[\left(e^{-rT} X - e^{-\int_0^T \beta_S(w) dw} S_T\right) \left(I_{A_2, B_1} + \frac{1-\alpha}{e^{-rT} D} e^{-\int_0^T \beta_V(w) dw} V_T I_{A_2, B_2}\right)\right], \quad (9)$$

where $I_{(\cdot)}$ is an indicator function. A_i and B_i ($i = 1, 2$) are given by

$$\begin{aligned} A_1 &= e^{-\int_0^T \beta_S(w) dw} S_T \geq e^{-rT} X, \\ A_2 &= e^{-\int_0^T \beta_S(w) dw} S_T < e^{-rT} X, \\ B_1 &= e^{-\int_0^T \beta_V(w) dw} V_T \geq e^{-rT} D^*, \\ B_2 &= e^{-\int_0^T \beta_V(w) dw} V_T < e^{-rT} D^*. \end{aligned} \quad (10)$$

A_1 and A_2 are the essential conditions for performing the European vulnerable call and put option on the maturity

date T , respectively. B_1 and B_2 indicate whether a credit loss occurs or not.

3. Pricing for Vulnerable Options in the Jump BFBM Environment

In this section, we derived closed-form pricing formulae for European vulnerable options in the BFBM environment with jumps by actuarial approach. Taking the long-range dependence and jumps in the underlying asset price and the firm value of the counterparty into consideration, we adopt jump bifractional Brownian motions to describe the dynamics of the underlying asset price and the firm value of the counterparty. As in [18], we divide the jump term in the underlying asset price into the individual jumps and the common jumps as well as the firm value of the counterparty.

Now, our pricing model is described in detail as follows.

3.1. Financial Model Description

Assumption 1. Let S_t be the underlying asset price. S_t is given by the following jump-diffusion process:

$$dS_t = S_t \left[(\mu_S(t) - \lambda_S^* \theta_S) dt + \sigma_S dB_S^{H,K}(t) + (J_S(t) - 1) dM_S(t) \right], \quad (11)$$

where $\mu_S(t)$, which is a determinate function of time t , is the instantaneous expected return of S_t , σ_S is its volatility, $B_S^{H,K}(t)$ is the BFBM, and $J_S(t) - 1$ is the percentage jump of S_t with mean $E(J_S(t) - 1) = \theta_S$. Specially, for different time, $J_S(t)$ is a sequence of lognormal, independent, and identical distributed with mean μ_1 and variance σ_1^2 by Merton [3]. Thus, $\theta_S = e^{\mu_1 + (\sigma_1^2/2)} - 1$. The jump term $(J_S(t) - 1) dM_S(t)$ describes discontinuous changes in the underlying asset price. $M_S(t)$ denotes a Poisson process with intensity λ_S^* . In our framework, jump term $M_S(t)$ and the jump intensity λ_S^* are both composed of two parts:

$$\begin{aligned} M_S(t) &= N_S(t) + N(t), \\ \lambda_S^* &= \lambda_S + \lambda. \end{aligned} \quad (12)$$

Moreover, sudden shocks occurred in the underlying asset price consist of two parts. One part is individual shocks which correspond to $N_S(t)$. The other part is common shocks linked to market factors, corresponding to $N(t)$ which affects the prices of the underlying asset and the firm value of the counterparty, where $N_S(t)$ with the intensity λ_S and $N(t)$ with the intensity λ are independent Poisson processes.

Assumption 2. Let V_t be the firm value of the counterparty. The dynamics of V_t is governed by the following jump-diffusion process:

$$dV_t = V_t \left[(\mu_V(t) - \lambda_V^* \theta_V) dt + \sigma_V dB_V^{H,K}(t) + (J_V(t) - 1) dM_V(t) \right], \quad (13)$$

where $\mu_V(t)$, λ_V^* , θ_V , σ_V , and $B_V^{H,K}(t)$ are the expected return of V_t , the jump intensity of the Poisson process $M_V(t)$, the mean percentage jump of V_t , the volatility of V_t , and the BFBM, respectively. $J_V(t) - 1$ is the percentage jump. For different time, $J_V(t)$ is a sequence of lognormal, independent, and identical distributed with mean μ_2 and variance σ_2^2 as in [3]. Then, $\theta_V = e^{\mu_2 + (\sigma_2^2/2)} - 1$. Similarly, $M_V(t)$ and λ_V^* are both expressed as

$$\begin{aligned} M_V(t) &= N_V(t) + N(t), \\ \lambda_V^* &= \lambda_V + \lambda, \end{aligned} \quad (14)$$

where $N_V(t)$, which denotes individual jumps, is a Poisson process with intensity λ_V and independent of $N(t)$ and $N_S(t)$.

Further, it is supposed that $B_S^{H,K}(t)$ and $B_V^{H,K}(t)$ are two correlated BFBM with correlation coefficient ρ_{SV} , these processes $N(t)$, $N_S(t)$, $N_V(t)$, $\ln J_S(t)$, $\ln J_V(t)$, $B_i^{H,K}(t)$, ($i \in S, V$) are mutually independent.

3.2. Pricing Formulae for Vulnerable Options. In order to derive the closed-form pricing formulae for European vulnerable options, we need the following lemma.

Lemma 1. The expectation yield rates $\beta_S(w)$ and $\beta_V(w)$, given by (6) and (7), respectively, satisfy the following equalities:

$$\int_0^T \beta_S(w) dw = \int_0^T \mu_S(w) dw, \quad (15)$$

$$\int_0^T \beta_V(w) dw = \int_0^T \mu_V(w) dw. \quad (16)$$

Proof. By applying bifractional Itô formula by Es-Sebaï and Tudor [49], it is easy to obtain the solution of equation (11):

$$S_T = S_0 e^{\int_0^T \mu_S(w) dw - \lambda_S^* \theta_S T - (1/2) \sigma_S^2 T^{2HK} + \sigma_S B_S^{H,K}(T) + \sum_{i=1}^{M_S(T)} \ln J_S(t_i)}, \quad (17)$$

where t_i denotes the i th jump time of $M_S(T)$. For different time, $\ln J_S(t_i)$ is independent normally distributed with mean μ_1 and variance σ_1^2 , as in [3].

Noting that $B_S^{H,K}(T) \sim N(0, T^{2HK})$, $E[e^{\sigma_S B_S^{H,K}(T)}] = e^{(1/2) \sigma_S^2 T^{2HK}}$, the mutual independence of $B_S^{H,K}(T)$, $M_S(T)$

and $\ln J_S(t_i)$, and the theory of Poisson distribution with intensity $\lambda_S^* T$, then we can get

$$\frac{E(S_T)}{S_0} = e^{\int_0^T \mu_S(w) dw - \lambda_S^* \theta_S T - (1/2) \sigma_S^2 T^{2HK}} E \left(\exp(B_S^{H,K}(T)) E \left[\exp \left(\sum_{i=1}^{M_S(T)} \ln J_S(t_i) \right) \right] \right)$$

$$\begin{aligned}
&= e \int_0^T \mu_s(w) dw - \lambda_s^* \theta_s T E \left[\exp \left(\sum_{i=1}^{M_s(T)} \ln J_s(t_i) \right) \right] \\
&= e \int_0^T \mu_s(w) dw - \lambda_s^* \theta_s T E \left[E \left[\exp \left(\sum_{i=1}^{M_s(T)} \ln J_s(t_i) \right) \middle| M_s(T) \right] \right] \\
&= e \int_0^T \mu_s(w) dw - \lambda_s^* \theta_s T \sum_{m=0}^{\infty} P(M_s(T) = m) E \left[\exp \left(\sum_{i=0}^m \ln J_s(t_i) \right) \middle| M_s(T) = m \right] \\
&= e \int_0^T \mu_s(w) dw - \lambda_s^* \theta_s T \sum_{m=0}^{\infty} P(M_s(T) = m) (E(J_s(t_i)))^m \\
&= e \int_0^T \mu_s(w) dw - \lambda_s^* (1 + \theta_s) T \sum_{m=0}^{\infty} \frac{(\lambda_s^* T)^m}{m!} (1 + \theta_s)^m \\
&= e \int_0^T \mu_s(w) dw.
\end{aligned} \tag{18}$$

Combing (6) and (18), the proof of (15) is completed. Similarly, we can also show that the equality (16) is satisfied. $\square$

Theorem 1. Assume that the underlying asset price S_t and the firm value of the counterparty V_t are given by (11) and (12). Thus, the explicit formulae for European vulnerable option prices are expressed as

$$\begin{aligned}
C_0^d &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_s T) P_{m_2-n}(\lambda_V T) \\
&\quad \cdot \left(S_0 e^{-\lambda_s^* \theta_s T + m_1 (\mu_1 + (\sigma_1^2/2))} N_2(a_1(m_1), a_2(m_2), \rho) - X e^{-rT} N_2(b_1(m_1), b_2(m_2), \rho) \right. \\
&\quad \left. + \frac{1-\alpha}{D} S_0 V_0 e^{(r - \lambda_s^* \theta_s - \lambda_V^* \theta_V)T + m_1 (\mu_1 + (\sigma_1^2/2)) + m_2 (\mu_2 + (\sigma_2^2/2)) + \rho \delta_s \delta_V} N_2(c_1(m_1), -c_2(m_2), -\rho) \right. \\
&\quad \left. - \frac{1-\alpha}{D} X V_0 e^{-\lambda_V^* \theta_V T + m_2 (\mu_2 + (\sigma_2^2/2))} N_2(d_1(m_1), -d_2(m_2), -\rho) \right),
\end{aligned} \tag{19}$$

$$\begin{aligned}
P_0^d &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_s T) P_{m_2-n}(\lambda_V T) \\
&\quad \cdot \left(X e^{-rT} N_2(-b_1(m_1), b_2(m_2), -\rho) - S_0 e^{-\lambda_s^* \theta_s T + m_1 (\mu_1 + (\sigma_1^2/2))} N_2(-a_1(m_1), a_2(m_2), -\rho) \right. \\
&\quad \left. + \frac{1-\alpha}{D} X V_0 e^{-\lambda_V^* \theta_V T + m_2 (\mu_2 + (\sigma_2^2/2))} N_2(-d_1(m_1), -d_2(m_2), \rho) \right. \\
&\quad \left. - \frac{1-\alpha}{D} S_0 V_0 e^{(r - \lambda_s^* \theta_s - \lambda_V^* \theta_V)T + m_1 (\mu_1 + (\sigma_1^2/2)) + m_2 (\mu_2 + (\sigma_2^2/2)) + \rho \delta_s \delta_V} N_2(-c_1(m_1), -c_2(m_2), \rho) \right),
\end{aligned} \tag{20}$$

where $P_n(\lambda T)$, $P_{m_1-n}(\lambda_s T)$, and $P_{m_2-n}(\lambda_V T)$ are the probability functions of $N(T)$, $N_s(T)$, and $N_V(T)$ throughout the paper, respectively, with $P_n(\lambda T) = ((\lambda T)^n/n!)e^{-\lambda T}$, P_{m_1-n}

$(\lambda_s T) = ((\lambda_s T)^{m_1-n}/(m_1-n!)e^{-\lambda_s T}$, $P_{m_2-n}(\lambda_V T) = ((\lambda_V T)^{m_2-n}/(m_2-n!)e^{-\lambda_V T}$. $N_2(\cdot)$ denotes the bivariate normal cumulative distribution function given by

$$\begin{aligned}
N_2(x, y, \rho) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^x \int_{-\infty}^y e^{-(1/2(1-\rho^2))(u^2-2\rho uv+v^2)} du dv, \\
a_1(m_1) &= b_1(m_1) + \delta_S, \\
a_2(m_2) &= b_2(m_2) + \rho\delta_S, \\
b_1(m_1) &= \frac{\ln(S_0/X) + (r - \lambda_S^* \theta_S)T + m_1\mu_1 - (\sigma_S^2 T^{2HK}/2)}{\delta_S}, \\
b_2(m_2) &= \frac{\ln(V_0/D^*) + (r - \lambda_V^* \theta_V)T + m_2\mu_2 - (\sigma_V^2 T^{2HK}/2)}{\delta_V}, \\
c_1(m_1) &= b_1(m_1) + \delta_S + \rho\delta_V, \\
c_2(m_2) &= b_2(m_2) + \delta_V + \rho\delta_S, \\
d_1(m_1) &= b_1(m_1) + \rho\delta_V, \\
d_2(m_2) &= b_2(m_2) + \delta_V, \\
\delta_S^2 &= \sigma_S^2 T^{2HK} + m_1\sigma_1^2, \\
\delta_V^2 &= \sigma_V^2 T^{2HK} + m_2\sigma_2^2, \rho = \frac{\rho_{SV}\sigma_S\sigma_V T^{2HK}}{\delta_S\delta_V}.
\end{aligned} \tag{21}$$

Proof. We only show the pricing formula (19) for a European vulnerable call option which is satisfied in this part. The corresponding put option price (20) can also be derived by the similar calculations.

For the sake of simplicity, (8) can be rewritten as

$$C_0^d = I_1 - I_2 + I_3 - I_4, \tag{22}$$

where I_1, I_2, I_3, I_4 are given by

$$I_1 = E \left[e^{-\int_0^T \beta_S(w)dw} S_T I_{A_1, B_1} \right],$$

$$I_2 = X e^{-rT} E[I_{A_1, B_1}],$$

$$\begin{aligned}
I_3 &= \frac{1-\alpha}{e^{-rT}D} E \left[e^{-\int_0^T \beta_S(w)dw} S_T e^{-\int_0^T \beta_V(w)dw} V_T I_{A_1, B_2} \right], \\
I_4 &= \frac{(1-\alpha)X}{D} E \left[e^{-\int_0^T \beta_V(w)dw} V_T I_{A_1, B_2} \right].
\end{aligned} \tag{23}$$

In order to calculate I_1, I_2, I_3, I_4 , we give some notations:

$$\begin{aligned}
S_T^{m_1} &= S_0 e^{\int_0^T \mu_S(w)dw - \lambda_S^* \theta_S T - (1/2)\sigma_S^2 T^{2HK} + \sigma_S B_S^{H,K}(T) + \sum_{i=1}^{m_1} \ln J_S(t_i)}, \\
V_T^{m_2} &= V_0 e^{\int_0^T \mu_V(w)dw - \lambda_V^* \theta_V T - (1/2)\sigma_V^2 T^{2HK} + \sigma_V B_V^{H,K}(T) + \sum_{i=1}^{m_2} \ln J_V(t_i)}, \\
z_1 &= \frac{\sigma_S B_S^{H,K}(T) + \sum_{i=1}^{m_1} \ln J_S(t_i) - m_1 \mu_1}{\delta_S}, \\
\delta_S^2 &= (\sigma_S)^2 T^{2HK} + m_1 \sigma_1^2,
\end{aligned}$$

$$\begin{aligned}
z_2 &= \frac{\sigma_V B_V^{H,K}(T) + \sum_{i=1}^{m_2} \ln J_V(t_i) - m_2 \mu_2}{\delta_V}, \\
\delta_V^2 &= (\sigma_V)^2 T^{2HK} + m_2 \sigma_2^2, \\
b_1(m_1) &= \frac{\ln(S_0/X) + (r - \lambda_S \theta_S)T + m_1 \mu_1 - (\sigma_S^2 T^{2HK}/2)}{\delta_S}, \\
b_2(m_2) &= \frac{\ln(V_0/D^*) + (r - \lambda_V \theta_V)T + m_2 \mu_2 - (\sigma_V^2 T^{2HK}/2)}{\delta_V}.
\end{aligned} \tag{24}$$

Now, we deal with each of these terms I_1, I_2, I_3, I_4 separately.

Suppose that conditional on $\mathcal{G}_T^{n, n_1, n_2} := \{N(T) = n, N_S(T) = n_1, N_V(T) = n_2\}$, the total jump times of S_T and V_T are denoted by $m_1 \triangleq n + n_1, m_2 \triangleq n + n_2$, for simplicity.

Applying the mutual independence of $N(t), N_S(t), N_V(t), \ln J_S(t), \ln J_V(t), B_i^{H,K}(t) (i \in S, V)$, equations (11) and (13) and Lemma 1, we have

$$\begin{aligned}
I_2 &= X e^{-rT} E(I_{A_1, B_1}) \\
&= X e^{-rT} E[E(I_{A_1, B_1} | N(T), N_S(T), N_V(T))] \\
&= X e^{-rT} \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_n(\lambda T) P_{n_1}(\lambda_S T) P_{n_2}(\lambda_V T) \\
&\quad \cdot E \left[I_{e^{-\int_0^T \beta_S(w) dw} S_T^{m_1} \geq e^{-rT} K, e^{-\int_0^T \beta_V(w) dw} V_T^{m_2} \geq e^{-rT} D^*} | N(T) = n, N_S(T) = n_1, N_V(T) = n_2 \right] \\
&= X e^{-rT} \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) E(I_{z_1 \geq -b_1(m_1), z_2 \geq -b_2(m_2)}).
\end{aligned} \tag{25}$$

Noting that $B_S^{H,K}(T) \sim N(0, T^{2HK}), B_V^{H,K}(T) \sim N(0, T^{2HK}), \text{cov}(B_S^{H,K}(T), B_V^{H,K}(T)) = \rho_{SV} T^{2HK}, \ln J_S(t_i)$ is independent normally distributed with mean μ_1 and variance σ_1^2 , then (z_1, z_2) obey standard bivariate normal distribution with the correlation $\rho = (\rho_{SV} \sigma_S \sigma_V T^{2HK} / \delta_S \delta_V)$. Therefore, we obtain

$$\begin{aligned}
I_2 &= X e^{-rT} \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) \\
&\quad (\lambda_V T) N_2(b_1(m_1), b_2(m_2), \rho).
\end{aligned} \tag{26}$$

Similar to the calculation of I_2 , we have

$$\begin{aligned}
I_1 &= E \left[E \left(e^{-\int_0^T \beta_S(w) dw} S_T I_{A_1, B_1} | N(T), N_S(T), N_V(T) \right) \right] \\
&= \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P_n(\lambda T) P_{n_1}(\lambda_S T) P_{n_2}(\lambda_V T) S_0 e^{-\lambda_S^* \theta_S T + m_1 (\mu_1 + (\sigma_1^2/2))} \\
&\quad \cdot E \left[e^{\delta_S z_1 - (\delta_S^2/2)} I_{z_1 \geq -b_1(m_1), z_2 \geq -b_2(m_2)} | N(T) = n, N_S(T) = n_1, N_V(T) = n_2 \right] \\
&= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) S_0 e^{-\lambda_S^* \theta_S T + m_1 (\mu_1 + (\sigma_1^2/2))} \\
&\quad \cdot \int_{-b_1(m_1)}^{+\infty} \int_{-b_2(m_2)}^{+\infty} e^{\delta_S z_1 - (\delta_S^2/2)} \varphi(z_1, z_2, \rho) dz_1 dz_2,
\end{aligned} \tag{27}$$

where $\varphi(z_1, z_2, \rho)$, the standard bivariate normal probability density function, is given by

$$\varphi(z_1, z_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}(1-\rho^2)(z_1^2+z_2^2-2\rho z_1 z_2)}. \quad (28)$$

Applying the method of undetermined coefficients to equation (27), the exponent of the integrand in (27) is of the following form:

$$-\frac{1}{2(1-\rho^2)}((z_1+a)^2+(z_2+b)^2-2\rho(z_1+a)(z_2+b))+M, \quad (29)$$

where $a = -\delta_S, b = -\rho\delta_S, M = 0$. Setting $a_1(m_1) = b_1(m_1) + \delta_S, a_2(m_2) = b_2(m_2) + \rho\delta_S$, then we obtain

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) S_0 e^{-\lambda_S^* \theta_S T + m_1(\mu_1 + (\sigma_1^2/2))} \\ &\quad \cdot \int_{-b_1(m_1)}^{+\infty} \int_{-b_2(m_2)}^{+\infty} \varphi(z_1 - \delta_S, z_2 - \rho\delta_S, \rho) dz_1 dz_2 \\ &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) S_0 e^{-\lambda_S^* \theta_S T + m_1(\mu_1 + (\sigma_1^2/2))} \cdot N_2(a_1(m_1), a_2(m_2), \rho). \end{aligned} \quad (30)$$

In order to calculate I_3 , we observe that $e^{-\int_0^T \beta_V(w) dw} V_T^{m_2} < e^{-rT} D^*$ is equivalent to $z_2 < -b_2(m_2)$, then we deduce

$$\begin{aligned} I_3 &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) \frac{1-\alpha}{D} S_0 V_0 e^{(r-\lambda_S^* \theta_S - \lambda_V^* \theta_V)T} \\ &\quad \cdot e^{m_1(\mu_1 + (\sigma_1^2/2)) + m_2(\mu_2 + (\sigma_2^2/2))} \int_{-b_1(m_1)}^{+\infty} \int_{-b_2(m_2)}^{-\infty} e^{\delta_S z_1 + \delta_V z_2 - (\delta^2 S/2)} \varphi(z_1, z_2, \rho) dz_1 dz_2 \\ &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) \frac{1-\alpha}{D} S_0 V_0 e^{(r-\lambda_S^* \theta_S - \lambda_V^* \theta_V)T} \\ &\quad \cdot e^{m_1(\mu_1 + (\sigma_1^2/2)) + m_2(\mu_2 + (\sigma_2^2/2)) + \rho\delta_S \delta_V} \int_{-b_1(m_1)}^{+\infty} \int_{-\infty}^{-b_2(m_2)} \varphi(z_1 - \delta_S - \rho\delta_V, z_2 - \delta_V - \rho\delta_S, \rho) dz_1 dz_2. \end{aligned} \quad (31)$$

Setting $c_1(m_1) = b_1(m_1) + \delta_S + \rho\delta_V$ and $c_2(m_2) = b_2(m_2) + \delta_V + \rho\delta_S$, then

$$\begin{aligned} I_3 &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) \frac{1-\alpha}{D} S_0 V_0 e^{(r-\lambda_S^* \theta_S - \lambda_V^* \theta_V)T} \\ &\quad \cdot e^{m_1(\mu_1 + (\sigma_1^2/2)) + m_2(\mu_2 + (\sigma_2^2/2)) + \rho\delta_S \delta_V} N(c_1(m_1), -c_2(m_2), -\rho). \end{aligned} \quad (32)$$

For the term I_4 , it is easy to obtain

$$\begin{aligned} I_4 &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) \frac{(1-\alpha)XV_0}{D} \\ &\quad \cdot e^{-\lambda_V^* \theta_V T + m_2(\mu_2 + (\sigma_2^2/2))} \int_{-b_1(m_1)}^{+\infty} \int_{-\infty}^{-b_2(m_2)} \varphi(z_1 - \rho\delta_V, z_2 - \delta_V, \rho) dz_1 dz_2. \end{aligned} \quad (33)$$

Setting $d_1(m_1) = b_1(m_1) + \rho\delta_V$ and $d_2(m_2) = b_2(m_2) + \delta_V$, then

$$I_4 = \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) \frac{(1-\alpha)XV_0}{D} \cdot e^{-\lambda_V^* \theta_V T + m_2(\mu_2 + (\sigma_V^2/2))} N(d_1(m), -d_2(n), -\rho). \quad (34)$$

Combining the calculated I_1, I_2, I_3, I_4 , the pricing formula (19) is derived. $\square$

3.3. Special Cases of the Pricing Formulae. Compared with many existing models such as Black–Scholes [1], Merton [3], Klein [15], and Tian et al. [18], our proposed framework not only adds jump risk and credit risk but also adds the long-range dependence of financial asset prices. Therefore, our model encompasses these existing models as special cases. The details are as follows.

Case 1. Black–Scholes model [1].

If there is no jump risk, credit risk, and long-range dependence in our model, i.e., $\lambda = \lambda_S = \lambda_V = 0, n = n_1 = n_2 = 0, D^* \rightarrow 0, H = (1/2)$ and $K = 1$, our model will reduce to

the Black–Scholes model. The formulae for European options under the Black–Scholes model are given by

$$C_0 = S_0 N(a) - Xe^{-rT} N(b), \quad (35)$$

$$P_0 = Xe^{-rT} N(-b) - S_0 N(-a), \quad (36)$$

where $a = b + \sigma_S \sqrt{T}, b = (\ln(S_0/X) + (r - (\sigma_S^2/2))T/\sigma_S \sqrt{T}), N(\cdot)$ denotes the standard normal cumulative distribution function.

It is easily seen that $a_1(0)$ and $b_1(0)$ reduce to $a = b + \sigma_S \sqrt{T}$ and $b = (\ln(S_0/X) + (r - (\sigma_S^2/2))T/\sigma_S \sqrt{T})$. Moreover, $a_2(0) \rightarrow +\infty, b_2(0) \rightarrow +\infty, c_2(0) \rightarrow +\infty, d_2(0) \rightarrow +\infty$ as $D^* \rightarrow 0$. Note that $N_2(z, -\infty, \rho) \rightarrow 0, N_2(z, +\infty, \rho) \rightarrow N(z)$, then formulae (19) and (20) can be simplified to formulae (35) and (36), respectively.

Case 2. Merton jump-diffusion model [3].

If there is no credit risk, i.e., $D^* \rightarrow 0$, and the underlying asset prices are governed by geometric Brownian motion with jumps, that is, $H = (1/2)$ and $K = 1$, then formula (19) can be rewritten as the following form:

$$\begin{aligned} C_0^d &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} \frac{(\lambda T)^n}{n!} \frac{(\lambda_S T)^{m_1-n}}{(m_1-n)!} \frac{(\lambda_V T)^{m_2-n}}{(m_2-n)!} e^{(-\lambda-\lambda_S-\lambda_V)T} \\ &\quad \cdot \left(S_0 e^{-\lambda_S^* \theta_S T + m_1(\mu_1 + (\sigma_1^2/2))} N_2(a_1(m_1)) - Xe^{-rT} N_2(b_1(m_1)) \right) \\ &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \frac{(\lambda T)^n}{n!} \frac{(\lambda_S T)^{m_1-n}}{(m_1-n)!} e^{(-\lambda-\lambda_S)T} \\ &\quad \cdot \left(S_0 e^{-\lambda_S^* \theta_S T + m_1(\mu_1 + (\sigma_1^2/2))} N_2(a_1(m_1)) - Xe^{-rT} N_2(b_1(m_1)) \right) \\ &= \sum_{i=0}^{\infty} \frac{(\lambda_S^* T)^i}{i!} e^{-\lambda_S^* T} \left(S_0 e^{-\lambda_S^* \theta_S T + m_1(\mu_1 + (\sigma_1^2/2))} N_2(a_1(i)) - Xe^{-rT} N_2(b_1(i)) \right), \end{aligned} \quad (37)$$

where equality (37) comes from the expression $S_0 e^{-\lambda_S^* \theta_S T + m_1(\mu_1 + (\sigma_1^2/2))} N_2(a_1(m_1)) - Xe^{-rT} N_2(b_1(m_1))$ which has no relationship with m_2 , $\sum_{m_2=n}^{\infty} ((\lambda_V T)^{m_2-n}/(m_2-n)!) e^{-\lambda_V T} = 1$, and the binomial theorem $(a+b)^n = \sum_{k=0}^n C_n^k a^k b^{n-k}$.

Thus, formula (37) is the valuation of European call option under the Merton jump-diffusion process. Similarly, the pricing formula for the put option is

$$P_0^d = \sum_{i=0}^{\infty} \frac{(\lambda_S^* T)^i}{i!} e^{-\lambda_S^* T} \left(Xe^{-rT} N_2(-b_1(i)) - S_0 e^{-\lambda_S^* \theta_S T + m_1(\mu_1 + (\sigma_1^2/2))} N_2(-a_1(i)) \right) \quad (38)$$

Case 3. Klein model [15].

If no jump occurs in the underlying asset prices as well as the firm value of the counterparty, that is,

$\lambda = \lambda_S = \lambda_V, n = n_1 = n_2 = 0$, and the underlying asset prices and the firm value of the counterparty both obey geometric Brownian motion, i.e., $H = (1/2)$ and $K = 1$, formulae (19)

and (20) for the European vulnerable options, which become the Klein's pricing formulae, are represented as follows:

$$C_0^d = S_0 N_2(a_1, a_2, \rho_{SV}) - X e^{-rT} N_2(b_1, b_2, \rho_{SV}) + \frac{1-\alpha}{D} S_0 V_0 e^{rT + \rho_{SV} \sigma_S \sigma_V T} N_2(c_1, -c_2, -\rho_{SV}) - \frac{1-\alpha}{D} X V_0 N_2(d_1, -d_2, -\rho_{SV}), \quad (39)$$

$$P_0^d = X e^{-rT} N_2(-b_1, b_2, -\rho_{SV}) - S_0 N_2(-a_1, a_2, -\rho_{SV}) + \frac{1-\alpha}{D} X V_0 N_2(-d_1, -d_2, \rho_{SV}) - \frac{1-\alpha}{D} S_0 V_0 e^{rT + \rho_{SV} \sigma_S \sigma_V T} N_2(-c_1, -c_2, \rho_{SV}), \quad (40)$$

where

$$\begin{aligned} a_1 &= b_1 + \sigma_S \sqrt{T}, \\ a_2 &= b_2 + \rho_{SV} \sigma_S \sqrt{T}, \\ b_1 &= \frac{\ln(S_0/X) + (r - (\sigma_S^2/2))T}{\sigma_S \sqrt{T}}, \\ b_2 &= \frac{\ln(V_0/D^*) + (r - (\sigma_V^2/2))T}{\sigma_V \sqrt{T}}, \\ c_1 &= b_1 + \sigma_S \sqrt{T} + \rho_{SV} \sigma_V \sqrt{T}, \\ c_2 &= b_2 + \sigma_V \sqrt{T} + \rho_{SV} \sigma_S \sqrt{T}, \\ d_1 &= b_1 + \rho_{SV} \sigma_V \sqrt{T}, \\ d_2 &= b_2 + \sigma_V \sqrt{T}. \end{aligned} \quad (41)$$

Case 4. Vulnerable options under jump-diffusion processes in [18].

When $H = (1/2)$, $K = 1$, our jump bifractional Brownian motion model for pricing vulnerable options will reduce to the jump-diffusion model in the Brownian motion environment which was studied by Tian et al. [18]. Here, the pricing formulae (19) and (20) for European vulnerable options can be restated as follows:

$$\begin{aligned} C_0^d &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) \\ &\cdot \left(S_0 e^{-\lambda_S^* \theta_S T + m_1 (\mu_1 + (\sigma_1^2/2))} N_2(\hat{a}_1(m_1), \hat{a}_2(m_2), \hat{\rho}) - X e^{-rT} N_2(\hat{b}_1(m_1), \hat{b}_2(m_2), \hat{\rho}) \right. \\ &+ \frac{1-\alpha}{D} S_0 V_0 e^{(r - \lambda_S^* \theta_S - \lambda_V^* \theta_V)T + m_1 (\mu_1 + (\sigma_1^2/2)) + m_2 (\mu_2 + (\sigma_2^2/2)) + \hat{\rho} \hat{\delta}_S \hat{\delta}_V} N_2(\hat{c}_1(m_1), -\hat{c}_2(m_2), -\hat{\rho}) \\ &\left. - \frac{1-\alpha}{D} X V_0 e^{-\lambda_V^* \theta_V T + m_2 (\mu_2 + (\sigma_2^2/2))} N_2(\hat{d}_1(m_1), -\hat{d}_2(m_2), -\hat{\rho}) \right), \end{aligned} \quad (42)$$

$$\begin{aligned} P_0^d &= \sum_{n=0}^{\infty} \sum_{m_1=n}^{\infty} \sum_{m_2=n}^{\infty} P_n(\lambda T) P_{m_1-n}(\lambda_S T) P_{m_2-n}(\lambda_V T) \\ &\cdot \left(X e^{-rT} N_2(-\hat{b}_1(m_1), \hat{b}_2(m_2), -\hat{\rho}) - S_0 e^{-\lambda_S^* \theta_S T + m_1 (\mu_1 + (\sigma_1^2/2))} N_2(-\hat{a}_1(m_1), \hat{a}_2(m_2), -\hat{\rho}) \right. \\ &+ \frac{1-\alpha}{D} X V_0 e^{-\lambda_V^* \theta_V T + m_2 (\mu_2 + (\sigma_2^2/2))} N_2(-\hat{d}_1(m_1), -\hat{d}_2(m_2), \hat{\rho}) \\ &\left. - \frac{1-\alpha}{D} S_0 V_0 e^{(r - \lambda_S^* \theta_S - \lambda_V^* \theta_V)T + m_1 (\mu_1 + (\sigma_1^2/2)) + m_2 (\mu_2 + (\sigma_2^2/2)) + \hat{\rho} \hat{\delta}_S \hat{\delta}_V} N_2(-\hat{c}_1(m_1), -\hat{c}_2(m_2), \hat{\rho}) \right), \end{aligned} \quad (43)$$

TABLE 1: Model comparisons.

Pricing models	Jump risk	Credit risk	Long-range dependence	Pricing formulae
Black–Scholes model [1]	—	—	—	(35), (36)
Merton model [3]	Poisson jump	—	—	(37), (38)
Klein model [15]	—	Structural approach	—	(39), (40)
Tian et al. pricing model [18]	Poisson jump	Structural approach	—	(42), (43)
Our proposed model	Poisson jump	Structural approach	Bifractional Brownian motion	(19), (20)

where

$$\begin{aligned}
\hat{a}_1(m_1) &= \hat{b}_1(m_1) + \hat{\delta}_S, \\
\hat{a}_2(m_2) &= \hat{b}_2(m_2) + \hat{\rho}\hat{\delta}_S, \\
\hat{b}_1(m_1) &= \frac{\ln(S_0/X) + (r - (\sigma_S^2/2) - \lambda_S^*\theta_S)T + m_1\mu_1}{\hat{\delta}_S}, \\
\hat{b}_2(m_2) &= \frac{\ln(V_0/D^*) + (r - (\sigma_V^2/2) - \lambda_V^*\theta_V)T + m_2\mu_2}{\hat{\delta}_V}, \\
\hat{c}_1(m_1) &= \hat{b}_1(m_1) + \hat{\delta}_S + \hat{\rho}\hat{\delta}_V, \\
\hat{c}_2(m_2) &= \hat{b}_2(m_2) + \hat{\delta}_V + \hat{\rho}\hat{\delta}_S, \\
\hat{d}_1(m_1) &= \hat{b}_1(m_1) + \hat{\rho}\hat{\delta}_V, \\
\hat{d}_2(m_2) &= \hat{b}_2(m_2) + \hat{\delta}_V, \\
\hat{\delta}_S^2 &= \sigma_S^2T + m_1\sigma_1^2, \\
\hat{\delta}_V^2 &= \sigma_V^2T^{2HK} + m_2\sigma_2^2, \\
\hat{\rho} &= \frac{\rho_{SV}\sigma_S\sigma_VT}{\hat{\delta}_S\hat{\delta}_V}.
\end{aligned} \tag{44}$$

In conclusion, the relationship among the models mentioned above is summarized in Table 1.

4. Concluding Remarks and Future Directions

Based on the consideration of the jump risk of financial asset prices, the credit risk of the counterparty and the long-range dependence of financial asset prices in options trading, a more realistic pricing model for European vulnerable options is proposed in this paper. Assuming that the underlying asset price and the firm value of the counterparty are both governed by jump-diffusion processes under the bifractional Brownian motion environment, in which these jump-diffusion processes are correlated and have common factors in jump process components, the total jumps in the two processes are divided into idiosyncratic jumps for every asset price and common ones that affect the prices of all assets. In our framework, we apply the actuarial approach to derive the closed-form formulae for the European

vulnerable options, which encompass many existing formulae as special cases.

This study also suggests several important directions for further research. First, it is rather interesting and meaningful to make empirical study for vulnerable options which could help to adjust the parameters in the pricing model to better capture the real data. Second is to take more practical assumptions into account based on the current model which would make the pricing results for vulnerable options accurate. For example, a pricing model for European vulnerable options under BFBM with stochastic interest rate, stochastic volatility, and jumps can be considered due to a number of empirical lines of evidence that the interest rate has the property of mean-reversion and constant volatility cannot capture the volatility smile or skew which can be captured by stochastic volatility models. Third, our model does not consider the constant periods of asset prices, which makes the application of the pricing model limited. Motivated by the aforementioned models in references [35–41] and the references therein, it is a good idea to construct a new pricing model for options under BFBM in subdiffusive regime.

Data Availability

The data used to support the findings of this research are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

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