Research Article

On the Circumference of 3-Connected Cubic Triangle-Free Plane Graphs

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Received 2 June 2021; Revised 27 July 2021; Accepted 2 August 2021; Published 14 August 2021

Academic Editor: M. M. Bhatti

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The circumference of a graph G is the length of a longest cycle in G , denoted by $\text{cir}(G)$. For any even number n , let $c(n) = \min\{\text{cir}(G) \mid G \text{ is a 3-connected cubic triangle-free plane graph with } n \text{ vertices}\}$. In this paper, we show that an upper bound of $c(n)$ is $n + 1 - 3\lfloor n/136 \rfloor$ for $n \geq 136$.

1. Introduction

Graph theory has applications in various research areas. In chemistry and biology, it gives an illustration of molecular structures in [1, 2], and it identifies interactions between humans and nature in [3]. Graph theory can be combined together with other branches of mathematics such as algebraic graph theory, matroid theory, graph labeling, and topological graph theory. Recently, it is combined with fixed point theory in [4, 5].

We consider only simple undirected graph. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G . The degree of a vertex v is the number of edges incident to v . A graph is cubic if all vertices have degree exactly three. A plane graph is a graph embedded in a plane without edge crossing. An xy -path of a graph G is a sequence of distinct vertices $x = v_0, v_1, v_2, \dots, v_n = y$ of G such that $v_i v_{i+1} \in E(G)$, for all $i \in \{0, 1, 2, \dots, n-1\}$. Similar to xy -path, a cycle of a graph G is defined as a sequence of distinct vertices $u_1, u_2, \dots, u_m, u_1$ of G such that $u_m u_1, u_j u_{j+1} \in E(G)$, for all $j \in \{1, 2, \dots, m-1\}$. The triangle is a cycle of length 3, and a graph is triangle-free if it contains no triangle. A graph G is connected if there is a uv -path for all distinct $u, v \in V(G)$. A graph G is k -connected if $|V(G)| \geq k$, and for any $X \subseteq V(G)$ of size $k-1$, a graph $G-X$ is connected. A k -leg fragment H is a connected plane graph such that there are exactly k vertices of H with

degree two, and the remaining vertices have degree three. A Hamilton cycle of a graph G is a cycle containing all vertices of G .

Finding Hamilton cycles in graphs is a famous topic in graph theory. In 1880, Tait [6] conjectured that a 3-connected cubic plane graph has a Hamilton cycle. Tutte [7] constructed the Tutte fragment, a triangle-free 3-leg fragment with 15 vertices, and showed the following statement.

Lemma 1 (see [7]). *Let T be the Tutte fragment in Figure 1. Then,*

- (i) *There is an xy -path (respectively, an xz -path) containing all vertices of H , and*
- (ii) *For each yz -path P , $|V(P)| \leq 14$, i.e., P misses at least one vertex from T .*

Using three Tutte fragments and one vertex, the Tutte graph is the first counterexample of Tait's conjecture with 46 vertices. Later, Holton and McKay [8] showed that a smallest counterexample of this conjecture is the Barnette-Horák-Lederburg graph, a 38-vertex triangle-free graph containing two Tutte fragments. Moreover, Grinberg graph [9] and Faulkner-Younger graph [10] are other counterexamples of this conjecture without any Tutte fragment.

For 4-connected graphs embedded in surfaces, Tutte [11] proved that every 4-connected plane graph has a Hamilton

FIGURE 1: The Tutte fragment.

cycle. Extending Tutte's result, Thomas and Yu [12] proved that every 4-connected projective plane graph has a Hamilton cycle. In the case of toroidal graph, Grünbaum [13] and Nash-Williams [14] independently conjectured that every 4-connected toroidal graph has a Hamilton cycle. For the closest result of this conjecture, Thomas and Yu [15] proved that every 5-connected toroidal graph has a Hamilton cycle. A closed knight's tour on the chessboard is an example of Hamilton cycle problem. Some results of a closed knight's tour are shown in [16, 17].

Define the circumference of a graph G as the length of a longest cycle in a graph G , denoted by $\text{cir}(G)$. Since Tait's conjecture is false, the generalized question is what is the minimum circumference of any 3-connected cubic (triangle-free) plane graph with n vertices? Then, let $b(n) = \min\{\text{cir}(G) \mid G \text{ is a 3-connected cubic plane graph with } n \text{ vertices}\}$ and $c(n) = \min\{\text{cir}(G) \mid G \text{ is a 3-connected cubic triangle-free plane graph with } n \text{ vertices}\}$.

Note that $b(n) \leq c(n)$, and n must be even because the sum of degrees of all vertices is even. For the exact value of $c(n)$, Holton and McKay [8] showed that $b(n) = c(n) = n$ if $n \leq 36$, and $b(38) = c(38) = 37$. Note that $c(n)$ is still unknown for $n \geq 40$.

For a lower bound of $b(n)$, Jackson [18] showed that an n -vertex 3-connected cubic graph has circumference $\Omega(n^{0.694})$ for sufficiently large n . Bilinski et al. [19] improved this to $\Omega(n^{0.753})$, and then, Liu et al. [20] improved to $\Omega(n^{0.8})$. This implies $b(n) \geq \Omega(n^{0.8})$ for sufficiently large n .

For an upper bound of $b(n)$ for all even numbers $n \leq 38$. Let $\{G_i\}$ be the sequence of graphs such that G_1 is the the Barnette–Horák–Lederburg graph, and G_{i+1} is obtained from G_i by replacing all vertices of G_i by triangle. Then, G_i is a $38(3^{i-1})$ -vertex graph with circumference $37(3^{i-1})$. Note that a longest cycle of G_i misses 3^{i-1} vertices. Let H be a graph obtained by G_i by replacing k vertices of G_i by triangle, where $k < |V(G_i)|$. Then, $\text{cir}(H) \leq |V(H)| - 3^{i-1}$. This gives $b(n) \leq n - 3^{i-1}$ if $38(3^{i-1}) \leq n \leq 38(3^i) - 2$.

For a triangle-free graph, Lu [21] constructed a $[(4k+7)(4k'+2)]$ -vertex graph, where $k \geq 2$ and $k' \geq 1$, with circumference $(4k+7)(4k'+2) - \min\{4k+8, 4k'-2\}$. If $k' = k+3$, then this graph implies $c(n) \leq n - \sqrt{n+49/4} + 5/2$ for infinitely many integers n . Next, Lo and Schmidt [22] constructed a $122m$ -vertex graph with circumference at most $120m$ for some positive integer m . This implies $c(n) \leq 60n/61$ for some n which are divisible by 122.

In this paper, we show that an upper bound of $c(136)$ is 133 in Theorem 1, and the graph $G(k)$, a $136k$ -vertex graph, has the circumference $133k+1$ in Lemma 4. In Theorem 2, we improve an upper bound of $c(n)$ to $n+1-3\lfloor n/136 \rfloor$, for all $n \geq 138$.

2. Main Results

First, we construct the 4-leg fragment H with 136 vertices as in Figure 2(a). Note that H is triangle-free and contains nine Tutte fragments. Next, for any positive integer i , we let H_i be a copy of H such that a_i, b_i, c_i , and d_i correspond to a, b, c , and d in H , respectively. Finally, we construct a graph $G(k)$ from the union of graphs $H_1, H_2, \dots, H_k$, by adding edges $b_k a_1, d_k c_1, b_i a_{i+1}$, and $d_i c_{i+1}$, for all $i \in \{1, 2, \dots, k-1\}$, as in Figure 3. Note that $G(k)$ is a 3-connected cubic triangle-free plane graph with $136k$ vertices.

By the construction of $G(k)$, a longest cycle C of $G(k)$, where $k \geq 2$, must pass through some fragments H_i . Since H_i is a 4-leg fragment, $C \cap H_i$ are either an xy -path, where $x, y \in \{a_i, b_i, c_i, d_i\}$, or a union of two disjoint paths where all end vertices of these paths are in $\{a_i, b_i, c_i, d_i\}$. In order to find the circumference of $G(k)$, it is necessary to show the maximum size of each path of H in Lemma 2 and of each union of two disjoint paths of H in Lemma 3.

Lemma 2. *Let H be the 4-leg fragment in Figure 2(a). Then,*

- (i) *Each ab -path in H contains at most 130 vertices*
- (ii) *Each ac -path (respectively, bd -path) in H contains at most 132 vertices*
- (iii) *Each ad -path (respectively, bc -path) in H contains at most 133 vertices*
- (iv) *Each cd -path in H contains at most 119 vertices*

Proof. Let M be the graph obtained from H by contracting all Tutte fragments into vertices $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$, and C_3 as in Figure 2(b). Note that $|V(M)| = 10$ and, the vertices a, b, c , and d of H belong to the fragments represented by A_3, B_2, C_2 , and C_3 , respectively. Let P be a longest xy -path, where $x, y \in \{a, b, c, d\}$, and let Q be a path of M corresponding to P . So, Q is the uv -path, where $u, v \in \{A_3, B_2, C_2, C_3\}$. Note that if Q passes through all vertices in $V(M) - \{w\}$, then $|V(H) - V(P)| \leq 10$, i.e., P may miss the vertex w and one vertex from each Tutte fragments. Otherwise, $|V(H) - V(P)| \geq 15$. Then, $|V(Q) - \{w\}|$ should be the maximum in M . Thus, we consider four cases as follows.

Case 1: P is an ab -path. Then, Q is an $A_3 B_2$ -path. If $w \in Q$, then either $C_2 \in Q$ or $C_3 \in Q$. Suppose $w \in Q$. Since there is only one $A_3 B_2$ -path missing only vertex w , the sequence of vertices of Q is $A_3, A_1, A_2, C_2, C_1, C_3, B_3, B_1$, and B_2 . This implies P passes through each fragment in this order. By Lemma 1, P misses vertex w and other five vertices from the fragments A_1, B_1, C_1, C_2 , and C_3 . Then, $|V(P)| = 130$.

FIGURE 2: (a) The 4-leg fragment H . (b) The graph M obtained from H by contracting all Tutte fragments into vertices.

FIGURE 3: The graph $G(k)$.

Case 2: P is an ac -path, or a bd -path. By the symmetry of H , we consider only ac -path. Since there is only one A_3C_2 -path containing all vertices of M , the sequence of vertices of Q is $A_3, A_2, A_1, w, B_1, B_2, B_3, C_3, C_1$, and C_2 . By Lemma 1, P misses four vertices from the fragments A_2, B_2, C_1 , and C_3 . Then, $|V(P)| = 132$.

Case 3: P is an ad -path, or a bc -path. By the symmetry of H , we consider only ad -path. Since there is only one A_3C_3 -path containing all vertices of M , the sequence of vertices of Q is $A_3, A_1, A_2, C_2, C_1, w, B_1, B_2, B_3$, and C_3 . By Lemma 1, P misses three vertices from the fragments A_1, B_2 , and C_2 . Then, $|V(P)| = 133$.

Case 4: P is a cd -path. Then, Q is a C_2C_3 -path. If $C_1 \in Q$, then either $\{A_1, A_2, A_3\} \subseteq Q$ or $\{B_1, B_2, B_3\} \subseteq Q$. Suppose $C_1 \in Q$. Since there is only one C_2C_3 -path missing only vertex C_1 , the sequence of vertices of Q is $C_2, A_2, A_3, A_1, w, B_1, B_2, B_3$, and C_3 . Note that Q misses the vertex C_1 so P misses 15 vertices from fragment C_1 . By Lemma 1, P misses other two vertices from the fragments A_3 and B_2 . Then, $|V(P)| = 119$. $\square$

For a union of two disjoint paths, we define a (wx, yz) -dual path as the disjoint union of a wx -path and a yz -path. Note that the intersection of an ad -path and a bc -path in H is not empty. By the symmetry of H , there are exactly two cases of (wx, yz) -dual path, where $\{w, x, y, z\} = \{a, b, c, d\}$ in H as follows.

Lemma 3. Let H be the 4-leg fragment in Figure 2(a). Then,

- (i) Each (ab, cd) -dual path in H contains at most 133 vertices
- (ii) Each (ac, bd) -dual path in H contains at most 134 vertices

Proof. Let M be the graph defined from H in Figure 2(b). Let D be the largest (wx, yz) -dual path in H such that $\{w, x, y, z\} = \{a, b, c, d\}$. Let D_M be the dual path corresponding to D . Without loss of generality, we consider two cases as follows:

Case 1: D is an (ab, cd) -dual path. Then, D_M is an (A_3B_2, C_2C_3) -dual path. Since there is exactly one dual path containing all vertices of M , the sequences of these two paths of D_M are $A_3, A_2, A_1, w, B_1, B_3, B_2$, and C_2, C_1, C_3 , respectively. By Lemma 1, D misses three vertices from the fragments A_2, B_3 , and C_1 . Then, $|V(D)| = 133$.

Case 2: D is an (ac, bd) -dual path. Then, D_M is an (A_3C_2, B_2C_3) -dual path. Without the loss of generality, there is only one dual path containing all vertices of M . Then, the sequences of these two paths of D_M are $A_3, A_2, A_1, w, C_1, C_2$, and B_2, B_1, B_3, C_3 , respectively. By Lemma 1, D misses two vertices from the fragments A_2 and B_1 . Then, $|V(D)| = 134$. $\square$

Since we obtain Lemma 2 and Lemma 3, we have $c(136) \leq 133$ by showing the following theorem.

Theorem 1. There is a 136-vertex 3-connected cubic triangle-free plane graph with circumference 133.

Proof. Let K be a graph obtained from the fragment H by adding edges ac and bd (see Figure 4(a)). Note that K is a 3-connected cubic triangle-free plane graph with 136 vertices. Let M' be the graph obtained from K by contracting all Tutte fragments into vertices $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$, and C_3 as in Figure 4(b).

Let C be a longest cycle of K satisfying the condition in each following case. $\square$

FIGURE 4: (a) The graph K . (b) The graph M' obtained from K by contracting all Tutte fragments into vertices.

Case 1. Neither ac nor bd is in C . Let D be the cycle of M' corresponding to C . Then, neither A_3C_2 nor B_2C_3 is in D . If $w \in V(D)$, then either $V(D) = \{A_1, A_2, A_3\}$ or $V(D) = \{B_1, B_2, B_3\}$. Suppose $w \in V(D)$. Since there is only one cycle containing all vertices of M' , the sequence of vertices of D is $w, A_1, A_3, A_2, C_2, C_1, C_3, B_3, B_2, B_1$, and w . By Lemma 1, C misses five vertices from the fragments A_3, B_2, C_1, C_2 , and C_3 . Then, $|V(C)| = 131$.

Case 2. Only one edge from $\{ac, bd\}$ is in C . Assume without loss of the generality that $ac \in E(C)$, but $bd \notin E(C)$. Then, $C - \{ac\}$ is an ac -path in H . By Lemma 2 (ii), $|V(C)| = |V(C - \{ac\})| = 132$.

Case 3. Both ac and bd are in C . Then, $C - \{ac, bd\}$ is an (ab, cd) -dual path in H . By Lemma 3(i), $|V(C)| = |V(C - \{ac, bd\})| = 133$.

Comparing the length of C in each case, we conclude that $\text{cir}(K) = 133$.

Recall that a graph $G(k)$ has $136k$ vertices. Then, we give the circumference of $G(k)$ as follows.

Lemma 4. *The circumference of $G(k)$ is $133k + 1$ for all positive integers k .*

Proof. For $k = 1$, we use the similar proof to Theorem 1 to show that $\text{cir}(G(1)) = 134$. Suppose $k \geq 2$. Let C be a longest cycle of $G(k)$ passing through all fragments $H_1, H_2, H_3, \dots, H_k$. Without loss of generality, we generate each case by the possible cases of $C \cap H_1$ as follows.

Case 1: $C \cap H_1$ is an xy -path, where $x \in \{a_1, c_1\}$ and $y \in \{b_1, d_1\}$.

Then, $C \cap H_i$ is an xy -path, where $x \in \{a_i, c_i\}$ and $y \in \{b_i, d_i\}$, for any $i \in \{2, 3, \dots, k\}$ (see Figure 5(a)). By Lemma 2, $C \cap H_i \leq 133$. This implies $|V(C)| \leq 133k$.

Case 2: $C \cap H_1$ is either a b_1d_1 -path or an a_1c_1 -path. Assume without loss of generality that $C \cap H_1$ is a b_1d_1 -path. Then, $C \cap H_k$ is an a_kc_k -path, and $C \cap H_i$ is an (a_ib_i, c_id_i) -dual path, for any $i \in \{2, 3, \dots, k-1\}$ (see Figure 5(b)). By Lemma 2(ii) and Lemma 3(i), $|V(C)| = 2(132) + 133(k-2) = 133k - 2$.

Case 3: $C \cap H_1$ is an (a_1c_1, b_1d_1) -dual path. Then, $C \cap H_i$ is an (a_ib_i, c_id_i) -dual path, for any $i \in \{2, 3, \dots, k\}$. Note that $b_ka_1, d_kc_1, b_ia_{i+1}, d_ic_{i+1} \in E(C)$, for all $i = 1, 2, \dots, k-1$ (see Figure 5(c)). By Lemma 3, $|V(C)| = 134 + 133(k-1) = 133k + 1$.

For the case that a longest cycle D does not intersect some fragments H_i , we can assume, without loss of generality, that $D \cap H_i \neq \emptyset$ if $1 \leq i \leq m$, otherwise $D \cap H_i = \emptyset$ (see Figure 5(d)). This case is similar to Case 2, so we can show that $|V(D)| = 2(132) + 133(m-2) = 133m - 2 \leq 133k - 2$.

Then, we conclude that a longest cycle in $G(k)$ is a cycle defined in Case 3 which implies that $\text{cir}(G(k)) = 133k + 1$. $\square$

The ladder, denoted by L_m , is a graph with $V(L_m) = \{u_i, v_i \mid i = 1, 2, \dots, m\}$ and $E(L_m) = \{u_i v_i \mid i = 1, 2, \dots, m\} \cup \{u_i u_{i+1}, v_i v_{i+1} \mid i = 1, 2, \dots, m-1\}$. Then, L_m is a triangle-free 4-leg fragment with $2m$ vertices. It is easy to show the following statements:

- (i) There is a u_1v_1 -path (respectively, u_mv_m -path) containing all vertices of L_m .
- (ii) If m is odd, then there are a longest u_1v_m -path (respectively, v_1u_m -path) containing all vertices of L_m , and a longest u_1u_m -path (respectively, v_1v_m -path) missing exactly one vertex from L_m .
- (iii) If m is even, then there are a longest u_1v_1 -path (respectively, u_mv_m -path) containing all vertices of L_m , and a longest u_1v_m -path (respectively, v_1u_m -path) missing exactly one vertex from L_m .
- (iv) For any x, y, z such that $\{x, y, z\} = \{v_1, u_m, v_m\}$ except for the case $(x, y, z) = (v_m, v_1, u_m)$, there is a (u_1x, yz) -dual path containing all vertices of L_m .

Combining a graph $G(k)$ and a ladder L_m , we have the main theorem as follows.

Theorem 2. *For any even number $n \geq 136$, there is an n -vertex 3-connected cubic triangle-free plane graph with circumference $n + 1 - 3\lfloor n/136 \rfloor$.*

Proof. For the case that n is the multiple of 136, it is shown in Lemma 4. Assume that n is not the multiple of

FIGURE 5: Cycles in Lemma 4. (a) A cycle C in Case 1. (b) A cycle C in Case 2. (c) A cycle C in Case 3. (d) A cycle D where it does not intersect some fragments H_i of $G(k)$.

136. Let $k = \lfloor n/136 \rfloor$ and $m = (n - 136k)/2$. Let G be an n -vertex graph with $V(G) = V(G_k) \cup V(L_m)$ and $E(G) = [E(G(k) - \{b_1 a_2, d_1 c_2\}) \cup E(L_m) \cup \{b_1 u_1, d_1 v_1, u_m a_2, v_m c_2\}]$ (see Figure 6). By this construction, it is easy to show that G is a 3-connected cubic triangle-free plane graph. Let C^* be a longest cycle of $G(k)$. Then, C^* is a cycle defined in Case 3 of Lemma 4 which misses $3k - 1$ vertices from $G(k)$. Since $G(k) - \{b_1 a_2, d_1 c_2\}$ is a subgraph of G , a longest cycle of G should miss at least $3k - 1$ vertices, i.e., $\text{cir}(G) \leq n - (3k - 1)$.

Define paths $P: b_1, u_1, u_2, \dots, u_m, a_2$, and $Q: c_1, v_1, v_2, \dots, v_m, d_2$. Let $C' = [C^* - \{b_1 a_2, d_1 c_2\}] \cup P \cup Q$. Then, C' is a cycle in G containing $n + 1 - 3k$ vertices. Hence, $\text{cir}(G) = n + 1 - 3k$. $\square$

3. Discussion and Open Problems

It can be implied from Theorem 1 and Theorem 2 that $c(136) \leq 133$, and $c(n) \leq n + 1 - 3\lfloor n/136 \rfloor$ if $n \geq 138$. This improves most of upper bound of $c(n)$ from [21, 22] except for some cases in [21]. Then, we will compare ours to that result as follows.

Theorem 3 (see [21]). *For any integer $k \geq 2$ and $k' \geq 1$, there is a 3-connected cubic triangle-free plane graph with $[(4k + 7)(4k' + 2)]$ vertices and circumference $(4k + 7)(4k' + 2) - \min\{4k + 8, 4k' - 2\}$.*

Putting appropriate k and k' , this theorem gives a better upper bound of $c(n)$ than ours in the following cases: $c(n) \leq n - 2$ if $n \in \{90, 114\}$, $c(n) \leq n - 6$ if $n \in \{150, 190, 230, 270, 310, 350, 390\}$, $c(n) \leq n - 10$ if $n \in \{210, 266, 322, 378, 434, 490\}$, $c(n) \leq n - 14$ if $n \in \{270, 342, 414, 486, 558, 630\}$, $c(n) \leq n - 16$ if $n \in \{330, 390, 450, 510, 570, 630, 690, 750, 810\}$, $c(n) \leq n - 18$ if $n \in \{418, 506, 594, 682, 770, 858, 946\}$, $c(n) \leq n - 20$ if $n \in \{494, 570, 646, 722, 798, 874, 950\}$, $c(n) \leq n - 22$ if $n \in \{598, 702, 806, 910, 1014\}$, $c(n) \leq n - 24$ if $n \in \{690, 782, 874, 966, 1058, 1150\}$, $c(n) \leq n - 26$ if $n \in \{930, 1050, 1170\}$, $c(n) \leq n - 28$ if $n \in \{918, 1026, 1134, 1242, 1350\}$, $c(n) \leq n - 30$ if $n \in \{1054, 1190, 1326, 1462\}$, $c(n) \leq n - 32$ if $n \in \{1178, 1320, 1462\}$, $c(n) \leq n - 34$ if $n \in \{1330, 1482\}$, $c(n) \leq n - 36$ if $n \in \{1470, 1638\}$, $c(n) \leq n - 38$ if $n = 1638$, $c(n) \leq n - 40$ if $n = 1794$, and $c(n) \leq n - 42$ if $n = 1978$.

FIGURE 6: The graph G defined in Theorem 2.

For an open problem, we define $c_P(n)$ (respectively, $c_T(n)$) as the minimum of $\text{cir}(G)$, where G is a 3-connected cubic triangle-free projective plane graph (respectively, toroidal graph) with n vertices. The Petersen graph, a 10-vertex graph, is a smallest 3-connected cubic triangle-free non-hamiltonian graph which can be embedded in both projective plane and torus. This implies that $c_P(n) = c_T(n) = n$ if $n \leq 8$, and $c_P(10) = c_T(10) = 9$. Since a plane graph is also a projective plane graph and a toroidal graph, we have the following inequality: $c_P(n) \leq c(n)$ and $c_T(n) \leq c(n)$. Then, we give the following questions.

Question 1. What is an upper bound of $c_P(n)$?

Question 2. What is an upper bound of $c_T(n)$?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by Ramkhamhaeng University.

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