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Research Article

Exact Artificial Boundary Conditions for Quasi-Linear Problems in Semi-Infinite Strips

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In this paper, the exact artificial boundary conditions for quasi-linear problems in semi-infinite strips are investigated. Based on the Kirchhoff transformation, the exact and approximate boundary conditions on a segment artificial boundary are derived. The error estimate for the finite element approximation with the artificial boundary condition is obtained. Some numerical examples show the efficiency of this method.

1. Introduction

The quasi-linear problems in semi-infinite strips have many physical applications in the field of magnetostatics or compressible flow around an obstacle in a channel. There have been many relevant works about quasi-linear problems in bounded domains, for example, the Galerkin approximations [1, 2], the finite element method [3, 4], and the mixed finite element method [5–7] for quasi-linear problems. One can refer to [8–10], for more related works.

The artificial boundary method [11, 12], which is also called coupling of the finite element method with natural boundary reduction [13–15] or the DtN method [16, 17], is a common method to deal with quasi-linear problems in unbounded domains. In the last decade, artificial boundaries of various shapes have been derived for quasi-linear problems in unbounded domains. Circular [18, 19] and elliptical [20] artificial boundaries are for two-dimensional problems, spheroidal artificial boundaries [18] are for three-dimensional problems, and circular arc artificial boundaries [21] are for problems in concave angle domains.

The purpose of this paper is to propose an artificial boundary method of using a segment artificial boundary for quasi-linear problems in semi-infinite strips. The segment artificial boundary we proposed in this paper is different with the circular artificial boundary in [18]. We also obtain an error estimate in Section 3, which was not discussed in [18].

Let Ω be a strip, and b is the width of the channel Ω. The boundary of domain Ω is decomposed into three disjoint parts: Γ_W, Γ_N, and Γ_S (see Figure 1). We introduce a Cartesian coordinate system (x_1, x_2), such that the ray Γ_S coincides with the x_1-axis.

We consider the following quasi-linear problem:

$$\begin{cases} -\left(\dfrac{\partial}{\partial x_1}\left(a(x,u)\dfrac{\partial u}{\partial x_1}\right)+\dfrac{\partial}{\partial x_2}\left(a(x,u)\dfrac{\partial u}{\partial x_2}\right)\right)=f, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}=0, & \text{on } \Gamma_N \mathrm{U} \Gamma_S, \\ u=0, & \text{on } \Gamma_W, \\ u(x) \text{ is bounded}, & \text{as } x_1 \longrightarrow +\infty, \end{cases} \tag{1}$$

where n is the unit exterior normal vector on Γ_N or Γ_S and $a(x,u)$ and f are two given functions.

Suppose that $\partial a/\partial s$ and $\partial^2 a/\partial s^2$ are continuous, and $a(\cdot,\cdot)$ satisfies [1]

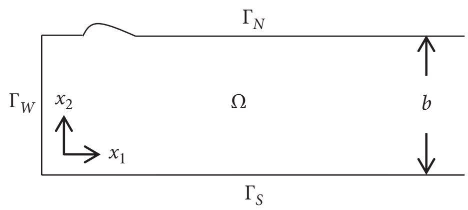

Figure 1: The illustration of domain Ω.

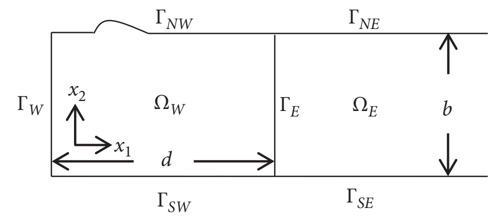

Figure 2: The illustration of domain Ω_W and Ω_E.

$$0 < C_0 \le a(x,u) \le C_1, \quad \forall u \in \mathbb{R}, \text{and for almost all } x \in \Omega, \tag{2}$$

where C_0 and C_1 are two positive constants, and

$$\begin{aligned} &|a(x,u) - a(x,v)| \le C_L |u - v|, \\ &\forall u, v \in \mathbb{R}, \text{and for almost all } x \in \Omega, \end{aligned} \tag{3}$$

where $C_L > 0$ is a positive constant. We also assume that $f \in L^2(\Omega)$ has compact support, i.e., there exists a constant $d_0 > 0$, such that

$$\operatorname{supp} f \subset \Omega_{d_0} = \{x \in \Omega | x_1 \le d_0\}. \tag{4}$$

Additionally, we suppose that

$$a(x,u) \equiv \tilde{a}(u), \quad \text{when } x_1 \ge d_0. \tag{5}$$

The rest of the paper is organized as follows. In Section 2, we derive the exact artificial boundary condition on a segment. In Section 3, we discuss the finite element approximation and a new error estimate. In Section 4, we give some numerical examples to show the efficiency of the method. The conclusions are given in Section 5.

2. Exact Quasi-Linear Artificial Boundary Condition

We introduce a segment artificial boundary $\Gamma_E = \{(x_1, x_2) | x_1 = d, \quad 0 \le x_2 \le b\}$ to enclose $\operatorname{supp} f$, which divides Ω into a bounded domain Ω_W and an unbounded domain Ω_E (see Figure 2).

Then, the original problem (1) can be described in the coupled form:

$$\begin{cases} -\left(\dfrac{\partial}{\partial x_1}\left(a(x,u)\dfrac{\partial u}{\partial x_1}\right) + \dfrac{\partial}{\partial x_2}\left(a(x,u)\dfrac{\partial u}{\partial x_2}\right)\right) = f, & \text{in } \Omega_W, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{NW} \cup \Gamma_{SW}, \\ u = 0, & \text{on } \Gamma_W, \end{cases} \tag{6}$$

$$\begin{cases} -\left(\dfrac{\partial}{\partial x_1}\left(\tilde{a}(u)\dfrac{\partial u}{\partial x_1}\right) + \dfrac{\partial}{\partial x_2}\left(\tilde{a}(u)\dfrac{\partial u}{\partial x_2}\right)\right) = 0, & \text{in } \Omega_E, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{NE} \cup \Gamma_{SE}, \\ u(x) \text{ is bounded}, & \text{as } x_1 \longrightarrow +\infty, \end{cases} \tag{7}$$

$$\begin{aligned} &u(x) \text{ and } \tilde{a}(u)\frac{\partial u}{\partial n} \\ &\text{are continuous on the artificial boundary } \Gamma_E, \end{aligned} \tag{8}$$

where $\Gamma_{NW} = \Gamma_N \cap \Omega_W$, $\Gamma_{SW} = \Gamma_S \cap \Omega_W$, $\Gamma_{NE} = \Gamma_N \cap \Omega_E$, and $\Gamma_{SE} = \Gamma_S \cap \Omega_E$.

We introduce the Kirchhoff transformation [22]:

$$w(x) = \int_0^{u(x)} \tilde{a}(\xi) d\xi, \text{ for } x \in \Omega_E. \tag{9}$$

Since $\tilde{a}(u)$ is a positive function, transformation (9) is invertible. Notice that

$$\nabla w = \tilde{a}(u) \nabla u. \tag{10}$$

Then, quasi-linear problem (1) can be transformed into a linear problem as follows:

$$\begin{cases} -\Delta w = 0, & \text{in } \Omega_E, \\ \dfrac{\partial w}{\partial n} = 0, & \text{on } \Gamma_{NE} \cup \Gamma_{SE}, \\ w(x) \text{ is bounded}, & \text{as } x_1 \longrightarrow +\infty. \end{cases} \tag{11}$$

By the natural boundary reduction [13–15], we know that the solution of problem (11) has the Fourier series expansion:

$$w(x_1, x_2) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{(d - x_1)(n\pi/b)} \cos \frac{n\pi x_2}{b}, \tag{12}$$

where

$$a_n = \frac{2}{b} \int_0^b w(d, x_2') \cos \frac{n\pi x_2'}{b} dx_2', \quad n = 0, 1, 2, \cdots. \tag{13}$$

We differentiate (12) with respect to x_1 and set $x_1 = d$ to obtain

$$\frac{\partial w}{\partial x_1}(x_1, x_2)|_{x_1 = d} = -\frac{2\pi}{b^2} \sum_{n=1}^{+\infty} n \int_0^b w(d, x_2') \cos \frac{n\pi x_2'}{b} \cos \frac{n\pi x_2}{b} dx_2' \tag{14}$$

Since

$$\tilde{a}(u) \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = -\frac{\partial w}{\partial x_1}, \tag{15}$$

we have the exact artificial boundary condition of u on Γ_E:

$$\tilde{a}(u)\frac{\partial u}{\partial n} = \frac{2\pi}{b^2}\sum_{n=1}^{+\infty} n \int_0^b \left(\int_0^{u(d,x_2')} \tilde{a}(\xi)\mathrm{d}\xi\right) \cos\frac{n\pi x_2'}{b}\cos\frac{n\pi x_2}{b}\mathrm{d}x_2' \triangleq \mathcal{K}u(d,x_2). \tag{16}$$

By the exact artificial boundary condition (16), we obtain

$$\begin{cases} -\left(\dfrac{\partial}{\partial x_1}\left(a(x,u)\dfrac{\partial u}{\partial x_1}\right) + \dfrac{\partial}{\partial x_2}\left(a(x,u)\dfrac{\partial u}{\partial x_2}\right)\right) = f, & \in \text{in } \Omega_W, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{NW} \mathrm{U} \Gamma_{SW}, \\ u = 0, & \text{on } \Gamma_W, \\ \tilde{a}(u)\dfrac{\partial u}{\partial n} = \mathcal{K}u(d,x_2), & \text{on } \Gamma_E. \end{cases} \tag{17}$$

Suppose $V = \{v \in H^1(\Omega_W) | v|_{\Gamma_W} = 0\}$; then, problem (17) is equivalent to the following variational problem:

$$\begin{cases} \text{Find } u \in V, \quad \text{such that,} \\ A(u;u,v) + B(u;u,v) = F(v), \quad \forall v \in V, \end{cases} \tag{18}$$

where

$$A(w;u,v) = \int_{\Omega_W} a(x,w)\left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right)\mathrm{d}x, \tag{19}$$

$$B(w;u,v) = \sum_{n=1}^{+\infty}\frac{2}{n\pi}\int_0^b\int_0^b \tilde{a}(w(d,x_2'))\frac{\partial u}{\partial x_2'} \cdot (d,x_2')\frac{\partial v}{\partial x_2}(d,x_2)\sin\frac{n\pi x_2'}{b}\sin\frac{n\pi x_2}{b}\mathrm{d}x_2'\mathrm{d}x_2, \tag{20}$$

$$F(v) = \int_{\Omega_W} f(x)v(x)\mathrm{d}x. \tag{21}$$

For any $s \in \mathbb{R}$, we have the following equivalent definition of Sobolev spaces $H^s(\Gamma_E)$ [23]:

$$\forall H^s(\Gamma_E) \Leftrightarrow v(d,x_2) = \frac{c_0}{2} + \sum_{n=1}^{+\infty} c_n \cos\frac{n\pi x_2}{b} \text{ and } \frac{c_0^2}{2} + \sum_{n=1}^{+\infty}\left(1+n^2\right)^s c_n^2 < \infty. \tag{22}$$

The norm of $H^s(\Gamma_E)$ can be defined as follows:

$$\|v(d,x_2)\|_{s,\Gamma_E} = \left[\frac{c_0^2}{2} + \sum_{n=1}^{+\infty}\left(1+n^2\right)^s c_n^2\right]^{1/2}. \tag{23}$$

Then, we obtain the following lemma.

Lemma 1. *The bilinear form $B(u;u,v)$ is symmetric, continuous, and semidefinite on $V \times V$.*

Proof. For $u, v \in V$, we suppose that

$$u(d,x_2') = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\frac{n\pi x_2'}{b}, \qquad v(d,x_2) = \frac{c_0}{2} + \sum_{n=1}^{+\infty} c_n \cos\frac{n\pi x_2'}{b}. \tag{24}$$

Taking the derivative with respect to x_2' and x_2, we obtain

$$\frac{\partial u}{\partial x_2'}(d,x_2') = \sum_{n=1}^{+\infty}\frac{n\pi}{b}a_n \sin\frac{n\pi x_2'}{b}, \qquad \frac{\partial v}{\partial x_2}(d,x_2) = \sum_{n=1}^{+\infty}\frac{n\pi}{b}c_n \sin\frac{n\pi x_2}{b}. \tag{25}$$

Then, we have

$$|B(u;u,v)| \le C\left(\sum_{n=1}^{+\infty} n a_n^2\right)^{1/2}\left(\sum_{n=1}^{+\infty} n c_n^2\right)^{1/2} \le C\|u\|_{1/2,\Gamma_E}\|v\|_{1/2,\Gamma_E} \le C\|u\|_{1,\Omega_W}\|v\|_{1,\Omega_W}. \tag{26}$$

Next, we show that $B(u;u,v)$ is semidefinite. For any given $v \in V$, we consider the auxiliary problem as follows:

$$\begin{cases} -\left(\dfrac{\partial}{\partial x_1}\left(a(x,u)\dfrac{\partial u}{\partial x_1}\right) + \dfrac{\partial}{\partial x_2}\left(a(x,u)\dfrac{\partial u}{\partial x_2}\right)\right) = 0, & \text{in } \Omega_E, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{NE} \mathrm{U} \Gamma_{SE}, \\ u = v, & \text{on } \Gamma_E, \\ u(x) \text{ is bounded}, & \text{as } x_1 \longrightarrow +\infty. \end{cases} \tag{27}$$

The solution u of the above problem satisfies

$$\tilde{a}(u)\frac{\partial u}{\partial n} = \mathcal{K}u(d,x_2). \tag{28}$$

We multiply (27) by u and integrate over Ω_E; then, we have

$$B(u;u,u) = \int_{\Omega_E} \tilde{a}(u)|\nabla u|^2 \mathrm{d}x \ge 0. \tag{29}$$

This completes the proof.

In practice, we have to truncate the infinite series in (16) by finite terms; let

$$\mathcal{K}^N u = \frac{2\pi}{b^2}\sum_{n=1}^{N} n\int_0^b\left(\int_0^{u(d,x_2')}\tilde{a}(\xi)d\xi\right)\cos\frac{n\pi x_2'}{b}\cos\frac{n\pi x_2}{b}\mathrm{d}x_2'. \tag{30}$$

Consider the approximation problem

$$\begin{cases} -\left(\frac{\partial}{\partial x_1}\left(a\left(x,u^N\right)\frac{\partial u^N}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(a\left(x,u^N\right)\frac{\partial u^N}{\partial x_2}\right)\right)=f, & \text{in } \Omega_W, \\ \frac{\partial u^N}{\partial n}=0, & \text{on } \Gamma_{NW}\text{U}\Gamma_{SW}, \\ u^N=0, & \text{on } \Gamma_W, \\ \widetilde{a}\left(u^N\right)\frac{\partial u^N}{\partial n}=\mathscr{K}^N u^N, & \text{on } \Gamma_E. \end{cases} \tag{31}$$

Problem (31) is equivalent to the following variational problem:

$$\begin{cases} \text{Find } u^N \in V, \text{ such that,} \\ A\left(u^N;u^N,v\right)+B_N\left(u^N;u^N,v\right)=F(v), \quad \forall v \in V, \end{cases} \tag{32}$$

where

$$B_N(w;u,v)=\sum_{n=1}^{N}\frac{2}{n\pi}\int_0^b\int_0^b \widetilde{a}\left(w\left(d,x_2'\right)\right) \frac{\partial u}{\partial x_2'}\left(d,x_2'\right)\frac{\partial v}{\partial x_2}\left(d,x_2\right)\sin\frac{n\pi x_2'}{b}\sin \frac{n\pi x_2}{b}\mathrm{d}x_2'\mathrm{d}x_2. \tag{33}$$

Similar with Lemma 1, we have □

Lemma 2. *The bilinear form $B_N(u;u,v)$ is symmetric, continuous, and semidefinite on $V\times V$.*

3. Finite Element Approximation

Suppose that $\mathscr{J}_h$ is a quasi-uniform and regular triangulation of Ω_W such that

$$\Omega_W = U_{K\in\mathscr{J}_h}K, \tag{34}$$

where K is a (curved) triangle and h is the maximal diameter of the triangles. Let

$$V_h=\left\{v_h\in V, v_h|K \text{is a linear polynomial}, \quad \forall K\in\mathscr{J}_h\right\}. \tag{35}$$

We consider the approximation problem of (32):

$$\begin{cases} \text{Find } u_h^N \in V_h, \text{ such that} \\ A\left(u_h^N;u_h^N,v_h\right)+B_N\left(u_h^N;u_h^N,v_h\right)=F\left(v_h\right), \quad \forall v_h \in V_h. \end{cases} \tag{36}$$

Theorem 1. *The variational problems (18), (32), and (36) are uniquely solvable.*

Proof. By (2), we have

$$\begin{aligned} |A(u;v,v)| &\geq C_0\|v\|_{1,\Omega_W}^2, \\ |A(u;u,v)| &\leq C_1\|u\|_{1,\Omega_W}\|v\|_{1,\Omega_W}. \end{aligned} \tag{37}$$

This means that $A(u;u,v)$ is coercive and bounded in V. From Lemma 1, we obtain that $A(u;u,v)+B(u;u,v)$ is also coercive and bounded in V. By (3), we get that $a(x,u)$ is uniformly Lipschitz continuous with respect to u. Under these conditions, referring to [1], we obtain that variational problem (18) has a unique solution $u\in V$, for all $f\in L^2(\Omega)$. It is easy to deduce that problems (32) and (36) are uniquely solvable in the same way.

We assume $u, u^N \in H^2(\Omega_W)$ and $u_h^N\in V_h$ are the solutions of problems (18), (32), and (36), respectively. We also suppose that

$$V_h \subset V\cap W^{1,2+\varepsilon}(\Omega_W) \quad \text{for some } \varepsilon\in(0,1). \tag{38}$$

Additionally, we require that $\{V_h\}_{h\longrightarrow 0}$ is a family of finite-dimensional subspaces of $V\cap C(\Omega_W)$, such that, for any

$$\begin{aligned} &v\in V\cap C(\Omega_W), \text{ there exists } \{v_h\}\text{: } v_h\in V_h, \\ &\left\|v-v_h\right\|\longrightarrow 0, \text{ as } h\longrightarrow 0, \end{aligned} \tag{39}$$

$$\left\|v_h\right\|_{1,2+\varepsilon,\Omega_W}\leq C(v), \quad \text{for any } h, \tag{40}$$

where $C(v)>0$ is independent of h.

Then, we obtain that the continuous piecewise polynomial spaces (35) satisfy condition (38). Moreover, if we assume $v_h=\Pi_h v$, where Π_h: $v\longrightarrow v_h$ is the interpolation operator, then, by (40), we obtain

$$\left\|v_h\right\|_{1,2+\varepsilon,\Omega_W}\leq\left\|\Pi_h v-v\right\|_{1,2+\varepsilon,\Omega_W}+\|v\|_{1,2+\varepsilon,\Omega_W}\leq C(v). \tag{41}$$

Following the convergence theory in [4, 15], we have the result as follows:

$$\lim_{h\longrightarrow 0}\left\|u_h^N-u^N\right\|_{1,\Omega_W}=0 \text{ and } u^N\subset V\cap W^{1,2+\varepsilon}(\Omega_W),\ \forall N\geq 0. \tag{42}$$

Furthermore, we have the following lemma. □

Lemma 3. *Suppose u is the solution of (18) and u^N is the solution of (32); we have*

$$\lim_{N\longrightarrow+\infty}\left\|u-u^N\right\|_{1,\Omega_W}=0. \tag{43}$$

Proof. From (2) and Theorem 2, we have

$$\begin{aligned} \left\|u^N\right\|_{1,\Omega_W}^2 &\leq C\left(A\left(u^N;u^N,u^N\right)+B\left(u^N;u^N,u^N\right)\right) \\ &= C\left(F\left(u^N\right)+B\left(u^N;u^N,u^N\right)-B_N\left(u^N;u^N,u^N\right)\right) \\ &\leq C\left(\|f\|_{0,\Omega_W}\|u\|^N_{1,\Omega_W}+\left|B\left(u^N;u^N,u^N\right)\right.\right. \\ &\quad \left.\left.-B_N\left(u^N;u^N,u^N\right)\right|\right). \end{aligned} \tag{44}$$

For $u^N\in V$, we suppose

$$w^N(x_1,x_2') = \int_0^{u^N(x_1,x_2')} \widetilde{a}(\xi)d\xi = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{(d_0-x_1)(n\pi/b)} \cos\frac{n\pi x_2'}{b}, \quad \forall x_1 > d_0,$$
$$u^N(d,x_2) = \frac{c_0}{2} + \sum_{n=1}^{+\infty} c_n \cos\frac{n\pi x_2}{b}. \tag{45}$$

Then,

$$\begin{aligned}&\left|B\left(u^N;u^N,u^N\right) - B_N\left(u^N;u^N,u^N\right)\right| \\ &= \left|\sum_{n=N+1}^{+\infty}\frac{2}{n\pi}\int_0^b\int_0^b \frac{\partial w^N}{\partial x_2'}(d,x_2')\frac{\partial u^N}{\partial x_2}(d,x_2)\sin\frac{n\pi x_2'}{b}\sin\frac{n\pi x_2}{b}\mathrm{d}x_2'\mathrm{d}x_2\right| \\ &= \left|\sum_{n=N+1}^{+\infty}\frac{n\pi}{2}e^{(d_0-d)(n\pi/b)}a_n c_n\right| \le Ce^{(d_0-d)(((N+1)\pi)/b)}\left(\sum_{n=N+1}^{+\infty} n a_n^2\right)^{1/2}\left(\sum_{n=N+1}^{\infty} n c_n^2\right)^{1/2} \\ &\le Ce^{(d_0-d)(((N+1)\pi)/b)}\left\|w^N\right\|_{1/2,\Gamma_{d_0}}\left\|u^N\right\|_{1/2,\Gamma_E} \le Ce^{(d_0-d)(((N+1)\pi)/b)}\left\|u^N\right\|_{1,\Omega_W}^2.\end{aligned} \tag{46}$$

From $d > d_0$, we have that $\{u^N\}$ is bounded in V. So, there exists a subsequence $\{u^{N_n}\}$, s.t. $u^{N_n} \rightharpoonup \overline{u} \in V$. Then, similar with Lemma 3,4 in [18], we obtain (43).

Then, we have the following convergence theorem. □

Theorem 2. *Let $u \in H^2(\Omega_W)$, and assumptions (38)–(40) be satisfied. Then,*

$$\lim_{h\longrightarrow 0, N\longrightarrow +\infty}\left\|u - u_h^N\right\|_{1,\Omega_W} = 0. \tag{47}$$

Next, we deduce the error estimates. We suppose that the solution u of problem (1) satisfies

$$u|_{\Omega_W} \in V \cap W^{k,2+\varepsilon}(\Omega_W), \quad \varepsilon > 0,\ k \ge 2. \tag{48}$$

For simplicity, we also define some notations as follows:

$$\begin{aligned}D(u;u,v) &\triangleq A(u;u,v) + B(u;u,v), \\ D_N\left(u^N;u^N,v\right) &\triangleq A\left(u^N;u^N,v\right) + B_N\left(u^N;u^N,v\right), \\ D_N\left(u_h^N;u_h^N,v_h\right) &\triangleq A\left(u_h^N;u_h^N,v_h\right) + B_N\left(u_h^N;u_h^N,v_h\right).\end{aligned} \tag{49}$$

Then, problems (18), (32), and (36) can be replaced by some simple forms, respectively. Moreover, we introduce the following bilinear form $D'(u;\cdot,\cdot)$ and $D_N'(u^N;\cdot,\cdot)$

$$\begin{aligned}D'(u;v,z) &= \int_{\Omega_W}\frac{\partial a}{\partial s}(x,u)v\nabla u\cdot\nabla z\mathrm{d}x + \int_{\Omega_W} a(x,u)\nabla v\cdot\nabla z\mathrm{d}x \\ &+ \int_0^b\int_0^b \frac{\partial\widetilde{a}}{\partial s}(u)v\frac{\partial u}{\partial x_2'}(d,x_2')\frac{\partial z}{\partial x_2}(d,x_2)\sum_{n=1}^{+\infty}\frac{2}{n\pi}\sin\frac{n\pi x_2'}{b}\sin\frac{n\pi x_2}{b}\mathrm{d}x_2'\mathrm{d}x_2 \\ &+ \int_0^b\int_0^b \widetilde{a}(u)\frac{\partial v}{\partial x_2'}(d,x_2')\frac{\partial z}{\partial x_2}(d,x_2)\sum_{n=1}^{+\infty}\frac{2}{n\pi}\sin\frac{n\pi x_2'}{b}\sin\frac{n\pi x_2}{b}\mathrm{d}x_2'\mathrm{d}x_2, \\ D_N'\left(u^N;v,z\right) &= \int_{\Omega_W}\frac{\partial a}{\partial s}\left(x,u^N\right)v\nabla u^N\cdot\nabla z\mathrm{d}x + \int_{\Omega_W} a\left(x,u^N\right)\nabla v\cdot\nabla z\mathrm{d}x \\ &+ \int_0^b\int_0^b \frac{\partial\widetilde{a}}{\partial s}\left(u^N\right)v\frac{\partial u^N}{\partial x_2'}(d,x_2')\frac{\partial z}{\partial x_2}(d,x_2)\sum_{n=1}^{N}\frac{2}{n\pi}\sin\frac{n\pi x_2'}{b}\sin\frac{n\pi x_2}{b}\mathrm{d}x_2'\mathrm{d}x_2 \\ &+ \int_0^b\int_0^b \widetilde{a}\left(u^N\right)\frac{\partial v}{\partial x_2'}(d,x_2')\frac{\partial z}{\partial x_2}(d,x_2)\sum_{n=1}^{N}\frac{2}{n\pi}\sin\frac{n\pi x_2'}{b}\sin\frac{n\pi x_2}{b}\mathrm{d}x_2'\mathrm{d}x_2.\end{aligned} \tag{50}$$

Suppose V' is the dual space of V. By (2) and continuity of $(\partial a/\partial s)(\cdot, u(\cdot))$, we obtain that $D'(u;\cdot,\cdot)$ is bounded on $V\times V$. Then, there exists an operator $T\colon V\longrightarrow V'$ such that

$$(Tv,z) = D'(u;v,z), \quad \forall v,z\in V. \tag{51}$$

Similar to Lemma 2.2 in [19], we have the lemma as follows:

Lemma 4. *The following inequality,*

$$(Tv,v) + K\left(\|v\|^2_{0,\Omega_W} + \|v\|^2_{1/2,\Gamma_E}\right) \geq C\|v\|^2_{1,\Omega_W}, \quad \forall v\in V, \tag{52}$$

holds, where $K\geq 0$ is a sufficient large constant.

We suppose that

$$D'(u;v,z) = 0, \quad \forall z\in V \Rightarrow v = 0. \tag{53}$$

Assume $I\colon V\longrightarrow V'$ is the canonical injection. Since V is compactly embedded in $L^2(\Omega_W)$, we obtain that the operator $J\colon V\longrightarrow V'$ defined by $J(v) = (I(v),0)$ is also compact. Then, we deduce that $T\colon V\longrightarrow V'$ is an isomorphism.

By conditions (19), (52), and (53) and Theorem 10.1.2 in [24], there exists $h_0\in(0,1]$, s.t. the following inf-sup condition is satisfied:

$$\sup_{z\in V_h}\frac{D'(u;v,z)}{\|z\|_{1,\Omega_W}} \geq \alpha_1\|v\|_{1,\Omega_W}, \quad \forall v\in V_h, \tag{54}$$

where $\alpha_1>0$ is a constant independent of $h\,(h<h_0)$.

We define the Galerkin projection with respect to $D'(u;\cdot,\cdot)$ and $P_h\colon V\longrightarrow V_h$:

$$D'(u;P_h v,z) = D'(u;v,z), \quad \forall z\in V_h. \tag{55}$$

Then, we obtain

$$\left\|v - P_h v\right\|_{1,p,\Omega_W} \leq C\inf_{v_h\in V_h}\left\|v - v_h\right\|_{1,p,\Omega_W} \leq Ch^{\sigma}, \tag{56}$$

where $2\leq p\leq\infty$, $0<\sigma<1$.

Lemma 5. *$u_h^N\in V_h$ is a solution of (36) if and only if the following equation,*

$$D_N'\left(u^N; u^N - u_h^N, v\right) = R\left(u^N; u_h^N, v\right), \quad \forall v\in V_h, \tag{57}$$

holds, where

$$\begin{aligned}
&R\left(u^N;u_h^N,v\right)\\
&\triangleq \int_{\Omega_W}\left(\int_0^1\left(\frac{\partial^2 a}{\partial s^2}\left(x,w_h^N\right)\nabla w_h^N\cdot\nabla v\right)(1-t)\mathrm{d}t\right)\left(d_h^N\right)^2\mathrm{d}x\\
&\quad + 2\int_{\Omega_W}\left(\int_0^1\left(\frac{\partial a}{\partial s}\left(x,w_h^N\right)\nabla d_h^N\cdot\nabla v\right)(1-t)\mathrm{d}t\right)d_h^N\mathrm{d}x\\
&\quad + \int_0^b\int_0^b\left(\int_0^1\frac{\partial^2\tilde a}{\partial s^2}\left(w_h^N\right)\frac{\partial w_h^N}{\partial x_2'}\frac{\partial v}{\partial x_2}\sum_{n=1}^N\frac{2}{n\pi}\sin\frac{n\pi x_2'}{b}\sin\frac{n\pi x_2}{b}\right)(1-t)\mathrm{d}t\left(d_h^N\right)^2\mathrm{d}x_2'\mathrm{d}x_2\\
&\quad + 2\int_0^b\int_0^b\left(\int_0^1\left(\frac{\partial\tilde a}{\partial s}\left(w_h^N\right)\frac{\partial d_h^N}{\partial x_2'}\frac{\partial v}{\partial x_2}\sum_{n=1}^N\frac{2}{n\pi}\sin\frac{n\pi x_2'}{b}\sin\frac{n\pi x_2}{b}\right)(1-t)\mathrm{d}t\right)d_h^N\mathrm{d}x_2'\mathrm{d}x_2,
\end{aligned} \tag{58}$$

with $w_h^N = u^N + t(u_h^N - u^N)$, $d_h^N = u_h^N - u^N$.

Proof. Suppose $\eta(t)\triangleq D_N(w_h^N;w_h^N,v)$. Then, by (32) and (36) and

$$\eta(1) = \eta(0) + \eta'(0) + \int_0^1\eta''(t)(1-t)\mathrm{d}t. \tag{59}$$

we can obtain the desired result.

Suppose

$$M_h \triangleq \left\{v\in V_h \mid \|v\|_{1,\infty,\Omega_W} \leq 1 + \left\|u^N\right\|_{1,\infty,\Omega_W}\right\}. \tag{60}$$

Then, from [19, 25], we have the following lemma. □

Lemma 6. *There exists a constant $C>0$ independent of h, such that*

$$\left|R\left(u^N;v,z\right)\right| \leq C\left(\left\|u^N - v\right\|^2_{1,\Omega_W} + \left\|u^N - v\right\|_{1,\Omega_W}\right)\|z\|_{1,\Omega_W}, \quad \forall v\in M_h, \quad \forall z\in V_h. \tag{61}$$

We denote a nonlinear mapping $\psi\colon V_h\longrightarrow V_h$, which satisfies that $\psi(v)$ is the unique solution of

$$D'(u;\psi(v),z) = D'(u;u,z) - R(u;v,z), \quad \forall z\in V_h, \tag{62}$$

for any given $v\in V_h$. Suppose

$$E_h \triangleq \left\{v\in V_h \mid \left\|v - P_h v\right\|_{1,\infty,\Omega_W} \leq Ch^{\sigma}\right\}. \tag{63}$$

Then, we obtain the lemma as follows.

Lemma 7. *The nonlinear mapping ψ is a continuous mapping from E_h to E_h.*

Proof. By (62), we have

$$D'(u;\psi(v)-\psi(v_n),z)=R(u;v_n,z)-R(u;v,z). \quad (64)$$

Combining (64) with (54), we deduce that the mapping ψ is continuous, i.e.,

$$\lim_{v_n\longrightarrow v}\psi(v_n)=\psi(v). \quad (65)$$

For any $v\in E_h$,

$$\|v\|_{1,\infty,\Omega_W}\le\left\|u^N-v\right\|_{1,\infty,\Omega_W}+\left\|u^N\right\|_{1,\infty,\Omega_W}, \quad (66)$$

$$\left\|u^N-v\right\|_{1,\infty,\Omega_W}\le\left\|u^N-P_hu^N\right\|_{1,\infty,\Omega_W}+\left\|P_hu^N-v\right\|_{1,\infty,\Omega_W}, \quad (67)$$

$$\begin{aligned}\left\|u^N-P_hu^N\right\|_{1,\infty,\Omega_W}\le&\left\|u^N-\Pi_hu^N\right\|_{1,\infty,\Omega_W}\\&+\left\|\Pi_hu^N-P_hu^N\right\|_{1,\infty,\Omega_W}.\end{aligned} \quad (68)$$

Since $\mathcal{J}_h$ is regular and quasi-uniform, according to [26], we have the following inverse inequality:

$$\|w\|_{1,\infty,\Omega_W}\le C\left(\log\frac{1}{h}\right)^{1/2}\|w\|_{1,\Omega_W},\quad\forall w\in V_h. \quad (69)$$

By the definition of E_h, (56), and (69), we obtain

$$\left\|u^N-v\right\|_{1,\infty,\Omega_W}\le 1. \quad (70)$$

This implies that $v\in M_h$. Under the definition of P_h, (62) can be rewritten as

$$D'\left(u^N;\psi(v)-P_hu^N,z\right)=-R\left(u^N;v,z\right),\quad\forall z\in V_h. \quad (71)$$

Then, by (54) and Lemmas 5 and 6, we obtain

$$\begin{aligned}\left\|\psi(v)-P_hu^N\right\|_{1,\Omega_W}&\le C\sup_{z\in V_h}\frac{\left|D'\left(u;\psi(v)-P_hu^N,z\right)\right|}{\|z\|_{1,\Omega_W}}\\&\le C\left(\left\|u^N-v\right\|_{1,\Omega_W}^2+\left\|u^N-v\right\|_{1,\Omega_W}\right)\\&\le C\left(\left\|u^N-P_hu^N\right\|_{1,\Omega_W}^2+\left\|P_hu^N-v\right\|_{1,\Omega_W}^2+\left\|u^N-P_hu^N\right\|_{1,\Omega_W}+\left\|P_hu^N-v\right\|_{1,\Omega_W}\right)\\&\le Ch^\sigma.\end{aligned} \quad (72)$$

This means that ψ: $E_h\longrightarrow E_h$. □

Theorem 3. *Suppose $u\in V\cap W^{k,2+\varepsilon}(\Omega_W)$ is a solution of problem (1), where $\varepsilon>0$, $k\ge 2$. We also assume that $u|_{\Gamma_{d_0}}\in H^{k-(1/2)}(\Gamma_{d_0})$ and u satisfies (53). With sufficiently small h, problem (36) has an approximate solution $u_h^N\in V_h$, such that*

$$\left\|u-u_h^N\right\|_{1,\Omega_W}\le C\left(h^\sigma+\frac{1}{(N+1)^{k-1}}e^{(d_0-d)(((N+1)\pi)/b)}\|u\|_{k-\frac{1}{2},\Gamma_{d_0}}\right). \quad (73)$$

Proof. By Brouwer's fixed-point theorem and Lemma 7, there exists $u_h^N\in V_h$, such that $\psi(u_h^N)=u_h^N$. From Lemma 5, we deduce that u_h^N is a solution of (36). Moreover, by (56) and $u_h^N\in E_h$, we have

$$\left\|u^N-u_h^N\right\|_{1,\Omega_W}\le\left\|u^N-P_hu^N\right\|_{1,\Omega_W}+\left\|P_hu^N-u_h^N\right\|_{1,\Omega_W}\le Ch^\sigma, \quad (74)$$

For any $u^N\in V$, according to Lemma 3, we obtain

$$\begin{aligned}
&\left|B\left(u^N;u^N,v\right)-B_N\left(u^N;u^N,v\right)\right|\\
&\le Ce^{(d_0-d)(((N+1)\pi)/b)}\left(\sum_{n=N+1}^{+\infty}\left(1+n^2\right)^{1/2}a_n^2\right)^{1/2}\left(\sum_{n=N+1}^{+\infty}\left(1+n^2\right)^{1/2}c_n^2\right)^{1/2}\\
&\le C\frac{1}{(N+1)^{k-1}}e^{(d_0-d)(((N+1)\pi)/b)}\left(\sum_{n=N+1}^{+\infty}\left(1+n^2\right)^{k-1/2}a_n^2\right)^{1/2}\left(\sum_{n=N+1}^{+\infty}\left(1+n^2\right)^{1/2}c_n^2\right)^{1/2}\\
&\le C\frac{1}{(N+1)^{k-1}}e^{(d_0-d)(((N+1)\pi)/b)}\|u\|_{k-(1/2),\Gamma_{d_0}}v_{1,\Omega_W}.
\end{aligned}\tag{75}$$

From (32), we have

$$\begin{aligned}
D\left(u^N;u^N,v\right)&=A\left(u^N;u^N,v\right)+B\left(u^N;u^N,v\right)\\
&=F(v)+B\left(u^N;u^N,v\right)-B_N\left(u^N;u^N,v\right).
\end{aligned}\tag{76}$$

Let $\eta(t)=D(u+t(u^N-u);u+t(u^N-u),v)$; then,

$$\int_0^1 D'\left(u+t\left(u^N-u\right);u^N-u,v\right)\mathrm{d}t=D\left(u^N;u^N,v\right)-D(u;u,v).\tag{77}$$

By (18), (52), and (53) and [24], we have

$$\begin{aligned}
\left\|u-u^N\right\|_{1,\Omega_W}&\le C\sup_{v\in V}\frac{\int_0^1 D'\left(u+t\left(u^N-u\right);u^N-u,v\right)dt}{\|v\|_{1,\Omega_W}}\\
&\le C\frac{\left|B\left(u^N;u^N,v\right)-B_N\left(u^N;u^N,v\right)\right|}{\|v\|_{1,\Omega_W}}\le C\frac{1}{(N+1)^{k-1}}e^{(d_0-d)(((N+1)\pi)/b)}\|u\|_{k-(1/2),\Gamma_{d_0}}.
\end{aligned}\tag{78}$$

Combining (74) with (78), we obtain

$$\begin{aligned}
\left\|u-u_h^N\right\|_{1,\Omega_W}&\le\left\|u-u^N\right\|_{1,\Omega_W}+\left\|u^N-u_h^N\right\|_{1,\Omega_W}\\
&\le C\left(h^\sigma+\frac{1}{(N+1)^{k-1}}e^{(d_0-d)(((N+1)\pi)/b)}\|u\|_{k-(1/2),\Gamma_{d_0}}\right).
\end{aligned}\tag{79}$$

This completes the proof. □

4. Numerical Examples

In this section, we computed some numerical examples by the method developed in Sections 2 and 3 to test the efficiency of the method.

Example 1. We take $\Omega=\{(x_1,x_2)|x_1>0,\quad 0<x_2<b\}$, $\Gamma_W=\{(0,x_2)|0<x_2<b\}$, $\Gamma_N=\{(x_1,b)|x_1>0\}$, $\Gamma_S=\{(x_1,0)|x_1>0\}$, $b=1$, and $a(x,u)=(1/(1+u^2))$. The exact solution of original problem is $u=\tan(\sum_{m=1}^{3}(1/m^2)\ e^{-(m\pi x_1/b)}\cos(m\pi x_2/b))$. Let $\Gamma_E=\{(x_1,x_2)|x_1=d,\quad 0<x_2<b\}$ be the artificial boundaries. Figure 3 shows the mesh h of subdomain Ω_W $(d=1)$. The numerical results are given in Table 1 and Figures 4 and 5.

From the numerical results, we can deduce that the finite element mesh, the location of artificial boundary, and the truncation terms of series can affect the numerical errors. It is obvious that our method is very effective.

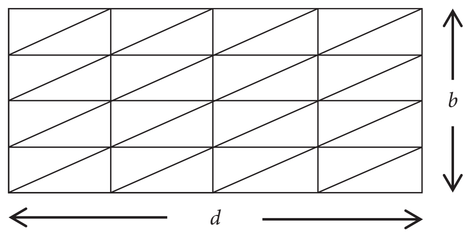

Figure 3: Mesh h of subdomain Ω_W.

Table 1: The errors with different meshes $(d = 1, N = 20)$.

Mesh	$L^2(\Omega_W)$ error	$L^\infty(\Omega_W)$ error
h	0.027864	0.059187
$h/2$	0.007744	0.026333
$h/4$	0.002033	0.009699
$h/8$	0.000504	0.002785

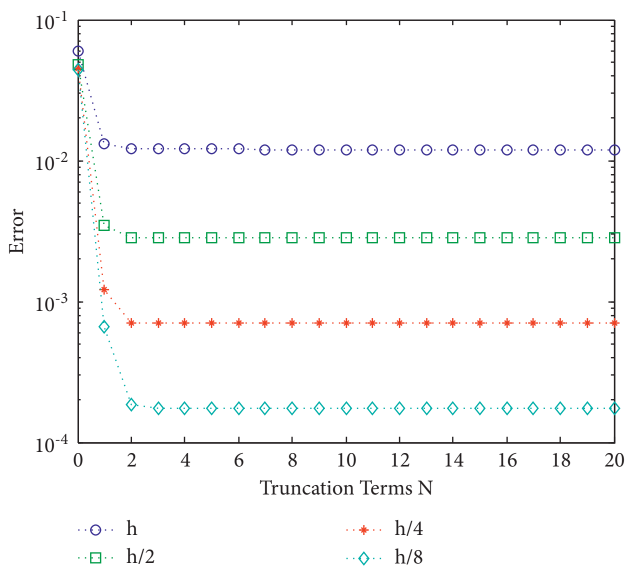

Figure 4: $L^\infty(\Gamma_E)$ errors with different $N\,(d = 1)$.

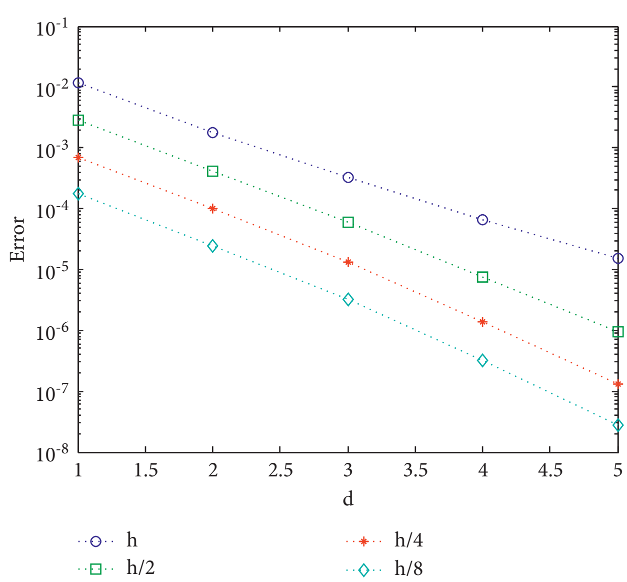

Figure 5: $L^\infty(\Gamma_E)$ errors with different $d\,(N = 20)$.

5. Conclusions

In this paper, we propose a method of artificial boundary conditions for quasi-linear problems in semi-infinite strips by using a segment artificial boundary. The exact and approximate artificial boundary conditions are given based on the Kirchhoff transformation. A new error estimate for the finite element approximation with the approximate artificial boundary condition is obtained. Finally, some numerical examples show the efficiency of this method. The quasi-linear problem, we considered in this paper, is a two-dimensional problem. Based on the proposed method, one can design some artificial boundary conditions for three-dimensional problem; we shall report on progress in some of these directions in a future publication.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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