Hindawi
Journal of Mathematics
Volume 2021, Article ID 1712685, 10 pages
https://doi.org/10.1155/2021/1712685

Research Article

Resistance Distance in Tensor and Strong Product of Path or Cycle Graphs Based on the Generalized Inverse Approach

Muhammad Shoaib Sardar,[1] **Xiang-Feng Pan**,[1] **Dalal Alrowaili**,[2] **and Imran Siddique**[3]

[1] *School of Mathematical Sciences, Anhui University, Hefei, Anhui 230601, China*
[2] *Department of Mathematics, College of Science, Jouf University, P.O. Box: 2014, Sakaka, Saudi Arabia*
[3] *Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan*

Correspondence should be addressed to Imran Siddique; imransmsrazi@gmail.com

Received 11 June 2021; Accepted 22 July 2021; Published 9 August 2021

Academic Editor: Ali Ahmad

Graph product plays a key role in many applications of graph theory because many large graphs can be constructed from small graphs by using graph products. Here, we discuss two of the most frequent graph-theoretical products. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two graphs. The Cartesian product $\mathscr{G}_1\square\mathscr{G}_2$ of any two graphs $\mathscr{G}_1$ and $\mathscr{G}_2$ is a graph whose vertex set is $V(\mathscr{G}_1\square\mathscr{G}_2) = V(\mathscr{G}_1)\times V(\mathscr{G}_2)$ and $(a_1,a_2)(b_1,b_2) \in E(\mathscr{G}_1\square\mathscr{G}_2)$ if either $a_1 = b_1$ and $a_2b_2 \in E(\mathscr{G}_2)$ or $a_1b_1 \in E(\mathscr{G}_1)$ and $a_2 = b_2$. The tensor product $\mathscr{G}_1 \times \mathscr{G}_2$ of $\mathscr{G}_1$ and $\mathscr{G}_2$ is a graph whose vertex set is $V(\mathscr{G}_1 \times \mathscr{G}_2) = V(\mathscr{G}_1)\times V(\mathscr{G}_2)$ and $(a_1,a_2)(b_1,b_2) \in E(\mathscr{G}_1 \times \mathscr{G}_2)$ if $a_1b_1 \in E(\mathscr{G}_1)$ and $a_2b_2 \in E(\mathscr{G}_2)$. The strong product $\mathscr{G}_1\boxtimes\mathscr{G}_2$ of any two graphs $\mathscr{G}_1$ and $\mathscr{G}_2$ is a graph whose vertex set is defined by $V(\mathscr{G}_1\boxtimes\mathscr{G}_2) = V(\mathscr{G}_1)\times V(\mathscr{G}_2)$ and edge set is defined by $E(\mathscr{G}_1\boxtimes\mathscr{G}_2) = E(\mathscr{G}_1\square\mathscr{G}_2)\cup E(\mathscr{G}_1 \times \mathscr{G}_2)$. The resistance distance among two vertices u and v of a graph $\mathscr{G}$ is determined as the effective resistance among the two vertices when a unit resistor replaces each edge of $\mathscr{G}$. Let P_n and C_n denote a path and a cycle of order n, respectively. In this paper, the generalized inverse of Laplacian matrix for the graphs $P_{n_1}\times C_{n_2}$ and $P_{n_1}\boxtimes P_{n_2}$ was procured, based on which the resistance distances of any two vertices in $P_{n_1}\times C_{n_2}$ and $P_{n_1}\boxtimes P_{n_2}$ can be acquired. Also, we give some examples as applications, which elucidated the effectiveness of the suggested method.

1. Introduction

Graph products [1] became an interesting area of research, and different types of products have been worked out in graph theory and other fields. The Cartesian product $\mathscr{G}_1\square\mathscr{G}_2$ of any two graphs $\mathscr{G}_1$ and $\mathscr{G}_2$ is a graph whose vertex set is $V(\mathscr{G}_1)\times V(\mathscr{G}_2)$ and two vertices (a_1,a_2) and (b_1,b_2) are adjacent in $\mathscr{G}_1\square\mathscr{G}_2$ if and only if either $a_1 = b_1$ and a_2 is adjacent to b_2 in $\mathscr{G}_2$, or $a_2 = b_2$ and a_1 is adjacent to b_1 in $\mathscr{G}_1$. The tensor product $\mathscr{G}_1 \times \mathscr{G}_2$ of $\mathscr{G}_1$ and $\mathscr{G}_2$ is a graph whose vertex set is the Cartesian product of $V(\mathscr{G}_1)\times V(\mathscr{G}_2)$ and distinct vertices (a_1,a_2) and (b_1,b_2) are adjacent in $\mathscr{G}_1\times\mathscr{G}_2$ if a_1 is adjacent to b_1 and a_2 is adjacent to b_2. The strong product $\mathscr{G}_1\boxtimes\mathscr{G}_2$ of graph $\mathscr{G}_1$ and $\mathscr{G}_2$ is a graph whose vertex set is $V(\mathscr{G}_1\square\mathscr{G}_2)$ and distinct vertices (a_1,a_2) and (b_1,b_2) are adjacent in $\mathscr{G}_1\boxtimes\mathscr{G}_2$ if either $a_1 = b_1$ and a_2 is adjacent to b_2, or $a_2 = b_2$ and a_1 is adjacent to b_1, or a_1 is adjacent to b_1 and a_2 is adjacent to b_2. It is the union of Cartesian product and tensor product. Sabidussi first proposed it in 1960 [2]. Let P_n and C_n be the path and the cycle graphs of order n, respectively. From the definition of tensor and strong product of graphs, the graphs $P_4\times C_4$ and $P_4\boxtimes P_4$ are depicted in Figure 1. The graph depicted in Figure 1(b) is also called a King's graph which is a strong product of two path graphs.

The resistance distance is a function of the distance in graphs, as suggested by Klein and Randic [3]. The resistance distance between any two vertices of a simple connected graph, G, is equal to the resistance between two equivalent points on an electrical network, constructed in such a way as (G), of each of the edge to replace a load resistance of 1 ohm. It is symbolized by r_{ij}, where $i, j \in V$. The computation of

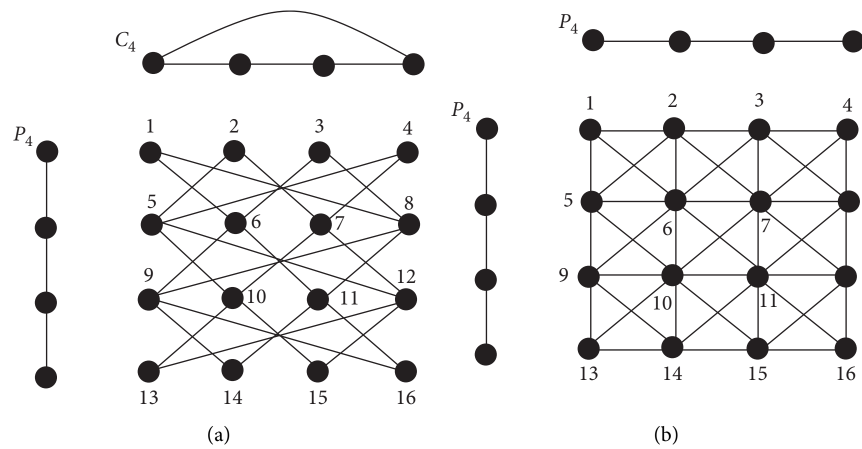

FIGURE 1: (a)$P_4 \times C_4$. (b)$P_4 \boxtimes P_4$.

resistance is relevant to a wide range of applications ranging from random walks [4], opinion formation [5], classical transport in disordered media [6], robustness of coupled oscillators network [7–9], first-passage processes [10], identifying the influential spreader node in a network [11], lattice Greens functions [12, 13], and resistance distance [3, 14] to graph theory [13, 15, 16]. At present, the resistance distance is a very suitable tool and internal graphic measurement to express the wave-like or fluid-like communication between two vertices [17]. It is also well studied in chemical and mathematical literature [3, 18–23].

Many kinds of formulae were attained for calculating the resistance distance, i.e., probabilistic formulae [4, 24], algebraic formulae [25–31], combinatorial formula [28], and so forth. Resistance distances have been procured for certain types of graphs, i.e., wheels and fans [18], cyclic graphs [32], some fullerene graphs [33], Cayley graphs [34], regular graphs [35, 36], pseudodistance regular graphs [37], and so forth. In recent years, the resistance distance of some graphic operations has been calculated (see [20, 38–41]).

In the present paper, we investigated the generalized inverse of Laplacian matrix for the graphs $P_{n_1} \times C_{n_2}$ and $P_{n_1} \boxtimes P_{n_2}$, based on which two-vertex resistances in $P_{n_1} \times C_{n_2}$ and $P_{n_1} \boxtimes P_{n_2}$ can be procured.

We ordered the paper in the following way. Section 2 covers some preliminary knowledge, i.e., basic definitions and necessary lemmas. In Section 3, we prove our main results, i.e., the generalized inverse of Laplacian matrix for tensor and strong product networks $P_{n_1} \times P_{n_2}$ and $P_{n_1} \boxtimes P_{n_2}$. In Section 4, as an application, we present a few examples. The final remarks are given in Section 5.

2. Preliminaries and Lemmas

Let $\mathscr{G}$ be a simple graph, and the vertex and edge sets of $\mathscr{G}$ are symbolized by $V(\mathscr{G}) = \{v_1, v_2, \ldots, v_n\}$ and $E(\mathscr{G}) = \{e_1, e_2, \ldots, e_m\}$, respectively. The adjacency matrix $A(\mathscr{G})$ of a graph $\mathscr{G}$ is an $n \times n$ matrix, whose element a_{uv} is one when there is an edge among vertex u and vertex v and zero when there is no edge between vertex u and vertex v. Let $D(\mathscr{G})$ be diagonal matrix with diagonal entries $d_{\mathscr{G}}(v_1), d_{\mathscr{G}}(v_2), \ldots, d_{\mathscr{G}}(v_n)$. For a graph $\mathscr{G}$, let $L(\mathscr{G}) = D(\mathscr{G}) - A(\mathscr{G})$ be a Laplacian matrix of order $n \times n$. The incidence matrix $B(\mathscr{G})$ of a graph $\mathscr{G}$ is an $n \times m$ matrix, where n and m are numbers of vertices and edges, respectively, such that $(B)_{ij}$ is 1 if the vertex v_i and edge e_j are incident and 0 otherwise. The identity matrix I_n is an $n \times n$ square matrix with 1s on main diagonal and 0s elsewhere.

Let $M = (m_{uv})_{i\times j}$ and $N = (n_{uv})_{l\times m}$ be the two matrices. The Kronecker product $M \otimes N$ is the $il \times jm$ matrix acquired from M by replacing each element m_{uv} by $m_{uv}N$ [42]. Let $M(1,1)(\mathscr{G})$ be the matrix acquired by removing the 1^{st} row and 1^{st} column of a matrix $M(\mathscr{G})$ of a graph $\mathscr{G}$, and matrix $B(1)(\mathscr{G})$ is equal to the 1^{st} row of an incidence matrix $B(\mathscr{G})$ of a graph $\mathscr{G}$. For example, considering a path graph P_3, the Laplacian matrix $L(P_3)$ is $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$; then, $L(1,1)(P_3) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. The incidence matrix $B(P_3)$ is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$; then, $B(1)(P_3) = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

Let M be a square matrix. A matrix $\mathscr{Z}$ is called a $\{1\}$-inverse of M if it satisfies $M\mathscr{Z}M = M$. $\{1\}$-inverse of M is represented as $M^{\{1\}}$. A matrix $\mathscr{Z}$ is a group inverse of a matrix M if it meets the following conditions [38]:

(i) $M\mathscr{Z}M = M$

(ii) $\mathscr{Z}M\mathscr{Z} = Z$

(iii) $\mathscr{Z}M = M\mathscr{Z}$

Let $M^{\#}$ symbolize a group inverse of M. If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$-inverse of M. Actually, $M^{\#}$ is equal to the Moore–Penrose inverse of M since M is symmetric [38].

The following lemma is used for computing the resistance distance.

Lemma 1 (see [38]). *Let L be a Laplacian matrix of a simple graph $\mathcal{G}$ with vertex set $\{1, 2, \ldots, n\}$. Then,*

$$r_{xy} = L^{\#}_{xx} + L^{\#}_{yy} - 2L^{\#}_{xy}. \tag{1}$$

Lemma 2 (see [38]). *For a nonsingular matrix $M = \begin{pmatrix} Q & N \\ R & P \end{pmatrix}$, if Q and P are nonsingular and $S = P - RQ^{-1}N$, then*

$$M^{-1} = \begin{pmatrix} Q^{-1} + Q^{-1}NS^{-1}RQ^{-1} & -Q^{-1}NS^{-1} \\ -S^{-1}RA^{-1} & S^{-1} \end{pmatrix} \tag{2}$$

is the Schur complement of Q in M.

C. Bu, in [38], stated the following expression.

Lemma 3 (see [38]). *Let $L = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ be a Laplacian matrix of a graph $\mathcal{G}$ and suppose each a column vector of L_2 is a zero vector or -1; then, the following matrix is a symmetric $\{1\}$-inverse of L:*

$$X = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & S^{\#} \end{pmatrix}, \tag{3}$$

where $S = L_3 - L_2^T L_1^{-1} L_2$.

The following expression, similar to Lemma 3, also holds for the Laplacian matrix of a simple graph. For more details, see [22, 39, 43].

Lemma 4. *If the Laplacian matrix of a simple graph $\mathcal{G}$ is partitioned as $L = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ and L_1 is nonsingular, then the following matrix is a symmetric $\{1\}$-inverse of L:*

$$X = \begin{pmatrix} L_1^{-1} + L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{pmatrix}, \tag{4}$$

where $S = L_3 - L_2^T L_1^{-1} L_2$.

3. Main Results

3.1. The Laplacian Generalized Inverse for Graph $P_{n_1} \times C_{n_1}$

Theorem 1 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be a path graph and a cycle graph with vertices n_1 and n_2, respectively. Then, the symmetric $\{1\}$-inverse of $L(\mathcal{G}_1 \times \mathcal{G}_1)$ is*

$$\begin{bmatrix} L_1^{-1} + L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{bmatrix}, \tag{5}$$

where

$$\begin{aligned} L_1 &= D(\mathcal{G}_2), \\ L_2 &= -B(1)(\mathcal{G}_1) \otimes A(\mathcal{G}_2), \\ L_3 &= 2\ D(1,1)(\mathcal{G}_1) \otimes I_{n_2} - A(1,1)(\mathcal{G}_1) \otimes A(\mathcal{G}_2), \\ S &= L_3 - \left[B(1)(\mathcal{G}_1)^T B(1)(\mathcal{G}_1)\right] \otimes \left[A(\mathcal{G}_2)(D(\mathcal{G}_2))^{-1} A(\mathcal{G}_2)\right]. \end{aligned} \tag{6}$$

Proof. Let $V(\mathcal{G}_1) = \{a_1, a_2, \ldots, a_{n_1}\}$ and $V(\mathcal{G}_2) = \{b_1, b_2, \ldots, b_{n_2}\}$. Then,

$$\{a_1, a_2, \ldots, a_{n_1}\} \times \{b_1, b_2, \ldots, b_{n_2}\} \tag{7}$$

is a partition of $V(\mathcal{G}_1 \times \mathcal{G}_2)$, where (a_1, b_1) and (a_2, b_2) are adjacent whenever (a_1, a_2) is an edge in $\mathcal{G}_1$ and (b_1, b_2) is an edge in $\mathcal{G}_2$. In $(\mathcal{G}_1 \times \mathcal{G}_2)$, $2n_2$ vertices are of degree 2 and $n_2(n_1 - 2)$ vertices are of degree 4. Label the vertices of $(\mathcal{G}_1 \times \mathcal{G}_2)$ like in Figure 1(a). According to partition (7), the Laplacian matrix of $\mathcal{G}_1 \times \mathcal{G}_2$ can be written as

$$\begin{bmatrix} D(\mathcal{G}_2) & -B(1)(\mathcal{G}_1) \otimes A(\mathcal{G}_2) \\ \left(-B(1)(\mathcal{G}_1)^T \otimes A(\mathcal{G}_2)\right. & 2\ D(1,1)(\mathcal{G}_1) \otimes I_{n_2} - A(1,1)(\mathcal{G}_1) \otimes A(\mathcal{G}_2) \end{bmatrix}. \tag{8}$$

We start with the calculation of S. For simplicity, let

$$\begin{aligned}
&L_1 = D(\mathscr{G}_2),\\
&L_2 = -B(1)(\mathscr{G}_1) \otimes A(\mathscr{G}_2),\\
&L_2^T = (-B(1)(\mathscr{G}_1))^T \otimes A(\mathscr{G}_2),\\
&L_3 = 2\ D(1,1)(\mathscr{G}_1) \otimes I_{n_2} - A(1,1)(\mathscr{G}_1) \otimes A(\mathscr{G}_2).
\end{aligned} \tag{9}$$

Due to Lemma 4, we have

$$\begin{aligned}
S &= \left[2\ D(1,1)(\mathscr{G}_1) \otimes I_{n_2} - A(1,1)(\mathscr{G}_1) \otimes A(\mathscr{G}_2)\right]\\
&\quad - [-B(1)(\mathscr{G}_1) \otimes A(\mathscr{G}_2)]^T [D(\mathscr{G}_2)]^{-1} [-B(1)(\mathscr{G}_1) \otimes A(\mathscr{G}_2)]\\
&= L_3 - \left[B(1)(\mathscr{G}_1)^T \otimes A(\mathscr{G}_2)\right]\left[D(\mathscr{G}_2)\right]^{-1}\left[B(1)(\mathscr{G}_1) \otimes A(\mathscr{G}_2)\right]\\
&= L_3 - \left[B(1)(\mathscr{G}_1)^T B(1)(\mathscr{G}_1)\right] \otimes \left[A(\mathscr{G}_2)[D(\mathscr{G}_2)]^{-1} A(\mathscr{G}_2)\right].
\end{aligned} \tag{10}$$

By using Lemma 4, the symmetric {1}-inverse of $L(\mathscr{G}_1 \times \mathscr{G}_2)$ is

$$\begin{bmatrix} L_1^{-1} + L_1^{-1} L_2 S^{\#} L_2^T L_1^{-1} & -L_1^{-1} L_2 S^{\#} \\ -S^{\#} L_2^T L_1^{-1} & S^{\#} \end{bmatrix}, \tag{11}$$

where

$$\begin{aligned}
&L_1 = D(\mathscr{G}_2),\\
&L_2 = -B(1)(\mathscr{G}_1) \otimes A(\mathscr{G}_2),\\
&L_3 = 2\ D(1,1)(\mathscr{G}_1) \otimes I_{n_2} - A(1,1)(\mathscr{G}_1) \otimes A(\mathscr{G}_2),\\
&S = L_3 - \left[B(1)(\mathscr{G}_1)^T B(1)(\mathscr{G}_1)\right] \otimes \left[A(\mathscr{G}_2)(D(\mathscr{G}_2))^{-1} A(\mathscr{G}_2)\right].
\end{aligned} \tag{12}$$

□

3.2. The Laplacian Generalized Inverse for Graph $P_{n_1} \boxtimes P_{n_2}$

Theorem 2. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two paths with n_1 and n_2 vertices, respectively, and let*

$$D = \begin{pmatrix} 5 & 0 & \dots & 0 \\ 0 & 8 & & \\ \vdots & \ddots & & \\ 8 & & & \\ 5 & & & \\ 5 & & & \\ 8 & & & \\ \ddots & & & \\ 8 & & & \\ 5 & & & \\ 3 & & & \\ 5 & & & \\ \ddots & & & \\ 5 & & & \\ 0 & 3 & & \end{pmatrix}_{(n_1-1)(n_2)\times(n_1-1)(n_2)}. \tag{13}$$

Then, the symmetric {1}-inverse of $L(\mathscr{G}_1 \times \mathscr{G}_2)$ is

$$\begin{bmatrix} L_1^{-1} + L_1^{-1} L_2 S^{\#} L_2^T L_1^{-1} & -L_1^{-1} L_2 S^{\#} \\ -S^{\#} L_2^T L_1^{-1} & S^{\#} \end{bmatrix}, \tag{14}$$

where

$$\begin{aligned}
&L_1 = 2\ D(\mathscr{G}_2) + I_{n_2} - A(\mathscr{G}_2),\\
&L_2 = -B(1)(\mathscr{G}_1) \otimes \left[A(\mathscr{G}_2) + I_{n_2}\right],\\
&L_3 = D - \left[I_{n_1-1} \otimes A(\mathscr{G}_2) + A(1,1)(\mathscr{G}_1) \otimes \left(A(\mathscr{G}_2) + I_{n_2}\right)\right],\\
&S = L_3 - \left[B(1)(\mathscr{G}_1)^T B(1)(\mathscr{G}_1)\right] \otimes \left(\left[A(\mathscr{G}_2) + I_{n_2}\right] L_1^{-1} \left[A(\mathscr{G}_2) + I_{n_2}\right]\right).
\end{aligned} \tag{15}$$

Proof. Let $V(\mathscr{G}_1) = \{a_1, a_2, \ldots, a_{n_1}\}$ and $V(\mathscr{G}_2) = \{b_1, b_2, \ldots, b_{n_2}\}$. Then,

$$V(\mathscr{G}_1 \boxtimes \mathscr{G}_2) = \{(a,b) | a \in V(\mathscr{G}_1), b \in V(\mathscr{G}_2)\} \tag{16}$$

and $E(\mathscr{G}_1 \boxtimes \mathscr{G}_2) = E(\mathscr{G}_1 \square \mathscr{G}_2) \cup E(\mathscr{G}_1 \times \mathscr{G}_2)$. The degree of vertices of $\mathscr{G}_1 \boxtimes \mathscr{G}_2$ is

$$\begin{aligned} d_{\mathscr{G}_1 \boxtimes \mathscr{G}_2}(a,b) &= d_{\mathscr{G}_1}(a) + d_{\mathscr{G}_2}(b) \\ &\quad + d_{\mathscr{G}_1}(a) \cdot d_{\mathscr{G}_2}(b), \quad (a,b) \in E(\mathscr{G}_1 \boxtimes \mathscr{G}_2). \end{aligned} \tag{17}$$

Label the vertices of $(\mathscr{G}_1 \boxtimes \mathscr{G}_2)$ like in Figure 1(b). According to partition (16), the Laplacian matrix of $\mathscr{G}_1 \boxtimes \mathscr{G}_2$ can be written as

$$\begin{bmatrix} 2\,D(\mathscr{G}_2) + I_{n_2} - A(\mathscr{G}_2) & -B(1)(\mathscr{G}_1) \otimes [A(\mathscr{G}_2) + I_{n_2}] \\ (-B(1)(\mathscr{G}_1))^T \otimes [A(\mathscr{G}_2) + I_{n_2}] & D - [I_{n_1 - 1} \otimes A(\mathscr{G}_2) + A(1,1)(\mathscr{G}_1) \otimes (A(\mathscr{G}_2) + I_{n_2})] \end{bmatrix}. \tag{18}$$

We start with the calculation of S. For simplicity, let

$$\begin{aligned} L_1 &= 2\,D(\mathscr{G}_2) + I_{n_2} - A(\mathscr{G}_2), \\ L_2 &= -B(1)(\mathscr{G}_1) \otimes [A(\mathscr{G}_2) + I_{n_2}], \\ L_2^T &= (-B(1)(\mathscr{G}_1))^T \otimes [A(\mathscr{G}_2) + I_{n_2}], \\ L_3 &= D - [I_{n_1 - 1} \otimes A(\mathscr{G}_2) + A(1,1)(\mathscr{G}_1) \otimes (A(\mathscr{G}_2) + I_{n_2})]. \end{aligned} \tag{19}$$

Due to Lemma 4, we have

$$\begin{aligned} S &= L_3 - (B(1)(\mathscr{G}_1)^T \otimes [A(\mathscr{G}_2) + I_{n_2}]) L_1^{-1} (B(1)(\mathscr{G}_1) \otimes [A(\mathscr{G}_2) + I_{n_2}]) \\ &= L_3 - [B(1)(\mathscr{G}_1)^T B(1)(\mathscr{G}_1)] \otimes ([A(\mathscr{G}_2) + I_{n_2}] L_1^{-1} [A(\mathscr{G}_2) + I_{n_2}]). \end{aligned} \tag{20}$$

By using Lemma 4, the symmetric {1}-inverse of $L(\mathscr{G}_1 \boxtimes \mathscr{G}_2)$ is

$$\begin{bmatrix} L_1^{-1} + L_1^{-1} L_2 S^{\#} L_2^T L_1^{-1} & -L_1^{-1} L_2 S^{\#} \\ -S^{\#} L_2^T L_1^{-1} & S^{\#} \end{bmatrix}, \tag{21}$$

where

$$\begin{aligned} L_1 &= 2\,D(\mathscr{G}_2) + I_{n_2} - A(\mathscr{G}_2), \\ L_2 &= -B(1)(\mathscr{G}_1) \otimes [A(\mathscr{G}_2) + I_{n_2}], \\ L_3 &= D - [I_{n_1 - 1} \otimes A(\mathscr{G}_2) + A(1,1)(\mathscr{G}_1) \otimes (A(\mathscr{G}_2) + I_{n_2})], \\ S &= L_3 - [B(1)(\mathscr{G}_1)^T B(1)(\mathscr{G}_1)] \otimes ([A(\mathscr{G}_2) + I_{n_2}] L_1^{-1} [A(\mathscr{G}_2) + I_{n_2}]). \end{aligned} \tag{22}$$

□

4. Examples to Summarize the Main Results

Here, we discuss few examples to show that two-vertex resistances in graphs $P_{n_1} \times C_{n_2}$ and $P_{n_1} \times P_{n_2}$ can be procured by the proposed method.

Example 1. The resistance distance matrix for the graph $P_3 \times C_3$ (see Figure 2(a)).

The Laplacian matrix of $(P_3 \times C_3)$ is

$$\begin{bmatrix} D(C_3) & (-B(1)(P_3)) \otimes A(C_3) \\ (-B(1)(P_3))^T \otimes A(C_3) & 2\, D(1,1)(P_3) \otimes I_3 - A(1,1)(P_3) \otimes A(C_3) \end{bmatrix}. \tag{23}$$

From Theorem 1, we obtain

$$L^{\#}(P_3 \times C_3) = \begin{pmatrix} \frac{43}{72} & \frac{1}{72} & \frac{1}{72} & -\frac{5}{72} & \frac{7}{72} & \frac{7}{72} & \frac{1}{72} & -\frac{5}{72} & -\frac{5}{72} \\ \frac{1}{72} & \frac{43}{72} & \frac{1}{72} & \frac{7}{72} & -\frac{5}{72} & \frac{7}{72} & -\frac{5}{72} & \frac{1}{72} & -\frac{5}{72} \\ \frac{1}{72} & \frac{1}{72} & \frac{43}{72} & \frac{7}{72} & \frac{7}{72} & -\frac{5}{72} & -\frac{5}{72} & -\frac{5}{72} & \frac{1}{72} \\ -\frac{5}{72} & \frac{7}{72} & \frac{7}{72} & \frac{19}{72} & -\frac{5}{72} & -\frac{5}{72} & -\frac{11}{72} & \frac{1}{72} & \frac{1}{72} \\ \frac{7}{72} & -\frac{5}{72} & \frac{7}{72} & -\frac{5}{72} & \frac{19}{72} & -\frac{5}{72} & \frac{1}{72} & -\frac{11}{72} & \frac{1}{72} \\ \frac{7}{72} & \frac{7}{72} & -\frac{5}{72} & -\frac{5}{72} & -\frac{5}{72} & \frac{19}{72} & \frac{1}{72} & \frac{1}{72} & -\frac{11}{72} \\ \frac{1}{72} & -\frac{5}{72} & -\frac{5}{72} & -\frac{11}{72} & \frac{1}{72} & \frac{1}{72} & \frac{31}{72} & -\frac{11}{72} & -\frac{11}{72} \\ -\frac{5}{72} & \frac{1}{72} & -\frac{5}{72} & \frac{1}{72} & -\frac{11}{72} & \frac{1}{72} & -\frac{11}{72} & \frac{31}{72} & -\frac{11}{72} \\ -\frac{5}{72} & -\frac{5}{72} & \frac{1}{72} & \frac{1}{72} & \frac{1}{72} & -\frac{11}{72} & -\frac{11}{72} & -\frac{11}{72} & \frac{31}{72} \end{pmatrix}. \tag{24}$$

By using Lemma 1 and $L^{\#}(P_3 \times C_3)$, the resistance distance matrix of $P_3 \times C_3$ is

$$R^{\#}(P_3 \times C_3) = \begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} & 1 & \frac{2}{3} & \frac{2}{3} & 1 & \frac{7}{6} & \frac{7}{6} \\ \frac{7}{6} & 0 & \frac{7}{6} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{7}{6} & 1 & \frac{7}{6} \\ \frac{7}{6} & \frac{7}{6} & 0 & \frac{2}{3} & \frac{2}{3} & 1 & \frac{7}{6} & \frac{7}{6} & 1 \\ 1 & \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 1 & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & \frac{2}{3} & 1 \\ 1 & \frac{7}{6} & \frac{7}{6} & 1 & \frac{2}{3} & \frac{2}{3} & 0 & \frac{7}{6} & \frac{7}{6} \\ \frac{7}{6} & 1 & \frac{7}{6} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{7}{6} & 0 & \frac{7}{6} \\ \frac{7}{6} & \frac{7}{6} & 1 & \frac{2}{3} & \frac{2}{3} & 1 & \frac{7}{6} & \frac{7}{6} & 0 \end{pmatrix}, \tag{25}$$

where r_{ij} denotes the two-vertex resistance between vertices i and j.

Example 2. The resistance distance matrix for the graph $P_3 \boxtimes P_3$ (see Figure 2(b)).

The Laplacian matrix of $P_3 \boxtimes P_3$ is

$$\begin{bmatrix} 2\,D(P_3) + I_3 - A(P_3) & -B(1)(P_3) \times [A(P_3) + I_3] \\ (-B(1)(P_3))^T \times [A(P_3) + I_3] & D - [I_2 \otimes A(P_3) + A(1,1)(P_3) \otimes (A(P_3) + I_3)] \end{bmatrix}, \tag{26}$$

where $D = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$.

Based on Theorem 2, we obtain that

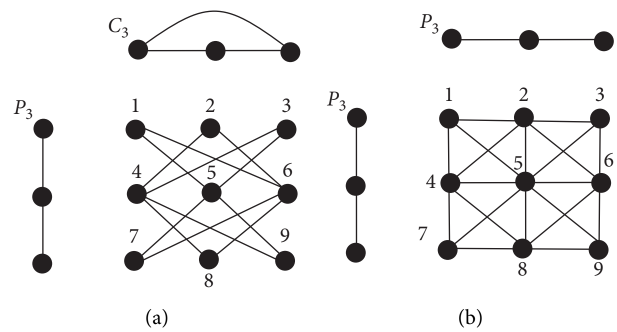

FIGURE 2: (a) $P_3 \times C_3$. (b) $P_3 \boxtimes P_3$.

$$L^{\#}(P_3 \boxtimes P_3) = \begin{pmatrix} \frac{728}{1779} & \frac{53}{468} & \frac{72}{1433} & \frac{227}{3276} & \frac{212}{4699} & -\frac{25}{3276} & -\frac{437}{16380} & -\frac{1}{36} & -\frac{857}{16380} \\ \frac{53}{468} & \frac{125}{468} & \frac{53}{468} & \frac{17}{468} & \frac{17}{468} & \frac{17}{468} & -\frac{19}{468} & -\frac{1}{36} & -\frac{19}{468} \\ \frac{72}{1433} & \frac{53}{468} & \frac{728}{1779} & -\frac{25}{3276} & \frac{212}{4699} & \frac{227}{3276} & -\frac{857}{16380} & -\frac{1}{36} & -\frac{437}{16380} \\ \frac{227}{3276} & \frac{17}{468} & -\frac{25}{3276} & \frac{346}{1931} & -\frac{25}{3276} & -\frac{13}{252} & -\frac{25}{3276} & -\frac{1}{36} & -\frac{277}{3276} \\ \frac{212}{4699} & \frac{17}{468} & \frac{212}{4699} & -\frac{25}{3276} & \frac{117}{1097} & -\frac{25}{3276} & -\frac{91}{2861} & -\frac{1}{36} & -\frac{91}{2861} \\ -\frac{25}{3276} & \frac{17}{468} & \frac{227}{3276} & -\frac{13}{252} & -\frac{25}{3276} & \frac{587}{3276} & -\frac{277}{3276} & -\frac{1}{36} & -\frac{25}{3276} \\ -\frac{437}{16380} & -\frac{19}{468} & -\frac{857}{16380} & -\frac{25}{3276} & -\frac{91}{2861} & -\frac{277}{3276} & \frac{511}{2001} & -\frac{1}{36} & -\frac{279}{2693} \\ -\frac{1}{36} & -\frac{1}{36} & -\frac{1}{36} & -\frac{1}{36} & -\frac{1}{36} & -\frac{1}{36} & -\frac{1}{36} & \frac{5}{36} & -\frac{1}{36} \\ -\frac{857}{16380} & -\frac{19}{468} & -\frac{437}{16380} & -\frac{277}{3276} & -\frac{91}{2861} & -\frac{25}{3276} & -\frac{279}{2693} & -\frac{1}{36} & \frac{511}{2001} \end{pmatrix}. \tag{27}$$

By using Lemma 1 and $L^{\#}(P_3 \boxtimes P_3)$, the resistance distance matrix of $P_3 \boxtimes P_3$

$$R^{\#}(P_3 \boxtimes P_3) = \begin{pmatrix} 0 & \frac{614}{1365} & \frac{28}{39} & \frac{614}{1365} & \frac{83}{195} & \frac{824}{1365} & \frac{28}{39} & \frac{824}{1365} & \frac{10}{13} \\ \frac{614}{1365} & 0 & \frac{614}{1365} & \frac{34}{91} & \frac{137}{455} & \frac{34}{91} & \frac{824}{1365} & \frac{6}{13} & \frac{824}{1365} \\ \frac{28}{39} & \frac{614}{1365} & 0 & \frac{824}{1365} & \frac{83}{195} & \frac{614}{1365} & \frac{10}{13} & \frac{824}{1365} & \frac{28}{39} \\ \frac{614}{1365} & \frac{34}{91} & \frac{824}{1365} & 0 & \frac{137}{455} & \frac{6}{13} & \frac{614}{1365} & \frac{34}{91} & \frac{824}{1365} \\ \frac{83}{195} & \frac{137}{455} & \frac{83}{195} & \frac{137}{455} & 0 & \frac{137}{455} & \frac{83}{195} & \frac{137}{455} & \frac{83}{195} \\ \frac{824}{1365} & \frac{34}{91} & \frac{614}{1365} & \frac{6}{13} & \frac{137}{455} & 0 & \frac{824}{1365} & \frac{34}{91} & \frac{614}{1365} \\ \frac{28}{39} & \frac{824}{1365} & \frac{10}{13} & \frac{614}{1365} & \frac{83}{195} & \frac{824}{1365} & 0 & \frac{614}{1365} & \frac{28}{39} \\ \frac{824}{1365} & \frac{6}{13} & \frac{824}{1365} & \frac{34}{91} & \frac{137}{455} & \frac{34}{91} & \frac{614}{1365} & 0 & \frac{614}{1365} \\ \frac{10}{13} & \frac{824}{1365} & \frac{28}{39} & \frac{824}{1365} & \frac{83}{195} & \frac{614}{1365} & \frac{28}{39} & \frac{614}{1365} & 0 \end{pmatrix}, \quad (28)$$

where r_{ij} denotes the two-vertex resistance between vertices i and j.

5. Conclusion

In this paper, we investigated the resistance distance in the tensor product of a path and a cycle as well as the strong product of two paths. First, we obtained the Laplacian matrix of these two kinds of product graphs. After calculation, we acquire the generalized inverse representations of the Laplacian matrices, and then, applying the generalized inverse theory of block matrices, we obtained the two-vertex resistances. Finally, we applied the above method to compute the resistance distance in graphs $P_3 \times C_3$ and $P_3 \boxtimes P_3$. We obtained the resistance distance between two pair of vertices in tensor and strong product of two classes of graphs. However, the resistance distance for some other graph products has not been solved yet. We recommend the readers to compute the resistance distance for other classes of graphs by using different graph products, i.e., zig zag product, modular product, co-normal product and lexicographical product.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Xiang-Feng Pan was supported by the University Natural Science Research Project of Anhui Province under Grant no. KJ2020A0001.

References

[1] R. Hammack, W. Imrich, and S. Klavzar, *Handbook of Product Graphs*, CRC Press, Boca Raton, FL, USA, 2016.
[2] G. Sabidussi, "Graph multiplication," *Mathematische Zeitschrift*, vol. 72, pp. 446–457, 1960.
[3] D. J. Klein and M. Randić, "Resistance distance," *Journal of Mathematical Chemistry*, vol. 12, no. 1, pp. 81–95, 1993.
[4] P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks*, Mathematical Association of America, Washington, DC, USA, 1984.
[5] F. Baumann, I. M. Sokolov, and M. Tyloo, "A Laplacian approach to stubborn agents and their role in opinion formation on influence networks," *Physica A: Statistical Mechanics and Its Applications*, vol. 557, Article ID 124869, 2020.

[6] S. Kirkpatrick, "Percolation and conduction," *Reviews of Modern Physics*, vol. 45, no. 4, pp. 574–588, 1973.

[7] T. W. Grunberg and D. F. Gayme, "Performance measures for linear oscillator networks over arbitrary graphs," *IEEE Transactions on Control of Network Systems*, vol. 5, no. 1, pp. 456–468, 2018.

[8] M. Tyloo, T. Coletta, and P. Jacquod, "Robustness of synchrony in complex networks and generalized Kirchhoff indices," *Physical Review Letters*, vol. 120, no. 8, Article ID 084101, 2018.

[9] M. Tyloo, L. Pagnier, and P. Jacquod, "The key player problem in complex oscillator networks and electric power grids: resistance centralities identify local vulnerabilities," *Science Advances*, vol. 5, no. 11, Article ID eaaw8359, 2019.

[10] S. Redner, *A Guide to First-Passage Processes*, Cambridge University Press, Cambridge, UK, 2001.

[11] P. Van Mieghem, K. Devriendt, and H. Cetinay, "Pseudoinverse of the Laplacian and best spreader node in a network," *Physical Review E*, vol. 96, Article ID 032311, 2017.

[12] S. Katsura and S. Inawashiro, "Lattice Green's functions for the rectangular and the square lattices at arbitrary points," *Journal of Mathematical Physics*, vol. 12, no. 8, pp. 1622–1630, 1971.

[13] K.. Woong, "Combinatorial Green's function of a graph and applications to networks," *Advances in Applied Mathematics*, vol. 46, pp. 417–423, 2011.

[14] W. Xiao and I. Gutman, "Resistance distance and Laplacian spectrum," *Theoretical Chemistry Accounts: Theory, Computation, and Modeling (Theoretica Chimica Acta)*, vol. 110, no. 4, pp. 284–289, 2003.

[15] L. Novak and A. Gibbons, *Hybrid Graph Theory and Network Analysis*, Cambridge University Press, Cambridge, UK, 2009.

[16] K. Yenoke and M. K. A. Kaabar, "The bounds for the distance two labelling and radio labelling of nanostar tree dendrimer," *TELKOMNIKA Telecommunication Computing Electronics and Control*, vol. 19, no. 5, pp. 1–8, 2021.

[17] D. J. Klein, "Resistance-distance sum rules," *Croatica Chemica Acta*, vol. 75, pp. 633–649, 2002.

[18] R. B. Bapat and S. Gupta, "Resistance distance in wheels and fans," *Indian Journal of Pure and Applied Mathematics*, vol. 41, no. 1, pp. 1–13, 2010.

[19] D. Babi?, D. J. Klein, I. Lukovits, S. Nikoli?, and N. Trinajsti?, "Resistance-distance matrix: a computational algorithm and its application," *International Journal of Quantum Chemistry*, vol. 90, no. 1, pp. 166–176, 2002.

[20] J. Cao, J. B. Liu, and S. Wang, "Resistance distances in corona and neighborhood corona networks based on Laplacian generalized inverse approach," *Journal of Algebra and Its Applications*, vol. 18, no. 3, Article ID 1950053, 2019.

[21] W. Fang, Y. Wang, J.-B. Liu, and G. Jing, "Maximum resistance-Harary index of cacti," *Discrete Applied Mathematics*, vol. 251, pp. 160–170, 2018.

[22] J.-B. Liu and J. Cao, "The resistance distances of electrical networks based on Laplacian generalized inverse," *Neurocomputing*, vol. 167, no. 167, pp. 306–313, 2015.

[23] V. G. Severino, "Resistance distance in complete *n*-partite graphs," *Discrete Appl. Math.*vol. 203, pp. 53–61, 2016.

[24] C. St and J. A. Nash-Williams, "Random walks and electric currents in networks," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 55, pp. 181–194, 1959.

[25] H. Chen and F. Zhang, "Resistance distance and the normalized Laplacian spectrum," *Discrete Applied Mathematics*, vol. 155, no. 5, pp. 654–661, 2007.

[26] D. J. Klein, "Graph geometry, graph metrics and Wiener," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 35, pp. 7–27, 1997.

[27] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and its Applications*, Wiley, New York, NY, USA, 1971.

[28] S. Seshu and M. B. Reed, *Linear Graphs and Electrical Networks*, Addison-Wesley, Reading, MA, USA, 1961.

[29] G. E. Sharpe and G. P. H. Styan, "A note-on equicofactor matrices," *Proceedings of the IEEE*, vol. 55, no. 7, pp. 1226-1227, 1967.

[30] G. Sharpe and B. Spain, "On the solution of networks by means of the equicofactor matrix," *IRE Transactions on Circuit Theory*, vol. 7, no. 3, pp. 230–239, 1960.

[31] G. Sharpe and G. Styan, "Circuit duality and the general network inverse," *IEEE Transactions on Circuit Theory*, vol. 12, no. 1, pp. 22–27, 1965.

[32] H. Zhang and Y. Yang, "Resistance distance and Kirchhoff index in circulant graphs," *International Journal of Quantum Chemistry*, vol. 107, no. 2, pp. 330–339, 2007.

[33] P. W. Fowler, "Resistance distances in Fullerene graphs," *Croatica Chemica Acta*, vol. 75, no. 2, pp. 401–408, 2002.

[34] X. Gao, Y. Luo, and W. Liu, "Resistance distances and the Kirchhoff index in cayley graphs," *Discrete Applied Mathematics*, vol. 159, no. 17, pp. 2050–2057, 2011.

[35] D. J. Klein, I. Lukovits, and I. Gutman, "On the definition of the hyper-wiener index for cycle-containing structures," *Journal of Chemical Information and Computer Sciences*, vol. 35, no. 1, pp. 50–52, 1995.

[36] I. Lukovits and S. Nikoli?, N. Trinajsti?, Resistance distance in regular graphs," *International Journal of Quantum Chemistry*, vol. 3, no. 71, pp. 306–313, 1999.

[37] S. Jafarizadeh, R. Sufiani, and M. A. Jafarizadeh, "Evaluation of effective resistances in pseudo-distance-regular resistor networks," *Journal of Statistical Physics*, vol. 1, no. 139, pp. 177–199, 2010.

[38] C. Bu, B. Yan, X. Zhou, and J. Zhou, "Resistance distance in subdivision-vertex join and subdivision-edge join of graphs," *Linear Algebra and Its Applications*, vol. 458, pp. 454–462, 2014.

[39] J.-B. Liu, X.-F. Pan, and F.-T. Hu, "The {1}-inverse of the Laplacian of subdivision-vertex and subdivision-edge coronae with applications," *Linear and Multilinear Algebra*, vol. 65, no. 1, pp. 178–191, 2017.

[40] X. Liu, J. Zhou, and C. Bu, "Resistance distance and Kirchhoff index of R-vertex join and R-edge join of two graphs," *Discrete Applied Mathematics*, vol. 187, pp. 130–139, 2015.

[41] Z. Li, X. Zheng, J. Li, and Y. Pan, "Resistance distance-based graph invariants and spanning trees of graphs derived from the strong prism of a star," *Applied Mathematics and Computation*, vol. 382, 2020.

[42] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1991.

[43] L. Sun, W. Wang, J. Zhou, and C. Bu, "Some results on resistance distances and resistance matrices," *Linear and Multilinear Algebra*, vol. 63, no. 3, pp. 523–533, 2015.