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*Research Article*

# Properties of Certain Classes of Holomorphic Functions Related to Strongly Janowski Type Function

**Bushra Kanwal**,[1] **Khalida Inayat Noor**,[1] **and Saqib Hussain**[2]

[1]*Mathematics Department, COMSATS University, Park Road, Islamabad, Pakistan*
[2]*Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus 22060, Abbottabad, Pakistan*

Correspondence should be addressed to Bushra Kanwal; bushrakanwal27pk@gmail.com

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Most subclasses of univalent functions are characterized with functions that map open unit disc ∇ onto the right-half plane. This concept was later modified in the literature with those mappings that conformally map ∇ onto a circular domain. Many researchers were inspired with this modification, and as such, several articles were written in this direction. On this note, we further modify this idea by relating certain subclasses of univalent functions with those that map ∇ onto a sector in the circular domain. As a result, conditions for univalence, radius results, growth rate, and several inclusion relations are obtained for these novel classes. Overall, many consequences of findings show the validity of our investigation.

## 1. Introduction

Let $A$ be the class of normalized analytic functions in the open unit disc $\nabla = \{z \in \mathbb{C}: |z| < 1\}$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (1)$$

where $g(0) = 0$ and $g'(0) = 1$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be two analytic functions in $A$. Then, $f(z)$ is said to be subordinate to $g(z)$, denoted by $f(z) \prec g(z)$, if there exist a Schwarz function $v(z)$ in $A$, under the conditions $v(0) = 0$ and $|v(z)| < 1$, such that $f(z) = g(v(z))$.

Let $S$ denote the subclass of $A$ of univalent functions in ∇ and $C, S^*$, and $K$ represent the usual subclasses of $S$ that are convex, star-like, and close to convex in ∇, respectively. A number of classes related with strongly star-like and strongly convex functions have been studied; for details, see [1–6]. The Janwoski-type function has been defined and studied in [7]. Here, in this study, we will define strongly Janowski-type function and will discuss some novel classes in relation with this function.

The strongly Janowski-type function is defined as

$$\varphi_\alpha(a, b; z) = \left(\frac{1 + az}{1 + bz}\right)^\alpha, \quad (2)$$

where $-1 \le b < a \le 1$ and $0 < \alpha \le 1$. It is easy to see that this function is univalent and convex in the unit disc ∇.

*Definition 1.* Let

$$\zeta(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \quad (3)$$

be analytic in ∇ such that $\zeta(0) = 1$. Then, $\zeta(z) \in P^\alpha[a, b]$ if and only if

$$\zeta(z) \prec \varphi_\alpha(a, b; z). \quad (4)$$

*Definition 2.* Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$. Then, $g(z) \in S^*_{(\alpha)}[a, b]$ if and only if

$$\frac{z g'(z)}{g(z)} \prec \varphi_\alpha(a, b; z), \quad (5)$$

and the function $g(z) \in C_{(\alpha)}[a,b]$ if and only if $zg'(z) \in S^*_{(\alpha)}[a,b]$.

It is obvious that, for $\alpha = 1$, we get the classes $P[a,b], S^*[a,b]$, and $C[a,b]$, introduced by Janowski in [7].

*Definition 3.* Let $\zeta(z) = 1 + \sum_{n=2}^{\infty} d_n z^n$. Then, $\zeta(z) \in P_{(m,\alpha)}[a,b]$, $m \geq 2$, if and only if

$$\zeta(z) = \left(\frac{m}{4} + \frac{1}{2}\right)\zeta_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)\zeta_2(z), \quad (6)$$

where $\zeta_1(z), \zeta_2(z) \in P^{\alpha}[a,b], -1 \leq b < a \leq 1$ and $0 < \alpha \leq 1$. Or equivalently

$$\zeta(z) = \frac{1}{2}\int_{-\pi}^{\pi}\left(\frac{1 + aze^{-\imath t}}{1 + bze^{-\imath t}}\right)^{\alpha} \mathrm{d}\mu(t), \quad (7)$$

where $\int_{-\pi}^{\pi} \mathrm{d}\mu(t) = 2$ and $\int_{-\pi}^{\pi} |\mathrm{d}\mu(t)| \leq m$.

It is easy to note that $P_{(m,1)}[a,b]$ is the well-known class $P_m[a,b]$ defined in [8]. For $\alpha = 1/2$, we have the class $P_{(m,(1/2))}[a,b]$.

*Definition 4.* Let $g(z)$ be an analytic function of form (1). Then, $g(z) \in V_{(m,\alpha)}[a,b]$ if and only if

$$\frac{(zg'(z))'}{g'(z)} \in P_{(m,\alpha)}[a,b], \quad (8)$$

where $m \geq 2, -1 \leq b < a \leq 1$, and $0 < \alpha \leq 1$.

For $\alpha = 1$, we get the class $V_m[a,b]$ introduced and studied by Noor [8]. Moreover, for $a = 1 - 2\beta$ and $b = -1$, we will get the class $V_m(\beta)$ defined by Padmanabhan and Parvatham in [9]. Also, if we take $\alpha = 1, a = 1$, and $b = -1$, then we get the class $V_m$ introduced by Paatero in [10].

*Definition 5.* Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$. Then, $g(z) \in T_{(m,\alpha)}[a,b;\beta]$ if and only if there exists a function $\psi(z) \in V_{(m,\alpha)}[a,b]$ such that

$$\frac{g'(z)}{\psi'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \quad (9)$$

or

$$\frac{g'(z)}{\psi'(z)} \in P^{\beta}[1,-1], \quad (10)$$

where $m \geq 2$, $-1 \leq b < a \leq 1$, $0 < \alpha \leq 1$, and $0 < \beta \leq 1$.

For $\alpha = 1$ and $\beta = 1$, we get the class $T_m[a,b]$ studied by Noor [11]. For $m = 2, a = 1, b = -1$, and $\alpha = \beta = 1$, we get the well-known class of close-to-convex functions.

*Definition 6.* Let $g(z) \in A$ be of form (1). Then, $g(z) \in Q^{\gamma}_{(m,\alpha)}[a,b;s,\beta]$ if and only if there exists a function $g_1(z) \in T_{(m,\alpha)}[a,b;\beta]$ such that

$$zg'(z) + sg(z) = (s+1)z\left(g_1'(z)\right)^{\gamma}, \quad (11)$$

where $0 < \gamma \leq 1, \Re\{s\} > 0, m \geq 2, -1 \leq b < a \leq 1, 0 < \alpha \leq 1$, and $0 < \beta \leq 1$.

*Remark 1*

(i) For $s = 0$ and $\gamma = 1$, we obtain

$$zg'(z) = zg_1'(z), \quad (12)$$

which implies $g(z) \in T_{m,\alpha}[a,b;\beta] = Q^1_{m,\alpha}[a,b;0,\beta]$.

(ii) As $s \longrightarrow \infty$, we have the class $Q^{\gamma}_{(m,\alpha)}[a,b;\infty,\beta]$, and $g(z) \in Q^{\gamma}_{(m,\alpha)}[a,b;\infty,\beta]$ implies

$$g(z) = z\left(g_1'(z)\right)^{\gamma}, \quad (13)$$

where $g_1(z) \in T_{(m,\alpha)}[a,b;\beta]$.

(iii) It can easily be seen that $g(z) \in Q^{\gamma}_{(m,\alpha)}[a,b;s,\beta]$ can be represented as the following integral:

$$g(z) = \frac{s+1}{z^s}\int_0^z t^{s-1}\left(t\left(g_1'(t)\right)^{\gamma}\right)\mathrm{d}t, \quad (14)$$

Some recent work on the classes related to our newly defined classes can be seen in [12, 13].

Throughout the present investigation, $0 < \alpha \leq 1, -1 \leq b < a \leq 1$, $0 < \beta \leq 1$, and $0 < \gamma \leq 1$, unless otherwise stated.

## 2. Preliminaries

**Lemma 1** (see [14]). *Let $\zeta(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ and $H(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ be two analytic function in $\nabla$ such that $\zeta(z) \prec H(z)$. If $H(z)$ is a convex univalent function in $\nabla$, then $|d_n| \leq |h_1|$, $\forall n \geq 1$.*

**Lemma 2.** *Let $\zeta(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \in P^{\alpha}[a,b]$. Then, $|d_n| \leq \alpha|a-b|$, $\forall n \geq 1$.*

*Proof.* Let $\zeta(z) \in P^{\alpha}[a,b]$. Then, by definition, $\zeta(z) \prec \varphi_{\alpha}(a,b;z)$. Now, with simple steps, we can show that $\Re\{\varphi_{\alpha}'[a,b;z]\} > 0$. Therefore, by Noshiro Warschawski theorem given in [15], $\varphi_{\alpha}[a,b;z]$ is univalent in $\nabla$. Moreover,

$$\Re\left[\frac{(z\varphi_{\alpha}'[a,b;z])'}{\varphi_{\alpha}'[a,b;z]}\right] = \Re\left[\frac{1 + \alpha(a-b)z - abz^2}{(1+az)(1+bz)}\right] \geq \frac{1 - \alpha(a-b)r - abr^2}{(1+ar)(1+br)}, \quad \text{where } |z| = r < 1. \quad (15)$$

Let $q(r) = 1 - \alpha(a-b)r - abr^2$. Then, $q(r)$ is decreasing on $(0,1)$, and therefore, $\Re[(z\varphi_{\alpha}'[a,b;z])'/\varphi_{\alpha}'[a,b;z]] > 0$. Hence, $\varphi_{\alpha}[a,b;z]$ is a convex function. Now, by using (15), we get the required result. □

*2.1. Special Cases.* Some special cases of Lemma 2 are stated as follows:

(i) For $a = c, b = 0, 0 < c \leq 1$, and $\alpha = 1/2$, we have $\zeta(z) \prec \sqrt{1 + cz}$ and $|d_n| \leq (1/2)|c|$

(ii) For $a = 1$ and $b = -1$, we get $\zeta(z) \prec ((1+z)/(1-z))^{\alpha}$ and $|d_n| \leq 2\alpha$

(iii) For $\alpha = (1/2)$ and $\zeta(z) \prec \sqrt{((1+az)/(1+bz))}$, we have $|d_n| \leq ((a-b)/2)$

**Lemma 3.** *$P^{\alpha}[a,b] \subset P(\varrho)$; that is, if $\zeta(z) \in P^{\alpha}[a,b]$, then $\zeta(z) \in P(\varrho)$, where $\varrho = ((1-a)/(1-b))^{\alpha}$.*

*Proof.* Let $\zeta(z) \in P^{\alpha}[a,b]$. Then, $\zeta(z) \prec ((1+az)/(1+bz))^{\alpha}$. This implies there exists a Schwarz function $v(z)$ such that $v(0) = 0$ and $|v(z)| < 1$:

$$\zeta(z) = \left(\frac{1+av(z)}{1+bv(z)}\right)^{\alpha}. \tag{16}$$

Now,

$$\begin{aligned}\Re(\zeta(z)) &= \Re\left(\frac{1+av(z)}{1+bv(z)}\right)^{\alpha}\\ &\geq \left(\frac{1-a|v(z)|}{1-b|v(z)|}\right)^{\alpha}\\ &> \left(\frac{1-a}{1-b}\right)^{\alpha}.\end{aligned} \tag{17}$$

Hence, we get the required result. □

**Lemma 4** (see [16]). *Let $p(z)$ be convex in $\nabla$ with $p(0) = 1$. Suppose also that $\varphi(z)$ is analytic in $\nabla$ with $\Re\{\varphi(z)\} \geq 0$, $z \in \nabla$. If $h(z)$ is analytic in $\nabla$ with $h(0) = 1$, then*

$$\{h(z) + \varphi(z)zh'(z)\} \prec p(z) \Rightarrow h(z) \prec p(z), \quad \text{in } \nabla. \tag{18}$$

**Lemma 5.** *Let $\zeta(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \in P_{(m,\alpha)}[a,b]$; then,*

*(i) $P_{(m,\alpha)}[a,b] \subset P_m(\varrho)$, where $\varrho = ((1-a)/(1-b))^{\alpha}$*

*(ii) $|d_n| \leq (m\alpha/2)(a-b)$*

*(iii) $(1/2\pi)\int_{\varrho}^{2\pi} |\zeta(re^{i\theta})|^2 d\theta \leq ((4 + (m^2\alpha^2(a-b)^2 - 4)r^2)/4(1-r^2))$*

*Proof.* The first result can be easily proved by using Lemma 3.

(ii) Let $\zeta(z) \in P_{(m,\alpha)}[a,b]$. Then, there exists two functions $\zeta_1(z), \zeta_2(z) \in P^{\alpha}[a,b]$ such that

$$\zeta(z) = \left(\frac{m}{4}+\frac{1}{2}\right)\zeta_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)\zeta_2(z), \quad \text{where } m \geq 2. \tag{19}$$

Now, let $\zeta_1(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ and $\zeta_2(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$; then,

$$1 + \sum_{n=1}^{\infty} d_n z^n = \left(\frac{m}{4}+\frac{1}{2}\right)\left(1 + \sum_{n=1}^{\infty} b_n z^n\right) - \left(\frac{m}{4}-\frac{1}{2}\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right), \tag{20}$$

which can be written as

$$\sum_{n=1}^{\infty} d_n z^n = \sum_{n=1}^{\infty} \left[\left(\frac{m}{4}+\frac{1}{2}\right)b_n - \left(\frac{m}{4}-\frac{1}{2}\right)c_n\right]z^n. \tag{21}$$

Comparing the $n$th terms of the equality and then by taking modulus on both sides, we obtain

$$|d_n| \leq \left(\frac{m}{4}+\frac{1}{2}\right)|b_n| + \left(\frac{m}{4}-\frac{1}{2}\right)|c_n|. \tag{22}$$

Using Lemma 2, we will get the required result.

(iii) Using Parseval's identity, we have

$$\frac{1}{2\pi}\int_{o}^{2\pi} \left|\zeta\left(re^{i\theta}\right)\right|^2 \mathrm{d}\theta = \sum_{n=0}^{\infty} |d_n|^2 r^2 n \leq 1 + \frac{m^2\alpha^2(a-b)^2 r^2}{4\left(1-r^2\right)}, \tag{23}$$

which gives the required result. □

**Corollary 1**

*(i) For $\alpha = 1$, we have $(1/2\pi)\int_{o}^{2\pi} |\zeta(re^{i\theta})|^2 d\theta \leq ((4 + [m^2(a-b)^2 - 4]r^2)/4(1-r^2))$*

*(ii) For $m = 2, a = 1$, and $b = -1$, we have $(1/2\pi)\int_{o}^{2\pi} |\zeta(re^{i\theta})|^2 d\theta \leq ((1 - r^2 + 4\alpha^2 r^2)/(1-r^2))$*

*(iii) For $\alpha = 1, m = 2, a = 1$, and $b = -1$, we have $(1/2\pi)\int_{o}^{2\pi} |\zeta(re^{i\theta})|^2 \leq ((1+3r^2)/(1-r^2))$, which is the result for the well-known class $P$ of Caratheodory functions*

**Lemma 6** (see [17]). *Let $g(z) \in A$ be of form (1). Then, $g(z) \in V_m(\varrho)$, where $m \geq 2$ and $0 \leq \varrho < 1$ if and only if*

*(i) There exists some $g_1(z) \in V_m$ such that*

$$g'(z) = (g_1(z))^{1-\varrho}, \quad \forall z \in \nabla. \tag{24}$$

*(ii) There exists two star-like functions $s_1(z)$ and $s_2(z)$ such that*

$$g'(z) = \frac{(s_1(z)/z)^{((m/4)+(1/2))(1-\varrho)}}{(s_2(z)/z)^{((m/4)-(1/2)(1-\varrho))}}. \tag{25}$$

**Lemma 7.** *Let $g(z)$ be of form (1). If $g(z) \in V_{(m,\alpha)}[a,b]$, then $g(z) \in V_m(\varrho)$, where $\varrho = ((1-a)/(1-b))^{\alpha}$.*

*Proof.* Let $g(z) \in V_{(m,\alpha)}[a,b]$. Then, there exist $\zeta_1(z), \zeta_2(z) \in P^{\alpha}[a,b]$, such that

$$\frac{(zg'(z))'}{g'(z)} = \left(\frac{m}{4}+\frac{1}{2}\right)\zeta_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)\zeta_2(z). \tag{26}$$

By using Lemma 3, $P^{\alpha}[a,b] \subset P(\varrho)$, where $\varrho = ((1-a)/(1-b))^{\alpha}$. Therefore, $\zeta_i(z) \in P(\varrho)$, for $i = 1, 2$. Hence, $g(z) \in V(\varrho)$. □

## 3. Main Results

In the following theorems, we demonstrate and analyze some novel features of newly defined classes using strongly Janowski-type functions.

**Theorem 1.** *Let $g(z)$ be of form (1) in the class $V_{(m,\alpha)}[a,b]$. Then, there exist two analytic functions $s_1(z), s_2(z) \in S^*_{(\alpha)}[a,b]$, such that*

$$zg'(z) = \frac{(s_1(z))^{(m/4)+(1/2)}}{(s_2(z))^{(m/4)-(1/2)}}. \tag{27}$$

*Proof.* Let $g(z) \in V_{(m,\alpha)}[a,b]$; then, there exist $\zeta_1(z), \zeta_2(z) \in P^{\alpha}[a,b]$, such that

$$\frac{(zg'(z))'}{g'(z)} = \left(\frac{m}{4}+\frac{1}{2}\right)\zeta_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)\zeta_2(z). \tag{28}$$

Then, by using the definition of $C_{(\alpha)}[a,b]$, we can say that there exist two functions $\gamma_1(z), \gamma_2(z) \in C_{(\alpha)}[a,b]$, such that

$$\frac{(z\gamma_1'(z))'}{\gamma_1'(z)} = \zeta_1(z),$$
$$\frac{(z\gamma_2'(z))'}{\gamma_2'(z)} = \zeta_2(z). \tag{29}$$

Substituting these values in equation (28) and after some simple steps and an integration, we obtain

$$g'(z) = \frac{(\gamma_1'(z))^{(m/4)+(1/2)}}{(\gamma_2'(z))^{(m/4)-(1/2)}}. \tag{30}$$

Since $\gamma_1(z), \gamma_2(z) \in C_{(\alpha)}[a,b]$, by the Alexander relation, there exist $s_1(z), s_2(z) \in S^*_{(\alpha)}[a,b]$, such that $s_1(z) = z\gamma_1'(z)$ and $s_2(z) = z\gamma_2'(z)$. By putting these values in equation (30), we get the required result. □

**Theorem 2.** *Let $g(z) \in V_{(m,\alpha)}[a,b]$; then, $g \in V_{(2,1)}(\varrho) = C(\varrho)$, for $|z| < r_o = ((m - \sqrt{m^2-4})/2)$, where $\varrho = ((1-a)/(1-b))^{\alpha}$.*

*Proof.* Let $g(z) \in V_{(m,\alpha)}[a,b]$; then, by Theorem 1, there exists two functions $g_1(z), g_2(z) \in C_{\alpha}[a,b]$ such that

$$zg'(z) = \frac{(zg_1'(z))^{(m/4)+(1/2)}}{(zg_2'(z))^{(m/4)-(1/2)}}. \tag{31}$$

Logarithmic differentiation of equation (31) yields

$$\frac{(zg'(z))'}{g'(z)} = \left(\frac{m}{4}+\frac{1}{2}\right)\frac{(zg_1'(z))'}{g_1'(z)} - \left(\frac{m}{4}-\frac{1}{2}\right)\frac{(zg_2'(z))'}{g_2'(z)}. \tag{32}$$

Since $g_1, g_2 \in C_{\alpha}[a,b] \subset C(\varrho)$, where $\varrho = ((1-a)/(1-b))^{\alpha}$, therefore, there exists $p_1(z), p_2(z) \in P(\varrho)$ such that $((zg_1'(z))'/g_1'(z)) = p_1(z)$ and $((zg_2'(z))'/g_2'(z)) = p_2(z)$. We have the well-known distortion result, for class $P(\varrho)$,

$$\frac{1-(1-2\varrho)r}{1+r} \le \operatorname{Re}(p_i(z)) \le |p_i(z)| \le \frac{1+(1-2\varrho)r}{1-r}. \tag{33}$$

So, by means of equation (32), we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{(zg'(z))'}{g'(z)} - \varrho\right\} &\ge \left(\frac{m}{4}+\frac{1}{2}\right)\left(\frac{1-(1-2\varrho)r}{1+r}\right) \\ &\quad - \left(\frac{m}{4}-\frac{1}{2}\right)\left(\frac{1+(1-2\varrho)r}{1-r}\right) - \varrho \\ &= \frac{1-\varrho}{1-r^2}\left[r^2 - mr + 1\right].\end{aligned} \tag{34}$$

Since $((1-\varrho)/(1-r^2)) \ge 0$, it follows that $\Re\{((zg'(z))'/g'(z)) - \varrho\} > 0$ for $|z| < r_o = ((m-\sqrt{m^2-4})/2)$. □

**Theorem 3.** *Let $g \in T_{(m,\alpha)}[a,b;\beta]$. Then, $f \in C(\varrho)$, for $|z| < r_1 = (m(1-\varrho) + 2\beta - \sqrt{(m^2-4)(1-\varrho)^2 + 4\beta(\beta + m(1-\varrho))})/2(1-\varrho)$, where $\varrho = ((1-a)/(1-b))^{\alpha}$.*

*Proof.* Let $g \in T_{(m,\alpha)}[a,b;\beta]$; then, there exists $\psi(z) \in V_{(m,\alpha)}[a,b]$, such that

$$\frac{g'(z)}{\psi'(z)} \in P^{\beta}[1,-1]. \tag{35}$$

On the contrary, we can say that

$$g'(z) = \psi'(z)h(z), \quad \text{where } h(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}. \tag{36}$$

Then, logarithmic differentiation yields

$$\frac{(g'(z))'}{g'(z)} = \frac{(\psi'(z))'}{\psi'(z)} + \frac{h'(z)}{h(z)}. \tag{37}$$

Then, employing equation (34) and the well-known distortion result,

$$\left|\frac{zh'(z)}{h(z)}\right| < \frac{2\beta r}{1-r^2}, \tag{38}$$

where $h(z) \in P^{\beta}[1,-1]$, we obtain

$$\begin{aligned}\Re\left\{\frac{(zg'(z))'}{g'(z)} - \varrho\right\} &\ge \left(\frac{1-\varrho}{1-r^2}\right)\left[r^2 - mr + 1\right] - \frac{2\beta r}{1-r^2} \\ &= \frac{r^2(1-\varrho) - r[m(-\varrho) + 2\beta] + (1-\varrho)}{1-r^2}.\end{aligned} \tag{39}$$

Now, let

$$\tau(r) = r^2(1-\varrho) - r[m(-\varrho) + 2\beta] + (1-\varrho). \tag{40}$$

Then,

$$\tau(0) = (1-\varrho) > 0, \quad \text{since } (0 \le \varrho \le 1),$$
$$\tau(1) = (1-\varrho)(1-m-2\beta) < 0, \quad \text{since } m \ge 2 \text{ and } 0 < \beta \le 1. \tag{41}$$

Therefore, $\tau(r)$ has a root in [0, 1]. Let $r_1$ be the root; then,

$$r_1 = \frac{m(1-\varrho) + 2\beta - \sqrt{(m^2-4)(1-\varrho)^2 + 4\beta[\beta + m(1-\varrho)]}}{2(1-\varrho)}. \tag{42}$$

Hence, $g \in C(\varrho)$, for $|z| < r_1$. □

**Corollary 2.** *For $\beta = 1, \alpha = 1, a = 1, b = -1$, and $m = 2$, we get the radius of convexity $r_1 = 2 - \sqrt{3}$ for the class of close-to-convex functions.*

**Corollary 3.** *By substituting $a = 1$ and $b = -1$, we get $\varrho = 0$; also, we have*

$$r_1 = \frac{m + 2\beta - \sqrt{(m^2-4) + 4\beta[\beta + m]}}{2}, \tag{43}$$

*which is the radius of convexity for the class $T_{(m,\alpha)}[1,-1,\beta]$.*

**Theorem 4.** *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_{(m,\alpha)}[a,b]$; then, for $z = re^{i\vartheta}$, $0 \le \vartheta_1 \le \vartheta_2 \le 2\pi$,*

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zg'(z))'}{g'(z)}\right\} d\vartheta > -\left(\frac{m}{2} - 1\right)(1-\varrho)\pi, \tag{44}$$

*where $\varrho = ((1-a)/(1-b))^{\alpha}$.*

*Proof.* Let $g(z) \in V_{(m,\alpha)}[a,b]$. Then, by using Lemma 7, we can say that $g(z) \in V_m(\varrho)$, where $\varrho = ((1-a)/(1-b))^{\alpha}$. Therefore, by using Lemma 6, there exist some $g_1(z) \in V_m$ such that

$$g'(z) = (g_1'(z))^{1-\varrho}. \tag{45}$$

By using Theorem 2.2 of [17], we have

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zg_1'(z))'}{g_1'(z)}\right\} d\vartheta > -\left(\frac{m}{2} - 1\right)\pi. \tag{46}$$

Combining equations (45) and (46), we get the required result. □

*Remark 2.* Goodman [18] defined the class of close-to-convex functions of order $\beta$ as follows.

A normalized analytic function $g(z)$ defined in (1) is said to be close to convex of order $\beta$ if, for $\beta \ge 0$, $z = re^{i\vartheta}$, $\vartheta_1 < \vartheta_2$, and $g'(z) \ne 0$,

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left(\frac{(zg'(z))'}{g'(z)}\right) d\vartheta > -\beta\pi. \tag{47}$$

Then, by the criteria defined by Kaplan in [19], we deduce that $g(z)$ is univalent if $\beta = 1$.

Hence, it is easy to note that the function $g(z) \in V_{(m,\alpha)}[a,b]$ is close to convex of order $((m/2)-1)(1-\varrho)$. Moreover, the function $g(z) \in V_{(m,\alpha)}[a,b]$ is univalent for $((m/2)-1)(1-\varrho) \le 1$ or $m \le 2((2-\varrho)/(1-\varrho))$.

Some noteworthy cases for the univalency of subclasses of $V_{(m,\alpha)}[a,b]$ are stated below.

(i) For $a = 1$ and $\varrho = 0$, we have $m \le 4$. So, in this case, $g(z) \in V_{(m,\alpha)}[1,b]$ is univalent for $2 \le m \le 4$.

(ii) For $a = 0, b = -1$, and $\alpha = 1$, we have $\varrho = (1/2)$ and $m \le 6$. So, $g(z) \in V_{(m,1)}[0,-1]$ is univalent, for $2 \le m \le 6$.

(iii) For $\alpha = (1/2), a = 0, b = -1$, and $\varrho = (1/\sqrt{2})$, $g(z) \in V_{(m,(1/2))}[0,-1]$ is univalent for $2 \le m \le ((4\sqrt{2}-2)/(\sqrt{2}-1))$.

(iv) For $a = (1/2), b = -(1/2)$, and $\alpha = 1$, we have $\varrho = 1/4$ and $g(z) \in V_{(m,1)}[(1/2),-(1/2)]$ is univalent for $2 \le m \le 5$.

Hence, for different values of parameters, we get different classes of analytic functions of bounded boundary rotations and modified limits for the univalency of functions. Since the class $V_m$ defined by Paatero in [10] is univalent for $2 \le m \le 4$, here we have discussed some classes of bounded boundary rotations, for which this limit has been improved.

**Theorem 5.** *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_{(m,\alpha)}[a,b]$; then, $b_n = O(1)n^{\sigma}$, where O(1) is a constant. $\sigma = ((m/2)+1)(1-\varrho) - 2$, $\varrho = ((1-a)/(1-b))^{\alpha}$, and $m \ge 2((1+\varrho)/(1-\varrho))$.*

The proof is straightforward by using Lemma 6 and Theorem 2.4 of [20].

Some noteworthy cases of Theorem 5 are stated as follows.

*Remark 3*

(i) For $a = 1$ and $\varrho = 0$, we have $b_n = O(1)n^{(m/2)-1}$. So, $m \ge 2$. For $m = 2$, we get a constant.

(ii) For $a = 0, b = -1$, and $\varrho = (1/2)^{\alpha}$, we have $b_n = O(1)n^{((m/2)+1)(1-(1/2)^{\alpha})-2}$ and $m \ge 2((2^{\alpha}+1)/(2^{\alpha}-1))$. Here, for $\alpha = 1$, we get $b_n = O(1)n^{(m/4)-(3/2)}$ and $m \ge 6$. For $m = 6$, we get a constant. Similarly, if we take $\alpha = 1/2$, we have $b_n = O(1)n^{((m/2)+1)(1-(1/\sqrt{2}))-2}$ and $m \ge 2((1+\sqrt{2})/(1-\sqrt{2}))$.

(iii) For $a = (1/2), b = -(1/2), \alpha = 1$, and $\varrho = (1/3)$, we have $b_n = O(1)n^{(m-4)/3}$ and $m \ge 4$. For $m = 4$, we get a constant.

(iv) For $a = (1/4), b = -(1/4), \alpha = 1$, and $\varrho = (3/5)$, we have $b_n = O(1)n^{(m-8)/5}$, and we get a constant for $m = 8$.

**Theorem 6.** *Let $g(z) \in Q^{\gamma}_{(m,\alpha)}[a,b;0,\beta]$; then, for $z = re^{\imath\vartheta}$ and $0 \le \vartheta_1 < \vartheta_2 \le 2\pi$, we have*

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zg'(z))'}{g'(z)}\right\} d\vartheta > -\gamma\left[\left(\frac{m}{2}-1\right)(1-\varrho)+\beta\right]\pi, \quad (48)$$

*where $\varrho = ((1-a)/(1-b))^{\alpha}$.*

*Proof.* Let $g \in Q^{\gamma}_{(m,\alpha)}[a,b;0,\beta]$; then, there exists $g_1(z) \in T_{(m,\alpha)}[a,b;\beta]$ such that

$$g'(z) = (g_1'(z))^{\gamma}. \quad (49)$$

Logarithmic differentiation of (49) and some manipulations yield

$$\frac{(zg'(z))'}{g'(z)} = \gamma\frac{(zg_1'(z))'}{g_1'(z)} + (1-\gamma). \quad (50)$$

For $0 \le \vartheta_1 < \vartheta_2 \le 2\pi$, we have

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zg'(z))'}{g'(z)}\right\} d\vartheta = \gamma\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zg_1'(z))'}{g_1'(z)}\right\} d\vartheta + (1-\gamma)(\vartheta_2-\vartheta_1). \quad (51)$$

Now, since $g_1(z) \in T_{(m,\alpha)}[a,b;\beta]$. thus, there exists $\psi(z) \in V_{(m,\alpha)}[a,b]$ such that

$$g_1'(z) = \psi'(z)h(z), \quad (52)$$

where $h(z) \in P^{\beta}[1,-1]$. Logarithmic differentiation of equation (52) and simple computation yield

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zg_1'(z))'}{g_1'(z)}\right\} d\vartheta = \int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(z\psi'(z))'}{\psi'(z)}\right\} d\vartheta + \int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zh'(z))}{h(z)}\right\} d\vartheta. \quad (53)$$

Since $\psi \in V_{(m,\alpha)}[a,b]$, so by Theorem 4, we have

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(z\psi'(z))'}{\psi'(z)}\right\} d\vartheta > -\left(\frac{m}{2}-1\right)(1-\varrho)\pi. \quad (54)$$

For $h(z) \in P^{\beta}[1,-1]$, we have

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zh'(z))}{h(z)}\right\} d\vartheta > -\beta\pi. \quad (55)$$

By combining equations (51) to (55), we will get the required result. □

From Theorem 6, we can conclude the following important results.

*Remark 4*

(i) The necessary condition for $g \in Q^{\gamma}_{(m,\alpha)}[a,b;0,\beta]$ is that the outward drawn normal on the image domain turns back at most $\gamma[((m/2)-1)(1-\varrho)+\beta]\pi$.

(ii) The function $g(z) \in Q^{\gamma}_{(m,\alpha)}[a,b;0,\beta]$ is univalent if $\gamma[((m/2)-1)(1-\varrho)+\beta]\pi < 1$, that is, $m < ((2(1-\gamma)[\beta-(1-\varrho)])/\gamma(1-\varrho))$.

(iii) The functions in the class $Q^{\gamma}_{(m,\alpha)}[a,b;0,\beta]$ are close to convex of order $\gamma[((m/2)-1)(1-\varrho)+\beta]$, that is, $Q^{\gamma}_{(m,\alpha)}[a,b;0,\beta] \subset K(\gamma[((m/2))-1(1-\varrho)]+\beta)$.

(iv) For $\gamma = 1$, we get $Q^{1}_{(m,\alpha)}[a,b;0,\beta] = T_{(m,\alpha)}[a,b;\beta]$; therefore, $g \in T_{(m,\alpha)}[a,b;\beta]$ implies

$$\int_{\vartheta_1}^{\vartheta_2} \Re\left\{\frac{(zg'(z))'}{g'(z)}\right\} d\vartheta > -\left[\left(\frac{m}{2}-1\right)(1-\varrho)+\beta\right]\pi, \quad (56)$$

where $0 \le \vartheta_1 < \vartheta_2 \le 2\pi$ and $\varrho = ((1-a)/(1-b))^{\alpha}$.

**Theorem 7.** *Let $g \in Q^{1}_{m,\alpha}[a,b;s,\beta]$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$; then, we have*

$$b_n = O(1)\frac{1}{(1-\delta)+\delta n}\cdot n^{(1-\varrho)((m/2)+1)}, \quad (57)$$

*where $\varrho = ((1-a)/(1-b))^{\alpha}$ and $\delta = 1/(s+1)$.*

*Proof.* Let $g \in Q^{1}_{m,\alpha}[a,b;s,\beta]$; then,

$$zg'(z) + sg(z) = (s+1)z(g_1'(z)), \quad (58)$$

where $g_1 \in T_{(m,\alpha)}[a,b;\beta]$. So, there exists some $\psi(z) \in V_{(m,\alpha)}[a,b]$ such that

$$\frac{g_1'(z)}{\psi'(z)} \in P^{\beta}[1,-1]. \quad (59)$$

Equation (58) can be written as

$$\frac{zg'(z)}{\psi'(z)} + s\frac{g(z)}{\psi'(z)} = (s+1)z\frac{g_1'(z)}{\psi'(z)}, \quad (60)$$

so we have

$$\frac{1}{s+1}\frac{g'(z)}{\psi'(z)} + \frac{s}{s+1}\frac{g(z)}{z\psi'(z)} = \frac{g_1'(z)}{\psi'(z)}. \quad (61)$$

Now, let $(1/(s+1)) = \delta$. Then, $(s/(s+1)) = 1-\delta$, so by using (59), we have

$$\delta\left(\frac{g'(z)}{\psi'(z)}\right) + (1-\delta)\frac{g(z)}{z\psi'(z)} \in P^{\beta}[1,-1]. \quad (62)$$

Let $h(z) \in P^{\beta}[1,-1]$ such that

$$\delta zg'(z) + (1-\delta)g(z) = z\psi'(z)h(z). \quad (63)$$

Since $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, it follows that $zg'(z) = z + \sum_{n=2}^{\infty} nb_n z^n$. So, by using the Cauchy theorem, we have

$$(1-\delta+\delta n)|b_n| \le \frac{1}{2\pi r^{n+1}}\int_0^{2\pi} |z\psi'(z)||h(z)| d\vartheta. \quad (64)$$

By using Lemma 6, equation (64) can be written as

$$(1-\delta+\delta n)\left|b_n\right| \leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} \left|\frac{(s_1(z))^{(m/4)+(1/2)}}{(s_2(z))^{(m/4)-(1/2)}}\right|^{(1-\varrho)} |h(z)| d\vartheta, \tag{65}$$

where $s_1, s_2 \in S^*$ and $\varrho = ((1-a)/(1-b))^{\alpha}$. Using the distortion result for $s_2$ given in [15], we have

$$(1-\delta+\delta n)\left|b_n\right| \leq \frac{1}{r^{n+1}}\left(\frac{4}{r}\right)^{((m/4)-(1/2))(1-\rho)} \cdot \left[\frac{1}{2\pi}\int_0^{2\pi}\left|s_1(z)\right|^{((m/4)+(1/2))(1-\rho)}|h(z)| d\vartheta\right]. \tag{66}$$

Then, by applying noted Hölder inequality, we have

$$(1-\delta+\delta n)\left|b_n\right| \leq \frac{1}{r^{n+1}}\left(\frac{4}{r}\right)^{((m/4)-(1/2))(1-\varrho)}\left[\frac{1}{2\pi}\int_0^{2\pi}\left|s_1(z)\right|^{2((m/4)+(1/2))(1-\varrho)} d\vartheta\right]^{(1/2)} \cdot \left[\frac{1}{2\pi}\int_0^{2\pi}|h(z)|^2 d\vartheta\right]^{(1/2)}. \tag{67}$$

Since $h(z) \in P^{\beta}[1,-1]$, by using Lemma 5, we obtain

$$\begin{aligned}(1-\delta+\delta n)\left|b_n\right| &\leq \frac{1}{r^{n+1}}\left(\frac{4}{r}\right)^{((m/4)-(1/2))(1-\varrho)}\left(\frac{1-r^2+4\beta^2 r^2}{1-r^2}\right)^{(1/2)}\left[\frac{1}{2\pi}\int_0^{2\pi}\left|s_1(z)\right|^{2((m/4)+(1/2))(1-\varrho)} d\vartheta\right]^{(1/2)} \\ &= r\cdot 2^{2((m/4)-(1/2))(1-\varrho)-(1/2)}(2\beta)\left(\frac{1}{1-r}\right)^{(1/2)}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{(1-r)^{2((m+2)/2)(1-\varrho)}} d\vartheta\right]^{(1/2)} \\ &= O(1)\left(\frac{1}{1-r}\right)^{((m/2)+1)(1-\varrho)},\end{aligned} \tag{68}$$

where $O(1)$ represents the constant term. By taking $r = 1-(1/n)$, we have the required result. □

Clearly, we see the following.

*Remark 5*

(i) For $\alpha = 1$, we get $b_n = O(1)(1/((1-\delta)+\delta n)) \cdot n^{((m/2)+1)((a-b)/(1-b))}$

(ii) For $\delta = 1$, $g \in T_{(m,\alpha)}[a,b;\beta]$ and $b_n = O(1)n^{((m/2)+1)(1-\varrho)-1}$

**Theorem 8.** *Let $0 < \gamma_1 < \gamma_2 < 1$. Then,*

$$Q^{\gamma_1}_{(m,\alpha)}[a,b;s,\beta] \subset Q^{\gamma_2}_{(m,\alpha)}[a,b;s,\beta]. \tag{69}$$

*Proof.* Let $g \in Q^{\gamma_1}_{(m,\alpha)}[a,b;s,\beta]$. Then, there exists $g_1 \in T_{(m,\alpha)}[a,b;\beta]$ such that

$$\begin{aligned} zg'(z)+sg(z) &= (s+1)z\left(g_1'(z)\right)^{\gamma_1} \\ &= (s+1)z\left(g_2'(z)\right)^{\gamma_2}, \end{aligned} \tag{70}$$

where $(g_1'(z))^{\gamma_1} = (g_2'(z))^{\gamma_2}$, so $(g_2'(z)) = (g_1'(z))^{\gamma_1/\gamma_2}$. Using the fact that $g_1(z) \in T_{(m,\alpha)}[a,b;\beta]$, this implies there exists $\psi_1(z) \in V_{(m,\alpha)}[a,b;\beta]$ such that

$$\frac{g_1'(z)}{\psi_1'(z)} = h(z), \tag{71}$$

where $h(z) \prec ((1+z)/(1-z))^{\beta}$. Now, consider $\psi_2'(z) = (\psi_1'(z))^{\gamma_1/\gamma_2}$. Then, by logarithmic differentiation and simple calculations, we obtain

$$\frac{(z\psi_2'(z))'}{\psi_2'(z)} = \left(1-\frac{\gamma_1}{\gamma_2}\right)+\frac{\gamma_1}{\gamma_2}\left[\frac{(z\psi_1'(z))'}{\psi_1'(z)}\right]. \tag{72}$$

Since $\psi_1(z) \in V_{(m,\alpha)}[a,b]$ implies $((z\psi_1'(z))'/\psi_1'(z)) \in P_{(m,\alpha)}[a,b]$, it can easily be shown that $P_{(m,\alpha)}[a,b]$ is a convex set and $(\gamma_1/\gamma_2) < 1$; therefore, $\psi_2(z) \in V_{(m,\alpha)}[a,b]$. Thus,

$$\frac{g_2'(z)}{\psi_2'(z)} = \left(\frac{g_1'(z)}{\psi_1'(z)}\right)^{\gamma_1/\gamma_2} = (h(z))^{\gamma_1/\gamma_2}, \tag{73}$$

where $h(z) \in P^{\beta}[1,-1]$. Since $(\gamma_1/\gamma_2) < 1$, therefore, $(h(z))^{\gamma_1/\gamma_2} \in P^{\beta}[1,-1]$. Hence, $g_2(z) \in T_{(m,\alpha)}[a,b;\beta]$. Thus, $g \in Q^{\gamma_2}_{(m,\alpha)}[a,b;s,\beta]$ which gives the required inclusion. □

**Theorem 9.** $Q^{\gamma}_{(2,1)}[1,-1;s,\beta] \subset Q^{\gamma}_{(2,1)}[1,-1;\infty,\beta]$.

*Proof.* Let $g \in Q^{\gamma}_{(2,1)}[1,-1;s,\beta]$; then, by Definition 5,

$$zg'(z) + sg(z) = (s+1)z\left(g_1'(z)\right)^{\gamma}, \tag{74}$$

where $g_1(z) \in T_{(2,1)}[1,-1;\beta]$; hence, there exists $\psi(z) \in V_{(2,1)}[1,-1] = C$ such that

$$g_1'(z) = \psi'(z)p(z), \tag{75}$$

where $p(z) \prec ((1+z)/(1-z))^{\beta}$. So, equation (74) can be written as

$$\begin{aligned} zg'(z) + sg(z) &= (s+1)(zp(z))^{\gamma}(\psi'(z))^{\gamma} \\ &= (s+1)zp_1(z)\psi_1'(z), \end{aligned} \tag{76}$$

where $p_1(z) = (p(z))^{\gamma}$ and $\psi_1'(z) = (\psi'(z))^{\gamma}$. Since $p(z) \in P^{\beta}[1,-1]$ and $\psi(z) \in C$, therefore, $p_1(z) \in P^{\beta}[1,-1]$ and $\psi_1(z) \in C$. Now, we define $G(z)$ such that

$$zG'(z) + sG(z) = (s+1)\psi_1(z), \tag{77}$$

where $\psi_1 \in C$; then,

$$G(z) = \frac{s+1}{z^s}\int_0^z t^{s-1}\psi_1(t)\mathrm{d}t. \tag{78}$$

$G(z)$ defined by equation (78) is convex [21]. Now, let

$$g(z) = h(z)zG'(z). \tag{79}$$

Then, by combining equations (76)–(79) and some simple computation, we obtain

$$h(z) + \frac{zh'(z)}{s + h_o(z)} = p_1(z), \tag{80}$$

where $h_o(z) = ((zG'(z))'/G'(z))$ and $\mathfrak{R}(h_o(z)) > 0$ since $G \in C$. Now, we write $h_1(z) = (1/(s + h_o(z)))$, $s \geq 0$; it is obvious that $\mathfrak{R}\{h_1(z)\} > 0$, so equation (80) can be written as

$$\{h(z) + h_1(z)zh'(z)\} = p_1(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}. \tag{81}$$

Therefore, by using Lemma 4, it follows that $h(z) \prec ((1+z)/(1-z))^{\beta}$, and hence, $g \in Q^{\gamma}_{(2,1)}[1,-1;\infty,\beta]$. □

## 4. Conclusion

In this study, taking into account the subordination technique, we have introduced certain new classes of analytic functions associated with strong Janowski-type functions. Furthermore, we have obtained several results, and their well-known special cases are apprehended in form of Remarks 2–5. We hope that this new methodology will stimulate futuristic research in the fascinating field of geometric function theory and differential subordination.

## Data Availability

No statstical data has been used in this paper.

## Conflicts of Interest

The authors declare no conflicts of interest.

## Authors' Contributions

The authors worked jointly and equally on this manuscript. All the authors approved the final version of the manuscript.

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## References

[1] M. Arif, K. I. Noor, M. Raza, and W. Haq, "Some properties of a generalized class of analytic functions related with janowski functions," *Abstract and Applied Analysis*, vol. 2012, Article ID 279843, 11 pages, 2012.

[2] M. Darus, S. Hussain, M. Raza, and J. Sokol, "On a subclass of starlike functions," *Results in Mathematics*, vol. 73, p. 12, 2018.

[3] R. Mendiratta, S. Nagpal, and V. Ravichandran, "On a subclass of strongly starlike functions associated with exponential function," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 38, no. 1, pp. 365–386, 2015.

[4] M. Nunokawa, J. Sokol, and K. Trabka-Wieclaw, "Some remarks on close to convex and strongly convex functions," *Mathematica Scandinavica*, vol. 116, no. 2, pp. 309–319, 2015.

[5] S. G. A. Shah, S. Noor, S. Hussain, A. Tasleem, A. Rasheed, and M. Darus, "Analytic functions related with starlikeness," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9924434, 5 pages, 2021.

[6] K. Sharma, N. E. Cho, and V. Ravichandran, "Sufficient conditions for strong starlikeness," *Bulletin of the Iranian Mathematical Society*, p. 23, 2020.

[7] W. Janowski, "Some extremal problems for certain families of analytic functions I," *Annales Polonici Mathematici*, vol. 28, no. 3, pp. 297–326, 1973.

[8] K. I. Noor, "Some radius of convexity problems for analytic functions of bounded boundary rotations," *Punjab University Journal of Mathematics*, vol. 21, pp. 71–81, 1988.

[9] K. Padmanabhan and R. Parvatham, "Properties of a class of functions with bounded boundary rotation," *Annales Polonici Mathematici*, vol. 31, pp. 311–323, 1975.

[10] V. Paatero, "Uber die konforme abbildungen von gebieten deren rander von beschrankter drehung sind," *Annales Academiae Scientiarum Fennicae. Series A*, vol. 33, pp. 1–77, 1931.

[11] K. I. Noor, "On some integral operators for certain families of analytic functions," *Tamkang Journal of Mathematics*, vol. 22, pp. 113–117, 1991.

[12] M. K. Aouf and T. Seoudy, "Certain class of Bi-bazilevic functions with bounded boundary rotation involving sălăgeăn operator," *Constructive Mathematical Analysis*, vol. 3, no. 4, pp. 139–149, 2020.

[13] T. Seoudy and M. K. Aouf, "Fekete-szegö problem for certain subclass of analytic functions with complex order defined by q-analogue of ruscheweyh operator," *Constructive Mathematical Analysis*, vol. 3, no. 1, pp. 36–44, 2020.

[14] W. Rogosinski, "On the coefficients of subordinate functions," *Proceedings of the London Mathematical Society*, vol. 48, pp. 48–82, 1943.

[15] A. W. Goodman, *Univalent Functions Vol I and II*, Polygonal Publishing House, Washington, New Jersey, 1983.

[16] S. S. Miller and P. T. Mocanu, "Differential subordinations, theory and applications," *Series of Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 225, Marcel Dekker, Newyork, NY, USA, 2000.

[17] D. A. Brannan, "On functions of bounded boundary rotation," *Proceedings of the Edinburgh Mathematical Society*, vol. 2, pp. 339–347, 1969.

[18] A. W. Goodman, "On close-to-convex functions of higher order," *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös Nominatae. Sectio Mathematica*, vol. 25, pp. 17–30, 1972.

[19] W. Kaplan, "Close to convex schlicht functions," *Michigan Mathematical Journal*, vol. 1, pp. 169–185, 1952.

[20] K. I. Noor, "On subclasses of close-to-convex functions of higher order," *International Journal of Mathematics and Mathematical Sciences*, vol. 15, no. 2, pp. 279–289, 1992.

[21] S. D. Barnardi, "Convex and starlike univalent functions," *Transactions of the American Mathematical Society*, vol. 135, pp. 429–446, 1969.