Research Article

Some δ -Tempered Fractional Hermite–Hadamard Inequalities Involving Harmonically Convex Functions and Applications

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The main objective of this paper is to obtain some new δ -tempered fractional versions of Hermite–Hadamard's inequality using the class of harmonic convex functions. In order to show the significance of the main results, we also discuss some interesting applications.

1. Introduction and Preliminaries

A function $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$f((1-\tau)v_1 + \tau v_2) \leq (1-\tau)f(v_1) + \tau f(v_2), \quad \forall v_1, v_2 \in I, \tau \in [0, 1]. \quad (1)$$

In recent years, several new extensions of classical convexity have been proposed in the literature. Iscan [1] introduced the notion of harmonically convex functions as follows.

A function $f: I \subset (0, \infty) \rightarrow \mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{v_1 v_2}{\tau v_1 + (1-\tau)v_2}\right) \leq (1-\tau)f(v_1) + \tau f(v_2), \quad \forall v_1, v_2 \in I, \tau \in [0, 1]. \quad (2)$$

Hermite–Hadamard's inequality is one of the most studied results pertaining to convexity property of the functions. This result of Hermite and Hadamard reads as follows.

Let $f: I = [v_1, v_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function; then,

$$f\left(\frac{v_1 + v_2}{2}\right) \leq \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} f(x) dx \leq \frac{f(v_1) + f(v_2)}{2}. \quad (3)$$

Iscan [1] extended the classical version of Hermite–Hadamard's inequality using the harmonic convexity property of the function.

Let $f: I = [v_1, v_2] \subset (0, \infty) \rightarrow \mathbb{R}$ be a harmonically convex function; then,

$$f\left(\frac{2v_1 v_2}{v_1 + v_2}\right) \leq \frac{v_1 v_2}{v_2 - v_1} \int_{v_1}^{v_2} f(x) dx \leq \frac{f(v_1) + f(v_2)}{2}. \quad (4)$$

The interrelation between theory of convex functions and theory of integral inequalities has attracted several inequality experts and as result several new versions of classical results have been obtained in the literature. For example, Sarikaya et al. [2] used the concepts of fractional calculus in obtaining the fractional analogue of Hermite–Hadamard's inequality. This idea attracted several researchers and a result number of new refined fractional analogues of classical inequalities have been obtained in the literature. For example, Gurbuz et al. [3] obtained some new refinements of integral inequalities using fractional integral operators of positive real order. İşcan and Wu [4] obtained fractional analogue of Hermite–Hadamard's inequality using the concept of harmonically convex functions Awan et al. [5] obtained conformable fractional Hermite–Hadamard's inequality using the harmonic convexity property of the functions. İftikhar et al. [6] obtained some local fractional Newton's type inequalities via generalized harmonic convex

functions. Recently, Sanli et al. [7] obtained some more new fractional Hermite–Hadamard type of inequalities using the harmonic convexity property of the functions.

In recent years, the classical concepts of fractional calculus have been extended and generalized using novel and innovative ideas. For instance, Meerschaert et al. [8] introduced the concepts where power laws are tempered by an exponential factor and showed that this exponential tempering has both mathematical and practical advantages. This inspired Mohammed et al. [9], and they obtained new generalizations of Hermite–Hadamard's inequality using tempered fractional integrals. Mubeen [10] and Sarikaya and Karaca [11] introduced the notion of δ -fractional integrals. Using the concepts of δ -fractional calculus, Lei et al. [12] obtained Hadamard δ -fractional inequalities of Fejer type using GA -s-convex functions. Luo et al. [13] obtained bounds related to multiparameterized δ -fractional integrals and discussed their applications as well. Awan et al. [14] obtained new fractional analogues of Hermite–Hadamard's inequality using δ -Appell's hypergeometric functions and the harmonic convexity property of the functions.

The objective of this paper is also to obtain some new δ -tempered fractional versions of Hermite–Hadamard's inequality using the class of harmonic convex functions. Before we proceed further, let us recall some previously known concepts.

Riemann–Liouville fractional integrals are defined as follows.

Definition 1 (see [15]). Let $\mathcal{F} \in L_1[\nu_1, \nu_2]$. Then, Riemann–Liouville integrals $J_{\nu_1+}^{\alpha} \mathcal{F}$ and $J_{\nu_2-}^{\alpha} \mathcal{F}$ of order $\alpha > 0$ with $\nu_1 \geq 0$ are defined by

$$J_{\nu_1+}^{\alpha} \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^x (x - \lambda)^{\alpha-1} \mathcal{F}(\lambda) d\lambda, \quad x > \nu_1, \quad (5)$$

$$J_{\nu_2-}^{\alpha} \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\nu_2} (\lambda - x)^{\alpha-1} \mathcal{F}(\lambda) d\lambda, \quad x < \nu_2,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx, \quad (6)$$

is the well-known Gamma function.

The concept of δ -Riemann–Liouville fractional integral is defined as follows: let $\mathcal{F}$ be piecewise continuous on $I^* = (0, \infty)$ and integrable on any finite subinterval of $I = [0, \infty]$. Then, for $\lambda > 0$, we consider δ -Riemann–Liouville fractional integral of $\mathcal{F}$ of order α

$${}_k J_{\nu_1+}^{\alpha} \mathcal{F}(x) = \frac{1}{\delta \Gamma_{\delta}(\alpha)} \int_{\nu_1}^x (x - \lambda)^{(\alpha/\delta)-1} \mathcal{F}(\lambda) d\lambda, \quad x > \nu_1, \delta > 0. \quad (7)$$

If $\delta \rightarrow 1$, then δ -Riemann–Liouville fractional integrals reduce to classical Riemann–Liouville fractional integral.

In [8], the authors have described a new variation on the fractional calculus as follows.

Definition 2. Let $\mathcal{F} \in L[\nu_1, \nu_2]$ with $\lambda \geq 0$ and $\alpha > 0$. Then, right- and left-tempered fractional integrals are defined as

$${}^{\tau} J_{\nu_1+}^{\alpha, \lambda} \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^x (x - \mu)^{\alpha-1} \exp^{-\lambda(x-\mu)} \mathcal{F}(\mu) d\mu, \quad x \in [\nu_1, \nu_2], \quad (8)$$

$${}^{\tau} J_{\nu_2-}^{\alpha, \lambda} \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\nu_2} (\mu - x)^{\alpha-1} \exp^{-\lambda(\mu-x)} \mathcal{F}(\mu) d\mu, \quad x \in [\nu_1, \nu_2].$$

We now introduce the δ -tempered fractional integrals.

Definition 3. Let $\mathcal{F} \in L[\nu_1, \nu_2]$ with $\lambda \geq 0, \alpha > 0$ and $\delta \geq 1$. Then, right- and left-tempered δ -fractional integrals are defined as

$${}^{\tau} J_{\nu_1+}^{\alpha, \lambda, \delta} \mathcal{F}(x) = \frac{1}{\delta \Gamma_{\delta}(\alpha)} \int_{\nu_1}^x (x - \mu)^{\alpha-1} \exp^{-(\lambda(x-\mu))^{\delta}/\delta} \mathcal{F}(\mu) d\mu, \quad x \in [\nu_1, \nu_2], \quad (9)$$

$${}^{\tau} J_{\nu_2-}^{\alpha, \lambda, \delta} \mathcal{F}(x) = \frac{1}{\delta \Gamma_{\delta}(\alpha)} \int_x^{\nu_2} (\mu - x)^{\alpha-1} \exp^{-(\lambda(\mu-x))^{\delta}/\delta} \mathcal{F}(\mu) d\mu, \quad x \in [\nu_1, \nu_2].$$

Definition 4. For the real numbers $\alpha > 0, \lambda \geq 1$ with $\delta \geq 1$, we define the λ -incomplete δ -gamma function by

$$\delta \gamma_{\lambda}(\alpha, x) = \int_0^x \mu^{\alpha-1} e^{-(\lambda \mu)^{\delta}/\delta} d\mu. \quad (10)$$

If $\lambda = 0$, it reduces the incomplete δ -gamma function:

$$\delta\gamma(\alpha, x) = \int_0^x \mu^{\alpha-1} e^{-\mu^{\delta/\delta}} d\mu. \quad (11)$$

Remark 1. For the real numbers $\alpha > 0, \lambda \geq 1$ with $\delta \geq 1$, we have

- (1) $\delta\gamma_{\lambda(v_2-v_1)}(\alpha, 1) = \int_0^1 \mu^{\alpha-1} e^{-(\lambda(v_2-v_1)\mu)^{\delta/\delta}} d\mu = 1/(v_2-v_1)^{\alpha} \delta\gamma_{\lambda}(\alpha, v_2-v_1)$
- (2) $\int_0^1 \delta\gamma_{\lambda(v_2-v_1)}(\alpha, x) dx = \delta\gamma_{\lambda}(\alpha, v_2-v_1)/(v_2-v_1)^{\alpha} - \delta\gamma_{\lambda}(\alpha+1, v_2-v_1)/(v_2-v_1)^{\alpha+1}$

Proof

- (1) The proof is straightforward, by using the change of variable technique $x = (v_2-v_1)\mu$.
- (2) To prove this, we use definition of λ -incomplete δ -gamma function:

$$\int_0^1 \delta\gamma_{\lambda(v_2-v_1)}(\alpha, x) dx = \int_0^1 \int_0^x \mu^{\alpha-1} e^{-(\lambda(v_2-v_1)\mu)^{\delta/\delta}} d\mu dx. \quad (12)$$

By changing the order of integration, we obtain

$$\begin{aligned} \int_0^1 \delta\gamma_{\lambda(v_2-v_1)}(\alpha, x) dx &= \int_0^1 \int_{\mu}^1 \mu^{\alpha-1} e^{-(\lambda(v_2-v_1)\mu)^{\delta/\delta}} d\mu dx \\ &= \int_0^1 (1-\mu) \mu^{\alpha-1} e^{-(\lambda(v_2-v_1)\mu)^{\delta/\delta}} d\mu \\ &= \int_0^1 \mu^{\alpha-1} e^{-(\lambda(v_2-v_1)\mu)^{\delta/\delta}} d\mu - \int_0^1 \mu^{\alpha} e^{-(\lambda(v_2-v_1)\mu)^{\delta/\delta}} d\mu. \end{aligned} \quad (13)$$

Using Remark 1 (1), we have

$$\int_0^1 \delta\gamma_{\lambda(v_2-v_1)}(\alpha, x) dx = \frac{\delta\gamma_{\lambda}(\alpha, v_2-v_1)}{(v_2-v_1)^{\alpha}} - \frac{\delta\gamma_{\lambda}(\alpha+1, v_2-v_1)}{(v_2-v_1)^{\alpha+1}}. \quad (14)$$

This completes the proof. $\square$

2. Results and Discussion

In this section, we discuss our main results.

2.1. A New Version of Hermite–Hadamard’s Inequality. We now derive a new δ -tempered fractional Hermite–Hadamard inequality via harmonically convex function.

Theorem 1. Let $f: [v_1, v_2] \rightarrow \mathbb{R}$ be an harmonically convex function on $[v_1, v_2]$ with $v_1 < v_2$; then,

$$f\left(\frac{2v_1v_2}{v_1+v_2}\right) \leq \left(\frac{v_1v_2}{v_2-v_1}\right)^{\alpha} \frac{\delta\Gamma_{\delta}(\alpha)}{2\delta\gamma_{\lambda(v_2-v_1)}(\alpha, 1)} \left[{}^{\tau}J_{(1/v_1)^{-}, \delta}^{\alpha, \lambda v_1 v_2} f \circ g\left(\frac{1}{v_2}\right) + {}^{\tau}J_{(1/v_2)^{+}, \delta}^{\alpha, \lambda v_1 v_2} f \circ g\left(\frac{1}{v_1}\right) \right] \leq \frac{f(v_1) + f(v_2)}{2}, \quad (15)$$

where $g(x) = 1/x$ and for all $\alpha > 0$ and $\lambda \geq 1$.

Proof. Since f is a harmonic convex function, then

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x) + f(y)}{2}. \quad (16)$$

This implies

$$2f\left(\frac{2v_1v_2}{v_1+v_2}\right) \leq f\left(\frac{v_1v_2}{(1-\tau)v_1 + \tau v_2}\right) + f\left(\frac{v_1v_2}{\tau v_1 + (1-\tau)v_2}\right). \quad (17)$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1} e^{-(\lambda(v_2-v_1)\tau)^{\delta/\delta}}$ and integrating with respect to τ on $[0, 1]$, we have

$$\begin{aligned} &2f\left(\frac{2v_1v_2}{v_1+v_2}\right) \int_0^1 \tau^{\alpha-1} e^{-(\lambda(v_2-v_1)\tau)^{\delta/\delta}} d\tau \\ &\leq \int_0^1 \tau^{\alpha-1} e^{-(\lambda(v_2-v_1)\tau)^{\delta/\delta}} f\left(\frac{v_1v_2}{(1-\tau)v_1 + \tau v_2}\right) d\tau + \int_0^1 \tau^{\alpha-1} e^{-(\lambda(v_2-v_1)\tau)^{\delta/\delta}} f\left(\frac{v_1v_2}{\tau v_1 + (1-\tau)v_2}\right) d\tau. \end{aligned} \quad (18)$$

This implies

$$\begin{aligned}
 & 2\delta\gamma_{\lambda(v_2-v_1)}(\alpha, 1) \left(\frac{2\nu_1\nu_2}{\nu_1+\nu_2} \right) \\
 & \leq \frac{(\nu_1\nu_2)^\alpha}{(\nu_2-\nu_1)^\alpha} \int_{1/\nu_2}^{1/\nu_1} \left(x - \frac{1}{\nu_2} \right)^{\alpha-1} e^{-(\lambda\nu_1\nu_2(x-(1/\nu_2)))^\delta/\delta} f\left(\frac{1}{x}\right) dx \\
 & \quad + \frac{(\nu_1\nu_2)^\alpha}{(\nu_2-\nu_1)^\alpha} \int_{1/\nu_2}^{1/\nu_1} \left(\frac{1}{\nu_1} - x \right)^{\alpha-1} e^{-(\lambda\nu_1\nu_2((1/\nu_1)-x))^\delta/\delta} f\left(\frac{1}{x}\right) dx.
 \end{aligned} \tag{19}$$

Thus, we have

$$\begin{aligned}
 f\left(\frac{2\nu_1\nu_2}{\nu_1+\nu_2}\right) & \leq \frac{(\nu_1\nu_2)^\alpha \delta \Gamma_\delta(\alpha)}{2(\nu_2-\nu_1)^\alpha \delta \gamma_{\lambda(v_2-v_1)}(\alpha, 1)} \\
 & \cdot \left[{}^\tau J_{(1/\nu_1)^-, \delta}^{\alpha, \lambda\nu_1\nu_2} f \circ g\left(\frac{1}{\nu_2}\right) + {}^\tau J_{(1/\nu_2)^+, \delta}^{\alpha, \lambda\nu_1\nu_2} f \circ g\left(\frac{1}{\nu_1}\right) \right].
 \end{aligned} \tag{20}$$

Also,

$$f\left(\frac{\nu_1\nu_2}{(1-\tau)\nu_1+\tau\nu_2}\right) \leq \tau f(\nu_1) + (1-\tau)\nu_2, \tag{21}$$

$$f\left(\frac{\nu_1\nu_2}{\tau\nu_1+(1-\tau)\nu_2}\right) \leq (1-\tau)f(\nu_1) + \tau f(\nu_2). \tag{22}$$

Adding (21) and (22), we have

$$f\left(\frac{\nu_1\nu_2}{(1-\tau)\nu_1+\tau\nu_2}\right) + f\left(\frac{\nu_1\nu_2}{\tau\nu_1+(1-\tau)\nu_2}\right) \leq f(\nu_1) + f(\nu_2). \tag{23}$$

Multiplying the above inequality by $\tau^{\alpha-1}e^{-(\lambda(v_2-v_1)\tau)^\delta/\delta}$ and integrating with respect to τ on $[0, 1]$, we have

$$\begin{aligned}
 & \int_0^1 \tau^{\alpha-1} e^{-(\lambda(v_2-v_1)\tau)^\delta/\delta} f\left(\frac{\nu_1\nu_2}{(1-\tau)\nu_1+\tau\nu_2}\right) d\tau + \int_0^1 \tau^{\alpha-1} e^{-(\lambda(v_2-v_1)\tau)^\delta/\delta} f\left(\frac{\nu_1\nu_2}{\tau\nu_1+(1-\tau)\nu_2}\right) d\tau \\
 & \leq [f(\nu_1) + f(\nu_2)] \int_0^1 \tau^{\alpha-1} e^{-(\lambda(v_2-v_1)\tau)^\delta/\delta} d\tau.
 \end{aligned} \tag{24}$$

This implies

$$\begin{aligned}
 & \left(\frac{\nu_1\nu_2}{\nu_2-\nu_1} \right)^\alpha \int_{1/\nu_2}^{1/\nu_1} \left(x - \frac{1}{\nu_2} \right)^{\alpha-1} e^{\frac{-(\lambda\nu_1\nu_2(x-(1/\nu_2)))^\delta}{\delta}} f\left(\frac{1}{x}\right) dx \\
 & \quad + \left(\frac{\nu_1\nu_2}{\nu_2-\nu_1} \right)^\alpha \int_{1/\nu_2}^{1/\nu_1} \left(\frac{1}{\nu_1} - x \right)^{\alpha-1} e^{\frac{-(\lambda\nu_1\nu_2((1/\nu_1)-x))^\delta}{\delta}} f\left(\frac{1}{x}\right) dx \\
 & \leq \delta \gamma_{\lambda(v_2-v_1)}(\alpha, 1) [f(\nu_1) + f(\nu_2)].
 \end{aligned} \tag{25}$$

This implies

$$\left(\frac{\nu_1\nu_2}{\nu_2-\nu_1} \right)^\alpha \frac{\delta \Gamma_\delta(\alpha)}{2\delta \gamma_{\lambda(v_2-v_1)}(\alpha, 1)} \left[{}^\tau J_{(1/\nu_1)^-, \delta}^{\alpha, \lambda\nu_1\nu_2} f \circ g\left(\frac{1}{\nu_2}\right) + {}^\tau J_{(1/\nu_2)^+, \delta}^{\alpha, \lambda\nu_1\nu_2} f \circ g\left(\frac{1}{\nu_1}\right) \right] \leq \frac{f(\nu_1) + f(\nu_2)}{2}. \tag{26}$$

Combining (20) and (26), we get the required inequality $\square$

(15)

Remark 2

- (1) For $\alpha = 1 = k$ and $\lambda = 0$ in (15), we obtain the classical Hermite–Hadamard’s type inequality for the notion of harmonic convexity
- (2) For $k = 1$ and $\lambda = 0$ in 2.1, we obtain the fractional version of Hermite–Hadamard’s type inequality for the notion of harmonic convexity

2.2. New Auxiliary Results. We now derive two new auxiliary results. These results will play a significant role in the development of our next results.

Lemma 1. Let $f: [\nu_1, \nu_2] \longrightarrow \mathbb{R}$ be L^1 function; then,

$$\begin{aligned} & \frac{f(\nu_1) + f(\nu_2)}{2} - \left(\frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \right)^\alpha \frac{\delta \Gamma_\delta(\alpha)}{2 \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1)} \left[{}^\tau J_{\left(\frac{1}{\nu_1}\right)^-, \delta}^{\alpha, \lambda \nu_1 \nu_2} f \circ g \left(\frac{1}{\nu_2} \right) + {}^\tau J_{\left(\frac{1}{\nu_2}\right)^+, \delta}^{\alpha, \lambda \nu_1 \nu_2} f \circ g \left(\frac{1}{\nu_1} \right) \right] \\ &= \frac{\nu_1 \nu_2 (\nu_2 - \nu_1)}{2 \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1)} \int_0^1 \frac{[\delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1 - \tau) - \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, \tau)]}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} f' \left(\frac{\nu_1 \nu_2}{(1 - \tau)\nu_1 + \tau\nu_2} \right) d\tau, \end{aligned} \quad (27)$$

for all $\alpha > 0, \lambda \geq 0$.

Proof. Consider

$$\begin{aligned} I &= \frac{\nu_1 \nu_2 (\nu_2 - \nu_1)}{2 \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1)} \int_0^1 \frac{[\delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1 - \tau) + \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, \tau)]}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} f' \left(\frac{\nu_1 \nu_2}{(1 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \\ &= \frac{\nu_1 \nu_2 (\nu_2 - \nu_1)}{2 \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1)} [I_1 - I_2]. \end{aligned} \quad (28)$$

Now,

$$\begin{aligned} I_1 &= \int_0^1 \frac{[\delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1 - \tau)]}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} f' \left(\frac{\nu_1 \nu_2}{(1 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \\ &= \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1) f(\nu_2) - \int_0^1 (1 - \tau)^{\alpha-1} e^{\frac{(\lambda(\nu_2 - \nu_1)(1 - \tau))^\delta}{\delta}} f \left(\frac{\nu_1 \nu_2}{(1 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \\ &= \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1) f(\nu_2) - \left(\frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \right)^\alpha \int_{1/\nu_2}^{1/\nu_1} \left(\frac{1}{\nu_1} - x \right)^{\alpha-1} e^{\frac{(\lambda(\nu_2 - \nu_1)((1/\nu_1) - x))^\delta}{\delta}} f \left(\frac{1}{x} \right) dx \\ &= \delta \gamma_\lambda(\nu_2 - \nu_1)(\alpha, 1) f(\nu_2) - \left(\frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \right)^\alpha \delta \Gamma_\delta(\alpha) {}^\tau J_{\left(\frac{1}{\nu_2}\right)^+, \delta}^{\alpha, \lambda \nu_1 \nu_2} f \circ g \left(\frac{1}{\nu_1} \right). \end{aligned} \quad (29)$$

Similarly,

$$\begin{aligned}
I_2 &= \int_0^1 \frac{\left[\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau) \right]}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} f' \left(\frac{\nu_1 \nu_2}{(1 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \\
&= -\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) f(\nu_1) + \int_0^1 \tau^{\alpha-1} e^{(\lambda(\nu_2 - \nu_1)\tau)^{\delta/\delta}} f \left(\frac{\nu_1 \nu_2}{(1 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \\
&= -\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) f(\nu_2) + \left(\frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \right)^{\alpha} \int_{1/\nu_2}^{1/\nu_1} \left(x - \frac{1}{\nu_2} \right)^{\alpha-1} e^{(\lambda(\nu_2 - \nu_1)(x - (1/\nu_2)))^{\delta/\delta}} f \left(\frac{1}{x} \right) dx \\
&= -\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) f(\nu_2) + \left(\frac{\nu_1 \nu_2}{\nu_2 - \nu_1} \right)^{\alpha} \delta \Gamma_{\delta}(\alpha) J_{\left(\frac{1}{\nu_1} \right)^-, \delta}^{\alpha, \lambda \nu_1 \nu_2} f \circ g \left(\frac{1}{\nu_2} \right).
\end{aligned} \tag{30}$$

Substituting the values of I_1 and I_2 in (28), we get the required result. $\square$

Lemma 2. Let $f: [\nu_1, \nu_2] \longrightarrow \mathbb{R}$ be L^1 function; then,

$$\begin{aligned}
&\frac{2^{\alpha-1}(\nu_1 \nu_2)^{\alpha} \delta \Gamma_{\delta}(\alpha)}{\delta \gamma_{\lambda}(\alpha, \nu_2 - \nu_1)} \left[{}^{\tau} J_{\left(\frac{1}{\nu_1 + \nu_2/2\nu_1 \nu_2} \right)^-, \delta}^{\alpha, 2\lambda \nu_1 \nu_2} f \circ g \left(\frac{1}{\nu_2} \right) + {}^{\tau} J_{\left(\frac{1}{\nu_1 + \nu_2/2\nu_1 \nu_2} \right)^+, \delta}^{\alpha, 2\lambda \nu_1 \nu_2} f \circ g \left(\frac{1}{\nu_1} \right) \right] - f \left(\frac{2\nu_1 \nu_2}{\nu_1 + \nu_2} \right) \\
&= \frac{(\nu_2 - \nu_1)^{\alpha}}{2\delta \gamma_{\lambda}(\alpha, \nu_2 - \nu_1)} \left[\int_0^1 \frac{2\nu_1 \nu_2 (\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} f' \left(\frac{2\nu_1 \nu_2}{(2 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \right. \\
&\quad \left. - \int_0^1 \frac{2\nu_1 \nu_2 (\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} f' \left(\frac{2\nu_1 \nu_2}{\tau\nu_1 + (2 - \tau)\nu_2} \right) d\tau \right],
\end{aligned} \tag{31}$$

where $\alpha > 0, \lambda \geq 1$.

Proof. Consider

$$\begin{aligned}
I &= \frac{(\nu_2 - \nu_1)^{\alpha}}{2\delta \gamma_{\lambda}(\alpha, \nu_2 - \nu_1)} \left[\int_0^1 \frac{2\nu_1 \nu_2 (\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} f' \left(\frac{2\nu_1 \nu_2}{(2 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \right. \\
&\quad \left. - \int_0^1 \frac{2\nu_1 \nu_2 (\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} f' \left(\frac{2\nu_1 \nu_2}{\tau\nu_1 + (2 - \tau)\nu_2} \right) d\tau \right] \\
&= \frac{(\nu_2 - \nu_1)^{\alpha}}{2\delta \gamma_{\lambda}(\alpha, \nu_2 - \nu_1)} [I_1 - I_2].
\end{aligned} \tag{32}$$

Using Remark 1 (1), we have

$$\begin{aligned}
I_1 &= \int_0^1 \frac{2\nu_1 \nu_2 (\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} f' \left(\frac{2\nu_1 \nu_2}{(2 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \\
&= -\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) f \left(\frac{2\nu_1 \nu_2}{\nu_1 + \nu_2} \right) + \int_0^1 \tau^{\alpha-1} e^{-(\lambda(\nu_2 - \nu_1)\tau)^{\delta/\delta}} f \left(\frac{2\nu_1 \nu_2}{(2 - \tau)\nu_1 + \tau\nu_2} \right) d\tau \\
&= \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) f \left(\frac{2\nu_1 \nu_2}{\nu_1 + \nu_2} \right) + \frac{(2\nu_1 \nu_2)^{\alpha}}{(\nu_2 - \nu_1)^{\alpha}} \int_{1/\nu_2}^{\nu_1 + \nu_2/2\nu_1 \nu_2} \left(x - \frac{1}{\nu_2} \right)^{\alpha-1} e^{-(2\nu_1 \nu_2 \lambda (x - (1/\nu_2)))^{\delta/\delta}} f \left(\frac{1}{x} \right) dx \\
&= -\delta \gamma_{\lambda}(\alpha, 1) f \left(\frac{2\nu_1 \nu_2}{\nu_1 + \nu_2} \right) + \frac{(2\nu_1 \nu_2)^{\alpha} \delta \Gamma_{\delta}(\alpha)}{(\nu_2 - \nu_1)^{\alpha}} J_{\left(\frac{1}{\nu_1 + \nu_2/2\nu_1 \nu_2} \right)^-, \delta}^{\alpha, 2\nu_1 \nu_2 \lambda} f \circ g \left(\frac{1}{\nu_2} \right).
\end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned}
 I_2 &= \int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1)\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} f'\left(\frac{2\nu_1\nu_2}{\tau\nu_1 + (2 - \tau)\nu_2}\right) d\tau \\
 &= \delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1) f\left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2}\right) - \int_0^1 \tau^{\alpha-1} e^{-\frac{(\lambda(\nu_2 - \nu_1)\tau)}{\delta}} f\left(\frac{2\nu_1\nu_2}{\tau\nu_1 + (2 - \tau)\nu_2}\right) d\tau \\
 &= \delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1) f\left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2}\right) - \frac{(2\nu_1\nu_2)^\alpha}{(\nu_2 - \nu_1)^\alpha} \int_{\nu_1+\nu_2/2\nu_1\nu_2}^{1/\nu_1} \left(\frac{1}{\nu_1} - x\right)^{\alpha-1} e^{-(2\nu_1\nu_2\lambda((1/\nu_1)-x))/\delta} f\left(\frac{1}{x}\right) dx \\
 &= \delta\gamma_{\lambda}(\alpha, 1) f^\circ g\left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2}\right) - \frac{(2\nu_1\nu_2)^\alpha \delta\Gamma_{(\alpha)}(\tau)}{(\nu_2 - \nu_1)^\alpha} J_{(\nu_1+\nu_2/2\nu_1\nu_2)^+, \delta}^{\alpha, 2\nu_1\nu_2\lambda} f^\circ g\left(\frac{1}{\nu_1}\right).
 \end{aligned} \tag{34}$$

Substituting the values of I_1 and I_2 in (32) and using Remark 1 (1), we get the required result. $\square$

previous section, we derive some new Hermite–Hadamard’s type of integral inequalities.

2.3. Some More New Hermite–Hadamard’s Type of Integral Inequalities. Now using auxiliary results obtained in the

Theorem 2. Let $f: [\nu_1, \nu_2] \longrightarrow \mathbb{R}$ be L^1 function. If $|f'|^q, q \geq 1$, is harmonically convex on $[\nu_1, \nu_2]$ with $\nu_1 < \nu_2$, then

$$\begin{aligned}
 &\left| \frac{f(\nu_1) + f(\nu_2)}{2} - \left(\frac{\nu_1\nu_2}{\nu_2 - \nu_1}\right)^\alpha \frac{\delta\Gamma_\delta(\alpha)}{2\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1)} \left[{}^\tau J_{(1/\nu_1)^-, \delta}^{\alpha, \lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_2}\right) + {}^\tau J_{(1/\nu_2)^+, \delta}^{\alpha, \lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_1}\right) \right] \right| \\
 &\leq \frac{\nu_1\nu_2(\nu_2 - \nu_1)}{2\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1)} \left(\int_0^1 \left| \frac{\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1 - \tau) - \delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, \tau)}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} \right|^p d\tau \right)^{1/p} \left(\frac{|f'(\nu_1)|^q + |f'(\nu_2)|^q}{2} \right)^{1/q}.
 \end{aligned} \tag{35}$$

Proof. Using Lemma 1, the harmonic convexity of $|f'|^q$, and Hölder’s inequality, we have

$$\begin{aligned}
 &\left| \frac{f(\nu_1) + f(\nu_2)}{2} - \left(\frac{\nu_1\nu_2}{\nu_2 - \nu_1}\right)^\alpha \frac{\delta\Gamma_\delta(\alpha)}{2\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1)} \left[{}^\tau J_{(1/\nu_1)^-, \delta}^{\alpha, \lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_2}\right) + {}^\tau J_{(1/\nu_2)^+, \delta}^{\alpha, \lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_1}\right) \right] \right| \\
 &\leq \frac{\nu_1\nu_2(\nu_2 - \nu_1)}{2\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1)} \int_0^1 \left| \frac{\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1 - \tau) - \delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, \tau)}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} \right| \left| f'\left(\frac{\nu_1\nu_2}{(1 - \tau)\nu_1 + \tau\nu_2}\right) \right| d\tau \\
 &\leq \frac{\nu_1\nu_2(\nu_2 - \nu_1)}{2\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1)} \left(\int_0^1 \left| \frac{\delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1 - \tau) - \delta\gamma_{\lambda(\nu_2-\nu_1)}(\alpha, \tau)}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} \right|^p d\tau \right)^{1/p} \\
 &\quad \times \left(\int_0^1 \left| f'\left(\frac{\nu_1\nu_2}{(1 - \tau)\nu_1 + \tau\nu_2}\right) \right|^q d\tau \right)^{1/q}
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\nu_1 \nu_2 (\nu_2 - \nu_1)}{2\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1)} \left(\int_0^1 \left| \frac{\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1 - \tau) - \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} \right|^p d\tau \right)^{1/p} \\
&\times \left(\int_0^1 [\tau |f'(\nu_1)|^q + (1 - \tau) |f'(\nu_2)|^q] d\tau \right)^{1/q} \\
&= \frac{\nu_1 \nu_2 (\nu_2 - \nu_1)}{2\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1)} \left(\int_0^1 \left| \frac{\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1 - \tau) - \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(1 - \tau)\nu_1 + \tau\nu_2]^2} \right|^p d\tau \right)^{1/p} \left(\frac{|f'(\nu_1)|^q + |f'(\nu_2)|^q}{2} \right)^{1/q}.
\end{aligned} \tag{36}$$

This completes our proof. $\square$

Theorem 3. Let $f: [\nu_1, \nu_2] \rightarrow \mathbb{R}$ be an harmonically convex function on $[\nu_1, \nu_2]$ with $\nu_1 < \nu_2$; then, we have

$$f\left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2}\right) \leq \left(\frac{\nu_1\nu_2}{\nu_2 - \nu_1}\right)^\alpha \frac{2^{\alpha-1}\delta\Gamma_\delta(\alpha)}{\delta\gamma_\lambda(\alpha, \nu_2 - \nu_1)} \left[\tau^{\alpha, 2\lambda\nu_1\nu_2}_{(\nu_1 + \nu_2/2\nu_1\nu_2)^-, \delta} f \circ g\left(\frac{1}{\nu_2}\right) + \tau^{\alpha, 2\lambda\nu_1\nu_2}_{(\nu_1 + \nu_2/2\nu_1\nu_2)^+, \delta} f \circ g\left(\frac{1}{\nu_1}\right) \right] \leq \frac{f(\nu_1) + f(\nu_2)}{2}, \tag{37}$$

for all $\alpha > 0, \lambda \geq 1$.

$$2f\left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2}\right) \leq f\left(\frac{2\nu_1\nu_2}{(2 - \tau)\nu_1 + \tau\nu_2}\right) + f\left(\frac{2\nu_1\nu_2}{\tau\nu_1 + (2 - \tau)\nu_2}\right). \tag{39}$$

Proof. Since f is an harmonically convex function, then we have

$$f\left(\frac{2xy}{x + y}\right) \leq \frac{f(x) + f(y)}{2}. \tag{38}$$

Multiplying both sides of the above inequality by $\tau^{\alpha-1} e^{-(\lambda(\nu_2 - \nu_1)\tau)^{\delta}/\delta}$ and integrating with respect to τ on $[0, 1]$, we have

This implies

$$\begin{aligned}
&2f\left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2}\right) \int_0^1 \tau^{\alpha-1} e^{-(\lambda(\nu_2 - \nu_1)\tau)^{\delta}/\delta} d\tau \\
&\leq \int_0^1 \tau^{\alpha-1} e^{-(\lambda(\nu_2 - \nu_1)\tau)^{\delta}/\delta} d\tau, f\left(\frac{2\nu_1\nu_2}{(2 - \tau)\nu_1 + \tau\nu_2}\right) d\tau + \int_0^1 \tau^{\alpha-1} e^{-(\lambda(\nu_2 - \nu_1)\tau)^{\delta}/\delta} d\tau, f\left(\frac{2\nu_1\nu_2}{\tau\nu_1 + (2 - \tau)\nu_2}\right) d\tau.
\end{aligned} \tag{40}$$

Thus, we have

$$\begin{aligned}
&2\delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) \left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2}\right) \\
&\leq \left(\frac{\nu_1\nu_2}{\nu_2 - \nu_1}\right)^\alpha \int_{1/\nu_2}^{\nu_1 + \nu_2/2\nu_1\nu_2} \left(x - \frac{1}{\nu_2}\right)^{\alpha-1} e^{\frac{-(\lambda 2\nu_1\nu_2(x - (1/\nu_2)))^\delta}{\delta}} f\left(\frac{1}{x}\right) dx + \left(\frac{\nu_1\nu_2}{\nu_2 - \nu_1}\right)^\alpha \\
&\int_{\nu_1 + \nu_2/2\nu_1\nu_2}^{1/\nu_1} \left(\frac{1}{\nu_1} - x\right)^{\alpha-1} e^{\frac{-(2\lambda\nu_1\nu_2((1/\nu_1) - x))^\delta}{\delta}} f\left(\frac{1}{x}\right) dx.
\end{aligned} \tag{41}$$

Using Remark 1 (1), we have

$$f\left(\frac{2\nu_1\nu_2}{\nu_1+\nu_2}\right) \leq \frac{2^{\alpha-1}(\nu_1\nu_2)^\alpha \delta \Gamma_\delta(\alpha)}{\delta \gamma_\lambda(\alpha, \nu_2 - \nu_1)} \left[{}^\tau J_{(\nu_1+\nu_2/2\nu_1\nu_2)^-, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_2}\right) + {}^\tau J_{(\nu_1+\nu_2/2\nu_1\nu_2)^+, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_1}\right) \right]. \quad (42)$$

Also,

$$f\left(\frac{2\nu_1\nu_2}{(2-\tau)\nu_1+\tau\nu_2}\right) + f\left(\frac{2\nu_1\nu_2}{\tau\nu_1+(2-\tau)\nu_2}\right) \leq f(\nu_1) + f(\nu_2). \quad (43)$$

Multiplying the above inequality by $\tau^{\alpha-1}e^{-(\lambda(\nu_2-\nu_1)\tau)^{\delta}/\delta}$ and integrating with respect to τ on $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 \tau^{\alpha-1} e^{-(\lambda(\nu_2-\nu_1)\tau)^{\delta}/\delta} f\left(\frac{2\nu_1\nu_2}{(2-\tau)\nu_1+\tau\nu_2}\right) d\tau \\ & + \int_0^1 \tau^{\alpha-1} e^{-(\lambda(\nu_2-\nu_1)\tau)^{\delta}/\delta} f\left(\frac{2\nu_1\nu_2}{\tau\nu_1+(2-\tau)\nu_2}\right) d\tau \end{aligned} \quad (44)$$

$$\leq [f(\nu_1) + f(\nu_2)] \int_0^1 \tau^{\alpha-1} e^{-(\lambda(\nu_2-\nu_1)\tau)^{\delta}/\delta} d\tau.$$

Using Remark 1 (1), we have

$$\begin{aligned} & \left(\frac{2\nu_1\nu_2}{\nu_2-\nu_1}\right)^\alpha \int_{1/\nu_2}^{\nu_1+\nu_2/2\nu_1\nu_2} \left(x - \frac{1}{\nu_2}\right)^{\alpha-1} e^{\frac{-(\lambda 2\nu_1\nu_2(x-(1/\nu_2)))^\delta}{\delta}} f\left(\frac{1}{x}\right) dx \\ & + \left(\frac{2\nu_1\nu_2}{\nu_2-\nu_1}\right)^\alpha \int_{\nu_1+\nu_2/2\nu_1\nu_2}^{1/\nu_1} \left(\frac{1}{\nu_1} - x\right)^{\alpha-1} e^{\frac{-(2\lambda\nu_1\nu_2((1/\nu_1)-x))^\delta}{\delta}} f\left(\frac{1}{x}\right) dx \\ & \leq \frac{\delta \gamma_\lambda(\alpha, \nu_2 - \nu_1)}{(\nu_2 - \nu_1)^\alpha} [f(\nu_1) + f(\nu_2)]. \end{aligned} \quad (45)$$

Thus, we have

$$\left(\frac{\nu_1\nu_2}{\nu_2-\nu_1}\right)^\alpha \frac{2^{\alpha-1} \delta \Gamma_\delta(\alpha)}{\delta \gamma_\lambda(\alpha, \nu_2 - \nu_1)} \left[{}^\tau J_{(\nu_1+\nu_2/2\nu_1\nu_2)^-, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_2}\right) + {}^\tau J_{(\nu_1+\nu_2/2\nu_1\nu_2)^+, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_1}\right) \right] \leq \frac{f(\nu_1) + f(\nu_2)}{2}. \quad (46)$$

Combining (42) and (46), we get the required inequality (37). $\square$

Theorem 4. If $f: [\nu_1, \nu_2] \longrightarrow \mathbb{R}$ is L^1 function, using the harmonic convexity property of $|f'|^q, q \geq 1$, on $[\nu_1, \nu_2]$ with $\nu_1 < \nu_2$, we have

$$\begin{aligned} & \left| \frac{2^{\alpha-1}(\nu_1\nu_2)^\alpha \delta \Gamma_\delta(\alpha)}{\delta \gamma_\lambda(\alpha, \nu_2 - \nu_1)} \left[{}^\tau J_{(\nu_1+\nu_2/2\nu_1\nu_2)^-, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_2}\right) + {}^\tau J_{(\nu_1+\nu_2/2\nu_1\nu_2)^+, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_1}\right) \right] - f\left(\frac{2\nu_1\nu_2}{\nu_1+\nu_2}\right) \right| \\ & \leq \frac{(\nu_2 - \nu_1)^\alpha}{2\delta \gamma_\lambda(\alpha, \nu_2 - \nu_1)} \left[\left[\frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln\left(\frac{ay + (2-y)\nu_2}{(2-y)\nu_1 + yb}\right) - \frac{4\nu_1^2\nu_2(1-y)}{((2-y)\nu_1 + yb)(ya + (2-y)\nu_2)} \right] \delta \gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1) |f'(\nu_1)|, \right. \\ & \quad \left. + \left[\frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln\left(\frac{(2-y)\nu_1 + yb}{ya + (2-y)\nu_2}\right) - \frac{4\nu_1\nu_2^2(1-y)}{((2-y)\nu_1 + yb)(ya + (2-y)\nu_2)} \right] \delta \gamma_{\lambda(\nu_2-\nu_1)}(\alpha, 1) |f'(\nu_2)| \right]. \end{aligned} \quad (47)$$

Proof. Using Lemma 2 and the harmonic convexity property of $|f'|$, we have

$$\left| \frac{2^{\alpha-1}(\nu_1\nu_2)^\alpha \delta \Gamma_\delta(\alpha)}{\delta \gamma_\lambda(\alpha, \nu_2 - \nu_1)} \left[{}^\tau J_{(\nu_1+\nu_2/2\nu_1\nu_2)^-, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_2}\right) + {}^\tau J_{(\nu_1+\nu_2/2\nu_1\nu_2)^+, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g\left(\frac{1}{\nu_1}\right) \right] - f\left(\frac{2\nu_1\nu_2}{\nu_1+\nu_2}\right) \right|$$

$$\begin{aligned}
&\leq \frac{(\nu_2 - \nu_1)^\alpha}{2\delta_{\gamma_\lambda}(\alpha, \nu_2 - \nu_1)} \left[\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1)\delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} \left| f' \left(\frac{2\nu_1\nu_2}{(2 - \tau)\nu_1 + \tau\nu_2} \right) \right| d\tau \right. \\
&\quad \left. + \int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1)\delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} \left| f' \left(\frac{2\nu_1\nu_2}{\tau\nu_1 + (2 - \tau)\nu_2} \right) \right| d\tau \right] \\
&\leq \frac{(\nu_2 - \nu_1)^\alpha}{2\delta_{\gamma_\lambda}(\alpha, \nu_2 - \nu_1)} \left[\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1)\delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} \left[\frac{\tau}{2} |f'(\nu_1)| + \frac{2 - \tau}{2} |f'(\nu_2)| \right] d\tau \right. \\
&\quad \left. + \int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1)\delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} \left[\frac{2 - \tau}{2} |f'(\nu_1)| + \frac{\tau}{2} |f'(\nu_2)| \right] d\tau \right].
\end{aligned} \tag{48}$$

Changing the order of integration, we have

$$\begin{aligned}
&= \frac{(\nu_2 - \nu_1)^\alpha}{2\delta_{\gamma_\lambda}(\alpha, \nu_2 - \nu_1)} \left[\left[\frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln \left(\frac{a\nu + (2 - \nu)\nu_2}{(2 - \nu)\nu_1 + \nu b} \right) - \frac{4\nu_1^2\nu_2(1 - \nu)}{((2 - \nu)\nu_1 + \nu b)(\nu a + (2 - \nu)\nu_2)} \right] \delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, 1) |f'(\nu_1)|, \right. \\
&\quad \left. + \left[\frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln \left(\frac{(2 - \nu)\nu_1 + \nu b}{\nu a + (2 - \nu)\nu_2} \right) - \frac{4\nu_1\nu_2^2(1 - \nu)}{((2 - \nu)\nu_1 + \nu b)(\nu a + (2 - \nu)\nu_2)} \right] \delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, 1) |f'(\nu_2)| \right].
\end{aligned} \tag{49}$$

This completes our proof. $\square$

Theorem 5. Let $f: [\nu_1, \nu_2] \longrightarrow \mathbb{R}$ be L^1 function. If $|f'|^q, q \geq 1$, is harmonically convex on $[\nu_1, \nu_2]$ with $\nu_1 < \nu_2$, then we have

$$\begin{aligned}
&\left| \frac{2^{\alpha-1}(\nu_1\nu_2)^\alpha \delta_{\gamma_\lambda}(\alpha)}{\delta_{\gamma_\lambda}(\alpha, \nu_2 - \nu_1)} \left[{}^\tau J_{(\nu_1 + \nu_2/2, \nu_2), \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g \left(\frac{1}{\nu_2} \right) + {}^\tau J_{(\nu_1 + \nu_2/2, \nu_1), \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^\circ g \left(\frac{1}{\nu_1} \right) \right] - f \left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2} \right) \right| \\
&\leq \frac{(\nu_2 - \nu_1)^\alpha}{2\delta_{\gamma_\lambda}(\alpha, \nu_2 - \nu_1)} \left[(M_1(\alpha, \lambda))^{1-(1/q)} (M_2(\alpha, \lambda) |f'(\nu_1)|^q + M_3(\alpha, \lambda) |f'(\nu_2)|^q)^{1/q} \right. \\
&\quad \left. + (M_4(\alpha, \lambda))^{1-(1/q)} (M_5(\alpha, \lambda) |f'(\nu_1)|^q + M_6(\alpha, \lambda) |f'(\nu_2)|^q)^{1/q} \right],
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
M_1(\alpha, \lambda) &= \delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, 1) \left(\frac{2\nu_1\nu_2}{(2 - \nu)\nu_1 + \nu b} - \frac{2\nu_1\nu_2}{\nu_1 + \nu_2} \right), \\
M_2(\alpha, \lambda) &= \delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, 1) \left[\frac{2\nu_1^2\nu_2}{\nu_2 - \nu_1} \left(\frac{1}{\nu_1 + \nu_2} - \frac{1}{(2 - \nu)\nu_1 + \nu b} \right) + \frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln \left(\frac{\nu_1 + \nu_2}{(2 - \nu)\nu_1 + \nu b} \right) \right], \\
M_3(\alpha, \lambda) &= \delta_{\gamma_\lambda(\nu_2 - \nu_1)}(\alpha, 1) \left[\frac{2\nu_1\nu_2^2}{\nu_2 - \nu_1} \left(\frac{1}{(2 - \nu)\nu_1 + \nu b} - \frac{1}{\nu_1 + \nu_2} \right) - \frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln \left(\frac{\nu_1 + \nu_2}{(2 - \nu)\nu_1 + \nu b} \right) \right],
\end{aligned}$$

$$\begin{aligned}
M_4(\alpha, \lambda) &= \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) \left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2} - \frac{2\nu_1\nu_2}{ya + (2-y)\nu_2} \right), \\
M_5(\alpha, \lambda) &= \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) \left[\frac{2\nu_1^2\nu_2}{\nu_2 - \nu_1} \left(\frac{1}{ya + (2-y)\nu_2} - \frac{1}{\nu_1 + \nu_2} \right) - \frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln \left(\frac{\nu_1 + \nu_2}{ya + (2-y)\nu_2} \right) \right], \\
M_6(\alpha, \lambda) &= \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, 1) \left[\frac{2\nu_1\nu_2^2\nu_2}{\nu_2 - \nu_1} \left(\frac{1}{\nu_1 + \nu_2} - \frac{1}{ya + (2-y)\nu_2} \right) + \frac{\nu_1\nu_2}{\nu_2 - \nu_1} \ln \left(\frac{\nu_1 + \nu_2}{ya + (2-y)a\nu_2} \right) \right].
\end{aligned} \tag{51}$$

Proof. Using Lemma 2, the harmonic convexity property of $|f'|$, and power mean inequality, we have

$$\begin{aligned}
& \left| \frac{2^{\alpha-1}(\nu_1\nu_2)^{\alpha} \delta \Gamma_{\delta}(\alpha)}{\delta \gamma_{\lambda}(\alpha, \nu_2 - \nu_1)} \left[{}^{\tau}J_{(\nu_1+\nu_2/2\nu_1\nu_2), \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^{\circ} g \left(\frac{1}{\nu_2} \right) + {}^{\tau}J_{(\nu_1+\nu_2/2\nu_1\nu_2)^+, \delta}^{\alpha, 2\lambda\nu_1\nu_2} f^{\circ} g \left(\frac{1}{\nu_1} \right) \right] - f \left(\frac{2\nu_1\nu_2}{\nu_1 + \nu_2} \right) \right| \\
& \leq \frac{(\nu_2 - \nu_1)^{\alpha}}{2\delta \gamma_{\lambda}(\alpha, \nu_2 - \nu_1)} \left[\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} \left| f' \left(\frac{2\nu_1\nu_2}{(2 - \tau)\nu_1 + \tau\nu_2} \right) \right| d\tau \right. \\
& \quad \left. + \int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} \left| f' \left(\frac{2\nu_1\nu_2}{\tau\nu_1 + (2 - \tau)\nu_2} \right) \right| d\tau \right] \\
& \leq \frac{(\nu_2 - \nu_1)^{\alpha}}{2\delta \gamma_{\lambda}(\alpha, \nu_2 - \nu_1)} \left[\left(\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} d\tau \right)^{1-(1/q)} \right. \\
& \quad \left(\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} \left| f' \left(\frac{2\nu_1\nu_2}{(2 - \tau)\nu_1 + \tau\nu_2} \right) \right|^q d\tau \right)^{1/q} \\
& \quad \left. + \left(\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} d\tau \right)^{1-(1/q)} \right] \\
& \quad \left(\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} \left| f' \left(\frac{2\nu_1\nu_2}{\tau\nu_1 + (2 - \tau)\nu_2} \right) \right|^q d\tau \right)^{1/q} \Bigg] \\
& \leq \frac{(\nu_2 - \nu_1)^{\alpha}}{2\delta \gamma_{\lambda}(\alpha, \nu_2 - \nu_1)} \left[\left(\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} d\tau \right)^{1-(1/q)} \right. \\
& \quad \left(\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[(2 - \tau)\nu_1 + \tau\nu_2]^2} \left[\frac{\tau}{2} |f'(\nu_1)|^q + \frac{2 - \tau}{2} |f'(\nu_2)|^q \right] d\tau \right)^{1/q} \\
& \quad \left. + \left(\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} d\tau \right)^{1-(1/q)} \right. \\
& \quad \left. \left(\int_0^1 \frac{2\nu_1\nu_2(\nu_2 - \nu_1) \delta \gamma_{\lambda}(\nu_2 - \nu_1)(\alpha, \tau)}{[\tau\nu_1 + (2 - \tau)\nu_2]^2} \left[\frac{2 - \tau}{2} |f'(\nu_1)|^q + \frac{\tau}{2} |f'(\nu_2)|^q \right] d\tau \right)^{1/q} \right].
\end{aligned} \tag{52}$$

Changing the order of integration and after simple calculation, we get the required result. $\square$

Corollary 1. All the assumption of Theorem 2 are true; for $\alpha = 1 = k$ and $\lambda = 0$, we have

$$\left| \frac{f(\nu_1) + f(\nu_2)}{2} - \frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} \frac{f(x)}{x^2} dx \right| \leq \frac{\nu_1 \nu_2 (\nu_2 - \nu_1)}{2} \times \left(\frac{\nu_1^{-2p}}{2(p+1)} \left[{}_2F_1 \left(2p, 1, p+2; \frac{\nu_1 - \nu_2}{2\nu_1} \right) + {}_2F_1 \left(2p, p+1, p+2; \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} \right) \right] \right)^{1/p} \left(\frac{|f'(\nu_1)|^q + |f'(\nu_2)|^q}{2} \right)^{1/q}. \quad (53)$$

3. Applications

In this section, we investigate some applications to special means and special functions.

3.1. Applications to Special Means. We now recall some special means for two nonnegative numbers.

- (1) The arithmetic mean: $A(\nu_1, \nu_2) = \nu_1 + \nu_2 / 2$
- (2) The geometric mean: $G(\nu_1, \nu_2) = \sqrt{\nu_1 \nu_2}$

- (3) The harmonic mean: $H(\nu_1, \nu_2) = 2\nu_1 \nu_2 / \nu_1 + \nu_2$
- (4) The logarithmic mean: $L(\nu_1, \nu_2) = \nu_2 - \nu_1 / \ln \nu_2 - \ln \nu_1$
- (5) The p -logarithmic mean: $L_p(\nu_1, \nu_2) = (\nu_2^{p+1} - \nu_1^{p+1} / (p+1)(\nu_2 - \nu_1))^{1/p}$, $p \in \mathbb{R} - \{-1, 0\}$

Proposition 1. Let $0 < \nu_1 < \nu_2$; then, we have

$$|A(\nu_1, \nu_2) - L^{-1}(\nu_1, \nu_2)| \leq G^2(\nu_1, \nu_2) \frac{(\nu_2 - \nu_1)}{2} \times \left(\frac{\nu_1^{-2p}}{2(p+1)} \left[{}_2F_1 \left(2p, 1, p+2; \frac{\nu_1 - \nu_2}{2\nu_1} \right) + {}_2F_1 \left(2p, p+1, p+2; \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} \right) \right] \right)^{1/p}. \quad (54)$$

Proof. The proof is simply obtained by using $f(x) = x$ in Corollary 1. $\square$

Proposition 2. Let $0 < \nu_1 < \nu_2$; then, we have

$$A(\nu_1^{n+2}, \nu_2^{n+2}) - G^2(\nu_1, \nu_2) L_p^p(\nu_1, \nu_2) \leq G^2(\nu_1, \nu_2) (n+2) \left(\frac{\nu_2 - \nu_1}{2} \right) \times \left(\frac{\nu_1^{-2p}}{2(p+1)} \left[{}_2F_1 \left(2p, 1, p+2; \frac{\nu_1 - \nu_2}{2\nu_1} \right) + {}_2F_1 \left(2p, p+1, p+2; \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} \right) \right] \right)^{1/p} \cdot A^{1/q}(\nu_1^{(n+1)q}, \nu_2^{(n+1)q}). \quad (55)$$

Proof. The proof is straightforward, by using $f(x) = x^{n+2}$ in Corollary 1. $\square$

3.2. Bessel Functions. Consider the function $J_\nu: \mathbb{R} \rightarrow [1, \infty)$, defined by $J_\nu(z) = 2^p \Gamma(p+1) z^{-\nu} I_\nu(z)$, $z \in \mathbb{R}$, and we recall modified Bessel functions of first kind I_ν and its first derivative formula, respectively:

Proposition 3. For $\nu > -1$, $\nu_1, \nu_2 \in \mathbb{R}$ with $0 < \nu_1 < \nu_2$, then we have

$$J_\nu = \sum_{n \geq 0} \frac{(z/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)}, \quad (56)$$

$$J'_\nu = \frac{z}{2(\nu+1)} J_{\nu+1}(z).$$

$$\left| \frac{J_\nu(\nu_2) - J_\nu(\nu_1)}{\nu_2 - \nu_1} \right| \leq \frac{\nu_1^3 J_{\nu+1}(\nu_1) + \nu_2^3 J_{\nu+1}(\nu_2)}{4(\nu+1)}. \quad (57)$$

For a particular case, we have

$$\left| \frac{\cosh(\nu_2) - \cosh(\nu_1)}{\nu_2 - \nu_1} \right| \leq \frac{\nu_1^2 \sinh(\nu_1) + \nu_2^2 \sinh(\nu_2)}{2}. \quad (58)$$

Proof. The proof is obtained by taking $x \rightarrow x^2 J'_\nu(x)$ and using it in Corollary 1. For the second inequality, we use relation $J_{-(1/2)}(z) = \cosh(z)$ and $J_{1/2}(z) = \sin(z)/z$. $\square$

3.3. q -Digamma Functions. Let $0 < q < 1$. Then, q -digamma function ψ_q is defined as

$$\begin{aligned} \psi_q(x) &= -\ln(1-q) + \ln q \sum_{u=0}^{\infty} \frac{q^{u-q}}{1-q^{u+z}} \\ &= -\ln(1-q) + \ln q \sum_{u=1}^{\infty} \frac{q^{uz}}{1-q^{uz}}. \end{aligned} \quad (59)$$

For $q > 1$ and $z > 1$, the q -digamma function ψ_q is defined as

$$\begin{aligned} \psi_q(x) &= -\ln(1-q) + \ln q \left[z - \frac{1}{2} - \sum_{u=0}^{\infty} \frac{q^{-(u+q)}}{1-q^{-(u+z)}} \right] \\ &= -\ln(1-q) + \ln q \left[z - \frac{1}{2} - \sum_{u=1}^{\infty} \frac{q^{-uq}}{1-q^{-uq}} \right]. \end{aligned} \quad (60)$$

Proposition 4. For $\nu_1, \nu_2 \in \mathbb{R}$, such that $\nu_1 < \nu_2$,

$$H^2(\nu_1, \nu_2) \psi'_q \left(\frac{2\nu_1 \nu_2}{\nu_1 + \nu_2} \right) \leq \frac{\psi_q(\nu_2) - \psi_q(\nu_1)}{\nu_2 - \nu_1} \leq \frac{\nu_1^2 \psi'_q(\nu_1) + \nu_2^2 \psi'_q(\nu_2)}{2}. \quad (61)$$

Proof. The proof is straightforward, by considering $f(x) = x^2 \psi'_q(x)$ and using Remark 2. $\square$

Proposition 5. For $\nu_1, \nu_2 \in \mathbb{R}$, such that $\nu_1 < \nu_2$,

$$\begin{aligned} & \left| \frac{\nu_1^2 \psi'_q(\nu_1) + \nu_2^2 \psi'_q(\nu_2)}{2} - \frac{\psi_q(\nu_2) - \psi_q(\nu_1)}{\nu_2 - \nu_1} \right| \leq \frac{\nu_1 \nu_2 (\nu_2 - \nu_1)}{2} \\ & \times \left(\frac{\nu_1^{-2p}}{2(p+1)} \left[{}_2F_1 \left(2p, 1, p+2; \frac{\nu_1 - \nu_2}{2\nu_1} \right) + {}_2F_1 \left(2p, p+1, p+2; \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} \right) \right] \right)^{1/p} \\ & \left(\left(2\nu_1 + \psi'_q(\nu_1) + \nu_1^2 \psi''_q(\nu_1) \right)^q + \left(2\nu_2 + \psi'_q(\nu_2) + \nu_2^2 \psi''_q(\nu_2) \right)^q \right)^{1/q}. \end{aligned} \quad (62)$$

Proof. The proof is straightforward, by considering $f(x) = x^2 \psi'_q(x)$ and using Corollary 1. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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