Research Article

Graphic Representation of a Dimensional Expansion of Triangular Fuzzy Number

Yong Sik Yun

Department of Mathematics, Jeju National University, Jeju 63243, Republic of Korea

Correspondence should be addressed to Yong Sik Yun; yunys@jejunu.ac.kr

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We calculate Zadeh's max-min composition operators for two 3-dimensional triangular fuzzy numbers. We prove that if the 3-dimensional result is limited to 2 dimensions, it is the same as the 2-dimensional result, which is shown as a graph. Since a 3-dimensional graph cannot be drawn, the value of the membership function is expressed with color density. We cut a 3-dimensional triangular fuzzy number by a perpendicular plane passing a vertex, and consider the cut plane as a domain. The value of the membership function for each point on the cut plane is also expressed with color density. The graph expressing the value of the membership function, defined in the plane as a 3-dimensional graph using the z-axis value instead of expressing with color density, is consistent with the results in the 2-dimensional case.

1. Introduction

Many results exist for Zadeh's max-min composition operators. The results for triangular fuzzy numbers are well known [1–5]. We have generalized 1-dimensional triangular fuzzy number to 2-dimensional triangular fuzzy number and calculated Zadeh's max-min composition operators for 2-dimensional fuzzy numbers [6]. In the 1-dimensional case, Zadeh's max-min composition operator can be calculated using α -cuts. By defining parametric operations between two region valued α -cuts, we obtained parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$ [7]. In the case of 3-dimensional fuzzy numbers, the α -cuts are subsets of $\mathbb{R}^3$. By defining parametric operations between two ellipsoids including interior valued α -cuts, we calculated Zadeh's max-min composition operators for two 3-dimensional fuzzy numbers [8]. This will help facilitate further study of triangular fuzzy matrices [1, 3, 9].

In this paper, we prove that Zadeh's max-min composition operators for two 3-dimensional triangular fuzzy numbers on $\mathbb{R}^3$ constitutes the generalization of Zadeh's max-min compositions for two 2-dimensional triangular fuzzy numbers on $\mathbb{R}^2$. In addition, by limiting the graph of

the 3-dimensional result to the 2-dimensional case, we prove that the result expressed as a graph is consistent with the graph of the 2-dimensional result.

2. Preliminaries

We define α -cut and α -set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$.

Definition 1 (see [10]). An α -cut of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} | \mu_A(x) \geq \alpha\}$ if $\alpha \in (0, 1]$ and $A_0 = \text{cl}\{x \in \mathbb{R} | \mu_A(x) > \alpha\}$. For $\alpha \in (0, 1)$, the set $A^\alpha = \{x \in X | \mu_A(x) = \alpha\}$ is said to be the α -set of the fuzzy set A , A^0 is the boundary of $\{x \in \mathbb{R} | \mu_A(x) = \alpha\}$, and $A^1 = A_1$.

Following Zadeh, Dubois, and Prade, the extension principle is defined as follows.

Definition 2 (see [11–13]). The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$, and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z=x*y} \min\{\mu_A(x), \mu_B(y)\}, \quad * = +, -, \cdot, /.$$

(1)

$$\mu_A(x, y) = \begin{cases} 1 - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}}, & b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2, \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

where $a, b > 0$ is called the 2-dimensional triangular fuzzy number and denoted by $(a, x_1, b, y_1)^2$.

Let $A = (a, x_1, b, y_1)^2$. Then, the α -cut A_α of a 2-dimensional triangular fuzzy number A is an interior of an ellipse in an xy -plane, including the boundary

$$A_\alpha = \left\{ (x, y) \in \mathbb{R}^2 \mid \left(\frac{x-x_1}{a(1-\alpha)} \right)^2 + \left(\frac{y-y_1}{b(1-\alpha)} \right)^2 \leq 1 \right\}. \quad (3)$$

Theorem 1 (see [7]). Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$, and $A^\alpha = \{(x, y) \in \mathbb{R}^2 \mid \mu_A(x, y) = \alpha\}$ be the α -set of A . Then, for all $\alpha \in (0, 1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\pi\}. \quad (4)$$

If A is a continuous convex fuzzy number defined on $\mathbb{R}^2$, then the α -cut A_α is a closed convex subset in $\mathbb{R}^2$.

Definition 4 (see [7]). Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$\begin{aligned} A^\alpha &= \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\pi\}, \\ B^\alpha &= \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\pi\}, \end{aligned} \quad (5)$$

be the α -sets of A and B , respectively. For $\alpha \in (0, 1)$, we define that the parametric addition $A(+)_p B$, parametric subtraction $A(-)_p B$, parametric multiplication $A(\cdot)_p B$, and parametric division $A(/)_p B$ of two fuzzy numbers A and B are fuzzy numbers that have their α -sets as follows:

- (1) $A(+)_p B$: $(A(+)_p B)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\pi\}$.
- (2) $A(-)_p B$: $(A(-)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\pi\}$, where

Definition 3 (see [7]). A fuzzy set A with a membership function:

$$\begin{aligned} x_\alpha(t) &= \begin{cases} f_1^\alpha(t) - g_1^\alpha(t + \pi), & \text{if } 0 \leq t \leq \pi, \\ f_1^\alpha(t) - g_1^\alpha(t - \pi), & \text{if } \pi \leq t \leq 2\pi, \end{cases} \\ y_\alpha(t) &= \begin{cases} f_2^\alpha(t) - g_2^\alpha(t + \pi), & \text{if } 0 \leq t \leq \pi, \\ f_2^\alpha(t) - g_2^\alpha(t - \pi), & \text{if } \pi \leq t \leq 2\pi. \end{cases} \end{aligned} \quad (6)$$

- (3) $A(\cdot)_p B$: $(A(\cdot)_p B)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\pi\}$.
- (4) $A(/)_p B$: $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \leq t \leq 2\pi\}$, where

$$\begin{aligned} x_\alpha(t) &= \frac{f_1^\alpha(t)}{g_1^\alpha(t + \pi)}, & (0 \leq t \leq \pi), \\ x_\alpha(t) &= \frac{f_1^\alpha(t)}{g_1^\alpha(t - \pi)}, & (\pi \leq t \leq 2\pi), \\ y_\alpha(t) &= \frac{f_2^\alpha(t)}{g_2^\alpha(t + \pi)}, & (0 \leq t \leq \pi), \\ y_\alpha(t) &= \frac{f_2^\alpha(t)}{g_2^\alpha(t - \pi)}, & (\pi \leq t \leq 2\pi). \end{aligned} \quad (7)$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_p B)^0 = \lim_{\alpha \rightarrow 0^+} (A(*)_p B)^\alpha$ and $(A(*)_p B)^1 = \lim_{\alpha \rightarrow 1^-} (A(*)_p B)^\alpha$, where $*$ = +, -, $\cdot$, /.

Theorem 2 (see [7]). Let $A = (a_1, x_1, b_1, y_1)^2$ and $B = (a_2, x_2, b_2, y_2)^2$ be two 2-dimensional triangular fuzzy numbers. Then, we have the following:

- (1) $A(+)_p B = (a_1 + a_2, x_1 + x_2, b_1 + b_2, y_1 + y_2)^2$.
- (2) $A(-)_p B = (a_1 + a_2, x_1 - x_2, b_1 + b_2, y_1 - y_2)^2$.
- (3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \leq t \leq 2\pi\}$, where
$$\begin{aligned} x_\alpha(t) &= x_1 x_2 + (x_1 a_2 + x_2 a_1)(1 - \alpha) \cos t + a_1 a_2 (1 - \alpha)^2 \cos^2 t, \\ y_\alpha(t) &= y_1 y_2 + (y_1 b_2 + y_2 b_1)(1 - \alpha) \sin t + b_1 b_2 (1 - \alpha)^2 \sin^2 t. \end{aligned} \quad (8)$$
- (4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \leq t \leq 2\pi\}$, where

$$\begin{aligned} x_\alpha(t) &= \frac{x_1 + a_1(1-\alpha)\cos t}{x_2 - a_2(1-\alpha)\cos t}, \\ y_\alpha(t) &= \frac{y_1 + b_1(1-\alpha)\sin t}{y_2 - b_2(1-\alpha)\sin t}. \end{aligned} \quad (9)$$

Therefore, $A(+)_p B$ and $A(-)_p B$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_p B$ and $A(/)_p B$ are not 2-dimensional triangular fuzzy numbers.

Theorem 3 (see [7]). *Parametric operations on $\mathbb{R}^2$ in Definition 4 are the generalization of Zadeh's extension principles on $\mathbb{R}$.*

$$\mu_A(x, y, z) = \begin{cases} 1 - \sqrt{\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2}}, & \text{if } b^2c^2(x-x_1)^2 + c^2a^2(y-y_1)^2 + a^2b^2(z-z_1)^2 \leq a^2b^2c^2, \\ 0, & \text{otherwise,} \end{cases} \quad (10)$$

where $a, b, c > 0$ is called the 3-dimensional triangular fuzzy number and denoted by $(a, x_1, b, y_1, c, z_1)^3$.

3. Zadeh's Max-Min Composition Operator for 3-Dimensional Triangular Fuzzy Numbers

We define parametric operations between two 3-dimensional triangular fuzzy numbers and calculate Zadeh's max-min composition operators for 3-dimensional triangular fuzzy numbers.

Definition 5. A fuzzy set A with a membership function:

Note that $\mu_A(x, y)$ is a cone in $\mathbb{R}^2$, but we cannot know the shape of $\mu_A(x, y, z)$ in $\mathbb{R}^3$. The α -cut A_α of a 3-dimensional triangular fuzzy number $A = (a, x_1, b, y_1, c, z_1)^3$ is the following set:

$$A_\alpha = \left\{ (x, y, z) \in \mathbb{R}^3 \left| \left(\frac{x-x_1}{a(1-\alpha)} \right)^2 + \left(\frac{y-y_1}{b(1-\alpha)} \right)^2 + \left(\frac{z-z_1}{c(1-\alpha)} \right)^2 \leq 1 \right. \right\}. \quad (11)$$

Definition 6. A 3-dimensional fuzzy number A defined on $\mathbb{R}^3$ is called a convex fuzzy number if, for all $\alpha \in (0, 1)$, the α -cuts

$$A_\alpha = \{ (x, y, z) \in \mathbb{R}^3 \mid \mu_A(x, y, z) \geq \alpha \}, \quad (12)$$

are convex subsets in $\mathbb{R}^3$.

Theorem 4 (see [8]). *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$, and $A^\alpha = \{ (x, y, z) \in \mathbb{R}^3 \mid \mu_A(x, y, z) = \alpha \}$ be the α -set of A . Then, for all $\alpha \in (0, 1)$, there exist continuous functions $f_1^\alpha(s), f_2^\alpha(s, t)$, and $f_3^\alpha(s, t)$ ($0 \leq s \leq 2\pi, -(\pi/2) \leq t \leq (\pi/2)$), such that*

$$A^\alpha = \left\{ (f_1^\alpha(s), f_2^\alpha(s, t), f_3^\alpha(s, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}. \quad (13)$$

Definition 7. Let A and B be two continuous convex fuzzy numbers defined on $\mathbb{R}^3$ and

$$\begin{aligned} A^\alpha &= \left\{ (f_1^\alpha(s), f_2^\alpha(s, t), f_3^\alpha(s, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}, \\ B^\alpha &= \left\{ (g_1^\alpha(s), g_2^\alpha(s, t), g_3^\alpha(s, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}, \end{aligned} \quad (14)$$

be the α -sets of A and B , respectively. For $\alpha \in (0, 1)$, we define that the parametric addition, parametric subtraction, parametric multiplication, and parametric division of two fuzzy numbers A and B are fuzzy numbers that have their α -sets as follows:

(1) Parametric addition, $A(+)_p B$:

$$(A(+)_p B)^\alpha = \left\{ (f_1^\alpha(s) + g_1^\alpha(s), f_2^\alpha(s, t) + g_2^\alpha(s, t), f_3^\alpha(s, t) + g_3^\alpha(s, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}. \quad (15)$$

(2) Parametric subtraction, $A(-)_p B$:

$$\begin{aligned} (A(-)_p B)^\alpha &= \left\{ (f_1^\alpha(s) - g_1^\alpha(s + \pi), f_2^\alpha(s, t) - g_2^\alpha(s + \pi, t), f_3^\alpha(s, t) - g_3^\alpha(s + \pi, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq \pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}, \\ (A(-)_p B)^\alpha &= \left\{ (f_1^\alpha(s) - g_1^\alpha(s - \pi), f_2^\alpha(s, t) - g_2^\alpha(s - \pi, t), f_3^\alpha(s, t) - g_3^\alpha(s - \pi, t)) \in \mathbb{R}^3 \mid \pi \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}. \end{aligned} \quad (16)$$

(3) Parametric multiplication, $A(\cdot)_p B$:

$$(A(\cdot)_p B)^\alpha = \left\{ (f_1^\alpha(s) \cdot g_1^\alpha(s), f_2^\alpha(s, t) \cdot g_2^\alpha(s, t), f_3^\alpha(s, t) \cdot g_3^\alpha(s, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}. \quad (17)$$

(4) Parametric division, $A(/)_p B$:

$$\begin{aligned} (A(/)_p B)^\alpha &= \left\{ \left(\frac{f_1^\alpha(s)}{g_1^\alpha(s + \pi)}, \frac{f_2^\alpha(s, t)}{g_2^\alpha(s + \pi, t)}, \frac{f_3^\alpha(s, t)}{g_3^\alpha(s + \pi, t)} \right) \in \mathbb{R}^3 \mid 0 \leq s \leq \pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}, \\ (A(/)_p B)^\alpha &= \left\{ \left(\frac{f_1^\alpha(s)}{g_1^\alpha(s - \pi)}, \frac{f_2^\alpha(s, t)}{g_2^\alpha(s - \pi, t)}, \frac{f_3^\alpha(s, t)}{g_3^\alpha(s - \pi, t)} \right) \in \mathbb{R}^3 \mid \pi \leq s \leq 2\pi, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \right\}. \end{aligned} \quad (18)$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_p B)^0 = \lim_{\alpha \rightarrow 0^+} (A(*)_p B)^\alpha$, and $(A(*)_p B)^1 = \lim_{\alpha \rightarrow 1^-} (A(*)_p B)^\alpha$, where $*$ = +, -, $\cdot$, /.

Theorem 5 (see [8]). Let $A = (a_1, x_1, b_1, y_1, c_1, z_1)^3$ and $B = (a_2, x_2, b_2, y_2, c_2, z_2)^3$ be two 3-dimensional triangular fuzzy numbers. Then, we have the following:

- (1) $A(+)_p B = (a_1 + a_2, x_1 + x_2, b_1 + b_2, y_1 + y_2, c_1 + c_2, z_1 + z_2)^3$.
- (2) $A(-)_p B = (a_1 + a_2, x_1 - x_2, b_1 + b_2, y_1 - y_2, c_1 + c_2, z_1 - z_2)^3$.
- (3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s, t), z_\alpha(s, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -(\pi/2) \leq t \leq (\pi/2)\}$, where

$$\begin{aligned} x_\alpha(s) &= x_1 x_2 + (x_1 a_2 + x_2 a_1)(1 - \alpha) \cos s + a_1 a_2 (1 - \alpha)^2 \cos^2 s, \\ y_\alpha(s, t) &= y_1 y_2 + (y_1 b_2 + y_2 b_1)(1 - \alpha) \sin s \cos t + b_1 b_2 (1 - \alpha)^2 \sin^2 s \cos^2 t, \\ z_\alpha(s, t) &= z_1 z_2 + (z_1 c_2 + z_2 c_1)(1 - \alpha) \sin s \sin t + c_1 c_2 (1 - \alpha)^2 \sin^2 s \sin^2 t. \end{aligned} \quad (19)$$

- (4) $(A(/)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s, t), z_\alpha(s, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -(\pi/2) \leq t \leq (\pi/2)\}$, where

$$\begin{aligned} x_\alpha(s) &= \frac{x_1 + a_1(1 - \alpha) \cos s}{x_2 - a_2(1 - \alpha) \cos s}, \\ y_\alpha(s, t) &= \frac{y_1 + b_1(1 - \alpha) \sin s \cos t}{y_2 - b_2(1 - \alpha) \sin s \cos t}, \\ z_\alpha(s, t) &= \frac{z_1 + c_1(1 - \alpha) \sin s \sin t}{z_2 - c_2(1 - \alpha) \sin s \sin t}. \end{aligned} \quad (20)$$

Therefore, $A(+)_p B$ and $A(-)_p B$ become 3-dimensional triangular fuzzy numbers, but $A(\cdot)_p B$ and $A(/)_p B$ are not 3-dimensional triangular fuzzy numbers.

Theorem 6. Parametric operations on $\mathbb{R}^3$ in Definition 7 are the generalization of parametric operations on $\mathbb{R}^2$ in Definition 4, which are the generalization of Zadeh's max-min composition operations on $\mathbb{R}$.

Proof. Consider two 3-dimensional triangular fuzzy numbers $A = (a_1, x_1, b_1, y_1, c_1, 0)^3$ and $B = (a_2, x_2, b_2, y_2, c_2, 0)^3$. By Theorem 5,

- (1) $A(+)_p B = (a_1 + a_2, x_1 + x_2, b_1 + b_2, y_1 + y_2, c_1 + c_2, 0)^3$.
- (2) $A(-)_p B = (a_1 + a_2, x_1 - x_2, b_1 + b_2, y_1 - y_2, c_1 + c_2, 0)^3$.
- (3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s, t), z_\alpha(s, t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, -(\pi/2) \leq t \leq (\pi/2)\}$, where

$$\begin{aligned}
x_\alpha(s) &= x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos s + a_1a_2(1-\alpha)^2\cos^2s, \\
y_\alpha(s, t) &= y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin s \cos t + b_1b_2(1-\alpha)^2\sin^2s \cos^2t, \\
z_\alpha(s, t) &= c_1c_2(1-\alpha)^2\sin^2s \sin^2t.
\end{aligned} \tag{21}$$

(4) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s, t), z_\alpha(s, t)) \in \mathbb{R}^3 | 0 \leq s \leq 2\pi, -(\pi/2) \leq t \leq (\pi/2)\}$, where

The intersections of these 3-dimensional triangular fuzzy numbers and $z = 0$ are as follows:

(1) $A(+)_p B$: note that

$$\begin{aligned}
x_\alpha(s) &= \frac{x_1 + a_1(1-\alpha)\cos s}{x_2 - a_2(1-\alpha)\cos s}, \\
y_\alpha(s, t) &= \frac{y_1 + b_1(1-\alpha)\sin s \cos t}{y_2 - b_2(1-\alpha)\sin s \cos t}, \\
z_\alpha(s, t) &= \frac{c_1(1-\alpha)\sin s \sin t}{-c_2(1-\alpha)\sin s \sin t} = -\frac{c_1}{c_2}.
\end{aligned} \tag{22}$$

$$\mu_{A(+)_p B}(x, y, z) = 1 - \sqrt{\left(\frac{x - x_1 - x_2}{a_1 + a_2}\right)^2 + \left(\frac{y - y_1 - y_2}{b_1 + b_2}\right)^2 + \left(\frac{z}{c_1 + c_2}\right)^2}. \tag{23}$$

If $z = 0$,

$$\mu_{A(+)_p B}(x, y, 0) = 1 - \sqrt{\left(\frac{x - x_1 - x_2}{a_1 + a_2}\right)^2 + \left(\frac{y - y_1 - y_2}{b_1 + b_2}\right)^2}. \tag{24}$$

Thus, the intersection is the 2-dimensional triangular fuzzy number $C = (a_1 + a_2, x_1 + x_2, b_1 + b_2, y_1 + y_2)^2$.

(2) $A(-)_p B$: note that

$$\mu_{A(-)_p B}(x, y, z) = 1 - \sqrt{\left(\frac{x - x_1 + x_2}{a_1 + a_2}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1 + b_2}\right)^2 + \left(\frac{z}{c_1 + c_2}\right)^2}. \tag{25}$$

If $z = 0$,

$$\mu_{A(-)_p B}(x, y, 0) = 1 - \sqrt{\left(\frac{x - x_1 + x_2}{a_1 + a_2}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1 + b_2}\right)^2}. \tag{26}$$

Thus, the intersection is the 2-dimensional triangular fuzzy number $D = (a_1 + a_2, x_1 - x_2, b_1 + b_2, y_1 - y_2)^2$.

(3) $A(\cdot)_p B$: if $z = 0$, $c_1c_2(1-\alpha)^2\sin^2s \sin^2t = 0$. Thus, the intersection is a fuzzy number E on $\mathbb{R}^2$ with α -cut, not α -set, $(A(\cdot)_p B)_\alpha = \{(x_\alpha(s), y_\alpha(s, 0)) \in \mathbb{R}^2 | 0 \leq s \leq 2\pi\}$, where

$$\begin{aligned}
x_\alpha(s) &= x_1x_2 + (x_1a_2 + x_2a_1)(1-\alpha)\cos s + a_1a_2(1-\alpha)^2\cos^2s, \\
y_\alpha(s, 0) &= y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin s \cos 0 + b_1b_2(1-\alpha)^2\sin^2s \cos^2 0 \\
&= y_1y_2 + (y_1b_2 + y_2b_1)(1-\alpha)\sin s + b_1b_2(1-\alpha)^2\sin^2s.
\end{aligned} \tag{27}$$

FIGURE 1: A^3 . FIGURE 4: $(A - B)^3$. FIGURE 2: B^3 . FIGURE 5: $(A \times B)(1/2)$. FIGURE 3: $(A + B)^3$.

- (4) $A(/)_p B$: if $z = 0$, $-(c_1/c_2) = 0$. Thus, the intersection is a fuzzy number F on $\mathbb{R}^2$ with α -cut, not α -set, $(A(/)_p B)_\alpha = \{(x_\alpha(s), y_\alpha(s, 0)) \in \mathbb{R}^2 | 0 \leq s \leq 2\pi\}$, where

$$\begin{aligned} x_\alpha(s) &= \frac{x_1 + a_1(1 - \alpha)\cos s}{x_2 - a_2(1 - \alpha)\cos s}, \\ y_\alpha(s, 0) &= \frac{y_1 + b_1(1 - \alpha)\sin s}{y_2 - b_2(1 - \alpha)\sin s}. \end{aligned} \quad (28)$$

FIGURE 6: $(A/B)(1/2)$.

FIGURE 8: $(A/B)^3$.

FIGURE 7: $(A \times B)^3$.

FIGURE 9: $A/2$.

FIGURE 10: $B/2$.

On the contrary, the intersection of 3-dimensional triangular fuzzy number $A = (a_1, x_1, b_1, y_1, c_1, 0)^3$ and $z = 0$ is $G = (a_1, x_1, b_1, y_1)^2$, and the intersection of 3-dimensional triangular fuzzy number $B = (a_2, x_2, b_2, y_2, c_2, 0)^3$ and $z = 0$ is $H = (a_2, x_2, b_2, y_2)^2$. For two 2-dimensional triangular fuzzy numbers G and H , the following results for parametric operations on $\mathbb{R}^2$ have been proven [7]:

FIGURE 11: $(A+B)/2$. FIGURE 14: B^2 . FIGURE 12: $(A-B)/2$. FIGURE 15: $(A+B)^2$. FIGURE 13: A^2 . FIGURE 16: $(A-B)^2$.

$$\begin{aligned}
 G(+)_p H &= C, \\
 G(-)_p H &= D, \\
 G(\cdot)_p H &= E, \\
 G(/)_p H &= F.
 \end{aligned}
 \tag{29}$$

The proof is complete. $\square$

FIGURE 17: $(A \times B)^2$. FIGURE 18: $(A/B)^2$.

FIGURE 19: Summary.

4. Conclusion

In this section, by limiting the graph of the 3-dimensional result to the 2-dimensional case, we prove that the result expressed as a graph is consistent with the graph of the 2-dimensional result.

For two 3-dimensional triangular fuzzy numbers $A = (6, 3, 8, 5, 4, 7)^3$ and $B = (4, 6, 6, 8, 3, 5)^3$, the values of membership function are expressed using color density in Figures 1 and 2, respectively. Zadeh's max-min operations expanded to 3 dimensions are expressed in Figures 3–6 as graphs.

The graph of $A(\cdot)B$ and $A(/)B$ for $\alpha = 1/2$ is Figures 5 and 6, respectively. In Figures 7 and 8, the graph for $A(\cdot)B$ and $A(/)B$ is the union of α ($0 \leq \alpha \leq 1$), respectively.

We cut the graph of $A^3, B^3, (A+B)^3$, and $(A-B)^3$ by a perpendicular plane passing a vertex (Figures 9–12). The values of the membership function of each point on the cross section are expressed with color density.

We cut two 3-dimensional triangular fuzzy numbers A^3 and B^3 by a perpendicular plane passing a vertex and considered the cut plane as a domain. The value of the membership function for each point on the cut plane is expressed with color density. Figures 13 and 14 are the graphs expressing the value of the membership function defined in the plane as a 3-dimensional graph using the z -axis value, instead of expressing with color density. In this way, restricting the three-dimensional results (Figures 3–6) to two dimensions, we express the value of the membership function in a three-dimensional graph without using color density (Figures 15–18).

Consider the 2-dimensional triangular fuzzy numbers A^2 and B^2 represented by Figures 13 and 14, respectively. The result of Zadeh's max-min operators expanded to 2 dimensions can be found in [7]. When the results are graphed, it becomes clear that the graphs are identical to graphs in Figures 15–18. The commands in Mathematica for ten figures on page 7 and 8 are provided in appendix. Figure 19 summarizes what has been mentioned. In conclusion, therefore, the 3-dimensional result is naturally expanded to the 3 dimensions while satisfying the 2-dimensional result as it is, and it can be applied in this form. This paper will further study the extended applications of triangular fuzzy numbers [14–18].

Appendix

(i) (A^3 :)

```
DensityPlot3D[1 - Sqrt[(x - 3)^2/6 + (y - 5)^2/8 + (z - 7)^2/4], {x, y, z} in Ellipsoid[{3, 5, 7}, {Sqrt[6], Sqrt[8], 2}], PlotPoints->100, ColorFunction->"SunsetColors", OpacityFunction->0.05, BoxRatios->{Sqrt[6], Sqrt[8], 2}, PlotLegends->Automatic]
```

(ii) ($(A+B)^3$:)

```
DensityPlot3D[1 - Sqrt[(x - 9)^2/10 + (y - 13)^2/14 + (z - 12)^2/7], {x, y, z} in Ellipsoid[{9, 13, 12}, {Sqrt[10], Sqrt[14], Sqrt[7]}], PlotPoints->100, ColorFunction->"SunsetColors", OpacityFunction->0.05, BoxRatios->{Sqrt[10], Sqrt[14], Sqrt[7]}, PlotLegends->Automatic]
```

(iii) ($(A * B)^{1/2}$:)

```
ParametricPlot3D[{18 + 24 Cos[s] + 6 (Cos[s])^2, 40 + 47 Sin[s] Cos[t] + 12 (Sin[s])^2 (Cos[t])^2, 35 + 41/2 Sin[s] Sin[t] + 3 (Sin[s])^2 (Sin[t])^2}, {s, 0, 2 Pi}, {t, -Pi/2, Pi/2}, Axes->True]
```

(iv) ($(A/B)^3$:)

```
g[a_] := ParametricPlot3D[{(3 + 6(1 - a) Cos[s])/(6 - 4(1 - a) Cos[s]), (5 + 8(1 - a) Sin[s] Cos[t])/(8 - 6(1 - a) Sin[s] Cos[t]), (7 + 4(1 - a) Sin[s] Sin[t])/(5 - 3(1 - a) Sin[s] Sin[t])}, {s, 0, 2 Pi}, {t, -Pi/2, Pi/2}, PlotStyle->Directive[RGBColor[0.2, 0.5 + a/2, 0.5 + a/2], Opacity[0.3]], BoxRatios->{1, 1, 1}];
```

```
tg = Table[g[i], {i, 0, 1.0, 0.01}];
```

```
Show[tg]
```

(v) ($A/2$:)

```
reg1 = ImplicitRegion[0 <= (x - 3)^2/6 + (y - 5)^2/8 + (z - 7)^2/4 <= 1 && z <= 7, {x, y, z}];
```

```
DensityPlot3D[1 - Sqrt[(x - 3)^2/6 + ((y - 5)^2/8 + ((z - 7)^2/4)], {x, y, z} in reg1, PlotPoints->100, ColorFunction->"SunsetColors", OpacityFunction->1, BoxRatios->{Sqrt[4], Sqrt[5], Sqrt[6]/2}, PlotLegends->Automatic]
```

(vi) ($(A+B)/2$:)

```
reg1 = ImplicitRegion[0 <= (x - 9)^2/10 + (y - 13)^2/14 + (z - 12)^2/7 <= 1 && z <= 12, {x, y, z}];
```

```
DensityPlot3D[1 - Sqrt[(x - 9)^2/10 + (y - 13)^2/14 + (z - 12)^2/7], {x, y, z} in reg1, PlotPoints->100, ColorFunction->"SunsetColors", OpacityFunction->1, BoxRatios->{Sqrt[10], Sqrt[14], Sqrt[7]}, PlotLegends->Automatic]
```

(vii) (A^2 :)

```
Plot3D[1 - Sqrt[(x - 3)^2/6 + (y - 5)^2/8], {x, y} in Ellipsoid[{3, 5}, {Sqrt[6], Sqrt[8]}], PlotPoints->50, ColorFunction->"SunsetColors", BoxRatios->{Sqrt[6], Sqrt[8], 1}, PlotLegends->Automatic]
```

(viii) ($(A+B)^2$:)

```
Plot3D[1 - Sqrt[(x - 9)^2/10 + (y - 13)^2/14], {x, y} in Ellipsoid[{9, 13}, {Sqrt[10], Sqrt[14]}], PlotPoints->50, ColorFunction->"SunsetColors", BoxRatios->{Sqrt[6], Sqrt[8], 1}, PlotLegends->Automatic]
```

(ix) ($(A * B)^2$:)

```
g[a_] := ParametricPlot3D[{18 + 48(1 - a) Cos[s] + 24(1 - a)^2 (Cos[s])^2, 40 + 94(1 - a) Sin[s] + 48(1 - a)^2 (Sin[s])^2, a}, {s, 0, 2 Pi}, PlotStyle->Directive[RGBColor[0.3, 0.5 + a, 0.5 + a], Opacity[0.3]], BoxRatios->{1, 1, 1}];
```

```
tg = Table[g[i], {i, 0, 1.0, 0.001}];
```

```
Show[tg]
```

(x) ($(A/B)^2$:)

```
g[a_] := ParametricPlot3D[{(3 + 3 Cos[s])/(6 - 2 Cos[s]), (5 + 4 Sin[s])/(8 - 3/2 Sin[s]), a}, {s, 0, 2 Pi}, PlotStyle->Directive[RGBColor[0.3, 0.5 + a, 0.5 + a], Opacity[0.3]], BoxRatios->{1, 1, 1}];
```

```
tg = Table[g[i], {i, 0, 1.0, 0.001}];
```

```
Show[tg]
```

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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