Research Article

New Improved Inequalities Associated with Ratios of Gamma Functions

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In this paper, we improve the estimate for the products $\prod_{k=1}^n (4k-3)/4k$ and $\prod_{k=1}^n (4k-1)/4k$, obtained by Cringanu "inequalities associated with ratios of gamma functions."

1. Introduction

Cringanu [1] proved the following inequalities for the Wallis ratio:

$$\frac{a}{\sqrt[4]{n^3}} \leq \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{b}{\sqrt[4]{n^3}}, \quad (1)$$

$$\frac{c}{\sqrt[4]{n}} \leq \frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)} < \frac{d}{\sqrt[4]{n}}, \quad (2)$$

for all $n \geq 1$, with the best constants $a = (1/4) = 0.25$, $b = 1/\Gamma(1/4) = 0.2758 \dots$, $c = (3/4) = 0.75$, and $d = 1/\Gamma(3/4) = 0.8160, \dots$

These inequalities are a consequence of the complete monotonicity on $(0, \infty)$ of the functions

$$x \longrightarrow \ln \frac{(1/\Gamma(1/4))^4}{x^3 (\Gamma(x + (1/4))/\Gamma(x+1)\Gamma(1/4))^4}, \quad (3)$$

$$x \longrightarrow \ln \frac{(1/\Gamma(3/4))^4}{x (\Gamma(x + (3/4))/\Gamma(x+1)\Gamma(3/4))^4},$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ for $x > 0$ is the gamma function (see, for example, [2–5]).

Since

$$\lim_{n \rightarrow \infty} \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} \sqrt[4]{n^3} = \frac{1}{\Gamma(1/4)}, \quad (4)$$

$$\lim_{n \rightarrow \infty} \frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)} \sqrt[4]{n} = \frac{1}{\Gamma(3/4)},$$

(see [1]), the following approximations as $n \rightarrow \infty$ are true:

$$\frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} \approx \frac{1}{\Gamma(1/4) \sqrt[4]{n^3}}, \quad (5)$$

$$\frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)} \approx \frac{1}{\Gamma(3/4) \sqrt[4]{n}}.$$

First, we concentrate on the approximation of $(1 \cdot 5, \dots, (4n-3)/4 \cdot 8, \dots, (4n))$ and introduce the relative error sequence w_n by the following relation for every integer $n \geq 1$:

$$\frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} = \frac{1}{\Gamma(1/4) \sqrt[4]{n^3}} \exp w_n. \quad (6)$$

A tool for measuring speed of convergence is a result by Mortici (lemma, [6]), who states that a sequence x_n converging to zero is the fastest possible when the difference $x_n - x_{n+1}$ is the fastest possible. More precisely, if there exists $\lim_{n \rightarrow \infty} n^k (x_n - x_{n+1}) = l$, then $\lim_{n \rightarrow \infty} n^{k-1} x_n = (l/k - 1)$.

Recent results using this lemma have been given in [7–12]. In our case of w_n , we have

$$w_n - w_{n+1} = \ln \frac{4n+4}{4n+1} - \frac{3}{4} \ln \frac{n+1}{n}, \quad (7)$$

and using the Maclaurin series of the function $w_n - w_{n+1}$ in the variable $(1/n)$, we obtain

$$w_n - w_{n+1} = -\frac{3}{32n^2} + O\left(\frac{1}{n^3}\right). \quad (8)$$

Then, $\lim_{n \rightarrow \infty} n^2 (w_n - w_{n+1}) = -(3/32)$ and so $\lim_{n \rightarrow \infty} n w_n = -(3/32)$.

Consequently, the following approximation is true: $w_n \approx -(3/32n)$ as $n \rightarrow \infty$.

Similarly, if $(3 \cdot 7, \dots, (4n-1)/(4 \cdot 8, \dots, (4n))) = (1/\Gamma(3/4)\sqrt[4]{n}) \exp v_n$, then $v_n - v_{n+1} = -(3/32n^2) + O(1/n^3)$ and so the following approximation is true: $v_n \approx -(3/32n)$ as $n \rightarrow \infty$.

2. The Main Results

Theorem 1. (i) For all integer $n \geq 1$, we have

$$\frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{a}{n^2}\right\} < \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)}, \quad (9)$$

with the best constant $a = (1/128)$.

(ii) For every $a < (1/128)$, there exists $n_a \in N, n \geq 1$, such that

$$\frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{a}{n^2}\right\}, \quad (10)$$

for all $n \geq n_a$.

Proof. We define the sequence (a_n) for $n \geq 1$,

$$a_n = \ln \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} + \ln \Gamma\left(\frac{1}{4}\right) + \frac{3}{4} \ln n + \frac{3}{32n} + \frac{a}{n^2}, \quad (11)$$

and so $a_{n+1} - a_n = f(n)$, where

$$f(n) = \ln \frac{4n+1}{4n+4} + \frac{3}{4} \ln(n+1) - \frac{3}{4} \ln n + \frac{3}{32(n+1)} - \frac{3}{32n} + \frac{a}{(n+1)^2} - \frac{a}{n^2}. \quad (12)$$

The derivative of the function f is equal to

$$f'(n) = \frac{P(n)}{32n^3(n+1)^3(4n+1)}, \quad (13)$$

where $P(n) = 6n^3(128a-1) + 3n^2(320a-1) + n(448a+3) + 64a$.

(i) If $a = (1/128)$, then $P(n) = (1/2)(9n^2 + 13n + 1) > 0$, for all $n \geq 1$, and then f is strictly increasing.

Since $f(\infty) = 0$, it follows that $f(n) < 0$, for all $n \geq 1$, so that (a_n) is strictly decreasing. The sequence (a_n) converges to zero, and then it follows that $a_n > 0$, for all $n \geq 1$, so that

$$\frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{128n^2}\right\} < \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)}, \quad (14)$$

for all $n \geq 1$.

(ii) If $a < (1/128)$, then there exists $n_a \in N, n \geq 1$, such that $P(n) < 0$ for all $n \geq n_a$, and then f is strictly decreasing on $[n_a, \infty)$. Since $f(\infty) = 0$, it follows that $f(n) > 0$, for all $n \geq n_a$, so that $(a_n)_{n \geq n_a}$ is strictly increasing. The sequence (a_n) converges to zero, and then it follows that $a_n < 0$, for all $n \geq n_a$, so that

$$\frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{a}{n^2}\right\}, \quad (15)$$

for all $n \geq n_a$.

Now, we find the constant n_a in some particular cases.

For example, if $a = (1/256) < (1/128)$, then $P(n) = -3n^3 + (3/4)n^2 + (19/4)n + (1/4) < 0$ for all $n \geq 2$ and so

$$\frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{128n^2}\right\} < \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{256n^2}\right\}, \quad (16)$$

for all $n \geq 2$.

Let us remark that a direct calculus shows that these inequalities hold for $n = 1$, and then

$$\frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{128n^2}\right\} < \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{256n^2}\right\}, \quad (17)$$

for all $n \geq 1$.

Now, if $a = (1/160) < (1/128)$, then $P(n) = -(6/5)n^3 + 3n^2 + (29/5)n + (2/5) < 0$ for all $n \geq 4$ and so

$$\frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{128n^2}\right\} < \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{160n^2}\right\}, \quad (18)$$

for all $n \geq 4$.

Let us remark that a direct calculus shows that these inequalities hold for $n = 3$, and then

$$\frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{128n^2}\right\} < \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{1}{160n^2}\right\}, \quad (19)$$

for all $n \geq 3$. $\square$

Remark 1. Let us remark that $(1/4\sqrt[4]{n^3}) < (1/\Gamma(1/4)\sqrt[4]{n^3}) \exp\{-(3/32n) - (1/128n^2)\}$, for all $n \geq 2$, and for $a \in (0, (1/128))$,

$$\frac{1}{\Gamma(1/4)\sqrt[4]{n^3}} \exp\left\{-\frac{3}{32n} - \frac{a}{n^2}\right\} < \frac{1}{\Gamma(1/4)\sqrt[4]{n^3}}, \quad \text{for all } n \geq 1, \quad (20)$$

which is shown to improve inequality (1).

Theorem 2. (i) For all integer $n \geq 1$, we have

$$\frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{a}{n^2}\right\} < \frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)}, \quad (21)$$

with the best constant $a = (1/128)$.

(ii) For every $a > (1/128)$, there exists $n_a \in \mathbb{N}$, $n_a \geq 1$, such that

$$\frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{a}{n^2}\right\}, \quad (22)$$

for all $n \geq n_a$.

Proof. We define the sequence (a_n) for $n \geq 1$,

$$a_n = \ln \frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)} + \ln \Gamma\left(\frac{3}{4}\right) + \frac{1}{4} \ln n + \frac{3}{32n} - \frac{a}{n^2}, \quad (23)$$

and so $a_{n+1} - a_n = f(n)$, where

$$f(n) = \ln \frac{4n+3}{4n+4} + \frac{1}{4} \ln(n+1) - \frac{1}{4} \ln n + \frac{3}{32(n+1)} - \frac{3}{32n} - \frac{a}{(n+1)^2} + \frac{a}{n^2}. \quad (24)$$

The derivative of the function f is equal to

$$f'(n) = \frac{Q(n)}{32n^3(n+1)^3(4n+3)}, \quad (25)$$

where $Q(n) = 6n^3(1 - 128a) + 3n^2(5 - 448a) + n(9 - 832a) - 192a$.

(i) If $a = (1/128)$, then $Q(n) = (1/2)(9n^2 + 5n - 3) > 0$, for all $n \geq 1$, and then f is strictly increasing.

Since $f(\infty) = 0$, it follows that $f(n) < 0$, for all $n \geq 1$, so that (a_n) is strictly decreasing. The sequence (a_n) converges to zero, and then it follows that $a_n > 0$, for all $n \geq 1$, so that

$$\frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{1}{128n^2}\right\} < \frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)}, \quad (26)$$

for all $n \geq 1$.

(ii) If $a > (1/128)$, then there exists $n_a \in \mathbb{N}$, $n \geq 1$, such that $Q(n) < 0$ for all $n \geq n_a$, and then f is strictly decreasing on $[n_a, \infty)$. Since $f(\infty) = 0$, it follows that $f(n) > 0$, for all $n \geq n_a$, so that $(a_n)_{n \geq n_a}$ is strictly increasing. The sequence (a_n) converges to zero, and then it follows that $a_n < 0$, for all $n \geq n_a$, so that

$$\frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{a}{n^2}\right\}, \quad (27)$$

for all $n \geq n_a$.

Now, we find the constant n_a in some particular cases. for all $n \geq 1$, and so
 For example, if $a = (1/64) > (1/128)$, then

$$Q(n) = -6n^3 - 6n^2 - 4n - 3 < 0, \quad (28)$$

$$\frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{1}{128n^2}\right\} < \frac{3 \cdot 7, \dots, (4n-1)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{1}{64n^2}\right\}, \quad (29)$$

for all $n \geq 1$.

Now, if $a = (1/120) > (1/128)$, then $Q(n) = -(1/15)(6n^3 - 57n^2 - 31n + 24) < 0$ for all $n \geq 10$ and so

$$\frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{1}{128n^2}\right\} < \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{1}{120n^2}\right\}, \quad (30)$$

for all $n \geq 10$.

Let us remark that a direct calculus shows that these inequalities hold for $n \in \{1, 2, \dots, 9\}$, and then

$$\frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{1}{128n^2}\right\} < \frac{1 \cdot 5, \dots, (4n-3)}{4 \cdot 8, \dots, (4n)} < \frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{1}{120n^2}\right\}, \quad (31)$$

for all $n \geq 1$. $\square$

Remark 2. Let us remark that $(3/4)\sqrt[4]{n} < (1/\Gamma(3/4)\sqrt[4]{n}) \exp\{-(3/32n) + (1/128n^2)\}$, for all $n \geq 1$, and for $a > (1/128)$,

$$\frac{1}{\Gamma(3/4)\sqrt[4]{n}} \exp\left\{-\frac{3}{32n} + \frac{a}{n^2}\right\} < \frac{1}{\Gamma(3/4)\sqrt[4]{n}}, \quad \text{for all } n > \frac{32a}{3}, \quad (32)$$

which is shown to improve inequality (2).

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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