Research Article

Nonlinear Jordan Derivable Mappings of Generalized Matrix Algebras by Lie Product Square-Zero Elements

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The aim of the paper is to give a description of nonlinear Jordan derivable mappings of a certain class of generalized matrix algebras by Lie product square-zero elements. We prove that under certain conditions, a nonlinear Jordan derivable mapping Δ of a generalized matrix algebra by Lie product square-zero elements is a sum of an additive derivation δ and an additive antiderivation f. Moreover, δ and f are uniquely determined.

1. Introduction

Throughout the paper, by an algebra we shall mean an algebra over a fixed unital commutative ring $\mathscr{R}$. Let $\mathscr{A}$ be a unital algebra, P be a fixed element of $\mathscr{A}$, $\Omega = \{A \in \mathscr{A}: A^2 = 0\}$, and $\Delta: \mathscr{A} \longrightarrow \mathscr{A}$ be an additive (resp., without assumption of additivity) mapping. For any $A, B \in \mathscr{A}$, denote the Jordan product (resp., Lie product) of A and B by $A \circ B = AB + BA$ (resp., $[A, B] = AB - BA$). For any $A \in \mathscr{A}$, if $2A = 0$, implying $A = 0$, then $\mathscr{A}$ is said to be a 2-torsion free algebra. Recall that Δ is called an additive derivation (resp., nonlinear derivable mapping) if $\Delta(AB) = \Delta(A)B + A\Delta(B)$ for all $A, B \in \mathscr{A}$. It is called an additive antiderivation (resp., nonlinear antiderivable mapping) if $\Delta(AB) = \Delta(B)A + B\Delta(A)$ for all $A, B \in \mathscr{A}$. It is called an additive Jordan derivation (resp., nonlinear Jordan derivable mapping) if $\Delta(A \circ B) = \Delta(A) \circ B + A \circ \Delta(B)$ for all $A, B \in \mathscr{A}$. It is called a P point derivable mapping (resp., P point nonlinear derivable mapping) if $\Delta(AB) = \Delta(A)B + A\Delta(B)$ for all $A, B \in \mathscr{A}$ with $AB = P$. Similarly, we have the definitions of P point Jordan derivable mapping and P point nonlinear Jordan derivable mapping. In this paper, if Δ is without assumption of additivity and satisfies

$$\Delta(A \circ B) = \Delta(A) \circ B + A \circ \Delta(B), \qquad (1)$$

for all $A, B \in \mathscr{A}$ with $[A, B] \in \Omega$, then we say Δ is a nonlinear Jordan derivable mapping of $\mathscr{A}$ by Lie product square-zero elements. Obviously, every additive derivation or additive antiderivation is an additive Jordan derivation. However, the inverse statement is not true in general (see [1]).

A natural and very interesting problem that we are dealing with is studying those conditions on a ring or an algebra such that every additive Jordan derivation or every nonlinear Jordan derivable map is an additive derivation.

In the past few decades, the research on this problem has attracted the attention of many mathematicians. For examples, Herstein in [2] proved that every additive Jordan derivation on a prime ring not of characteristic 2 is an additive derivation. Later, this result was extended by Cusack in [3] and Brešar and Vukman in [4] to the case of semiprime ring, respectively. Zhang and Yu in [5, 6] showed that every additive Jordan derivation on a nest algebra and a 2-torsion free triangular algebra is an inner derivation and an additive derivation, respectively. For some conclusions about P point Jordan derivable mapping, we refer the readers to [7–9] and references therein for more details. Lu in [10] showed that every nonlinear Jordan derivable mapping on a 2-torsion free semiprime ring is an additive derivation. Ashraf and Jabeen in [11] showed that every nonlinear Jordan triple derivable mapping on a 2-torsion

free triangular algebra is an additive derivation. In particular, Benkovič in [1] proved that every additive Jordan derivation from an upper triangular matrix algebra to its bimodule is a sum of an additive derivation and an additive antiderivation. Li et al. in [12] proved that under certain conditions every additive Jordan derivation on a generalized matrix algebra is a sum of an additive derivation and an additive antiderivation. For other similar results, we refer the readers to [13, 14] and references therein for more details.

Inspired by the above references, in this paper, we study the nonlinear Jordan derivable mapping of generalized

matrix algebras by Lie product square-zero elements, we get that under certain conditions, a nonlinear Jordan derivable mapping of a generalized matrix algebra by Lie product square-zero elements is a sum of an additive derivation and an additive antiderivation.

2. Generalized Matrix Algebras

Let $\mathscr{A}$ and $\mathscr{B}$ be two unital algebras with unit elements $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$, respectively, $\mathscr{M}$ be a faithful $(\mathscr{A}, \mathscr{B})$-bimodule, and $\mathscr{N}$ be a $(\mathscr{B}, \mathscr{A})$-bimodule. Then,

$$\mathscr{G} = \begin{pmatrix} \mathscr{A} & \mathscr{M} \\ \mathscr{N} & \mathscr{B} \end{pmatrix} = \left\{ \begin{pmatrix} A & M \\ N & B \end{pmatrix} : A \in \mathscr{A}, M \in \mathscr{M}, N \in \mathscr{N}, B \in \mathscr{B} \right\} \tag{2}$$

is a $\mathscr{R}$-algebra under matrix-like addition and multiplication, satisfying the following commutative diagrams:

$$\begin{array}{ccc} \mathscr{M} \otimes_{\mathscr{B}} \mathscr{N} \otimes_{\mathscr{A}} \mathscr{M} & \xrightarrow{\Phi_{\mathscr{M}\mathscr{N}} \otimes I_{\mathscr{M}}} & \mathscr{A} \otimes_{\mathscr{A}} \mathscr{M} \\ {\scriptstyle I_{\mathscr{M}} \otimes \Phi_{\mathscr{N}\mathscr{M}}} \downarrow & & \downarrow {\scriptstyle \cong} \\ \mathscr{M} \otimes_{\mathscr{B}} \mathscr{B} & \xrightarrow{\cong} & \mathscr{M} \end{array} \tag{3}$$

$$\begin{array}{ccc} \mathscr{N} \otimes_{\mathscr{A}} \mathscr{M} \otimes_{\mathscr{B}} \mathscr{N} & \xrightarrow{\Psi_{\mathscr{N}\mathscr{M}} \otimes I_{\mathscr{N}}} & \mathscr{B} \otimes_{\mathscr{B}} \mathscr{N} \\ {\scriptstyle I_{\mathscr{N}} \otimes \Phi_{\mathscr{M}\mathscr{N}}} \downarrow & & \downarrow {\scriptstyle \cong} \\ \mathscr{N} \otimes_{\mathscr{A}} \mathscr{A} & \xrightarrow{\cong} & \mathscr{N} \end{array} \tag{4}$$

where $\Phi_{\mathscr{M}\mathscr{N}} \colon \mathscr{M} \otimes_{\mathscr{B}} \mathscr{N} \longmapsto \mathscr{A}$ and $\Psi_{\mathscr{N}\mathscr{M}} \colon \mathscr{N} \otimes_{\mathscr{A}} \mathscr{M} \longmapsto \mathscr{B}$ are two bimodule homomorphisms. Such a $\mathscr{R}$-algebra $\mathscr{G}$ is called a generalized matrix algebra. For convenience in reading, we give the multiplication of generalized matrix algebra as follows:

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \begin{pmatrix} A' & M' \\ N' & B' \end{pmatrix} = \begin{pmatrix} AA' + \Phi_{\mathscr{M}\mathscr{N}}(M \otimes N') & AM' + MB' \\ NA' + BN' & BB' + \Psi_{\mathscr{N}\mathscr{M}}(N \otimes M') \end{pmatrix}, \tag{5}$$

for all $A, A' \in \mathscr{A}, M, M' \in \mathscr{M}, N, N' \in \mathscr{N}$, and $B, B' \in \mathscr{B}$. Furthermore, when $\mathscr{N} = 0$, then $\mathscr{G}$ is a triangular algebra. The most common examples of generalized matrix algebras are triangular algebras and full matrix algebras (see [15, 16] for details).

Considering a generalized matrix algebra $\mathscr{G}$, let 1 be the unit element of $\mathscr{G}$. Set

$$P_1 = \begin{pmatrix} 1_{\mathscr{A}} & 0 \\ 0 & 0 \end{pmatrix},$$

$$P_2 = 1 - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathscr{B}} \end{pmatrix}, \tag{6}$$

and $\mathscr{G}_{ij} = P_i \mathscr{G} P_j (1 \leq i, j \leq 2)$. It is clear that if $\Phi_{\mathscr{M}\mathscr{N}}(M \otimes N') = 0 = \Psi_{\mathscr{N}\mathscr{M}}(N \otimes M')$, then $\mathscr{G}$ can be represented as

$$\mathscr{G} = \mathscr{A} + \mathscr{M} + \mathscr{N} + \mathscr{B} = \mathscr{G}_{11} + \mathscr{G}_{12} + \mathscr{G}_{21} + \mathscr{G}_{22}. \tag{7}$$

Hence, for any $A \in \mathscr{G}$, A can be represented as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathscr{G}_{ij} (1 \leq i, j \leq 2)$, and

we can easily check that $A_{ij} B_{ji} = 0, \forall A_{ij} \in \mathscr{G}_{ij}$, and $B_{ji} \in \mathscr{G}_{ji} (1 \leq i \neq j \leq 2)$ (i.e., $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$).

3. Main Results

In this paper, our main result is the following theorem.

Theorem 1. *Let $\mathscr{G}$ be a 2-torsion free generalized matrix algebra and Δ be a nonlinear Jordan derivable mapping of $\mathscr{G}$ by Lie product square-zero elements. If $\mathscr{G}$ satisfies $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$, then there exist an additive derivation δ and an additive antiderivation f of $\mathscr{G}$, such that $\Delta(A) = \delta(A) + f(A)$ for all $A \in \mathscr{G}$. Moreover, δ and f are uniquely determined.*

In order to prove Theorem 1, we introduce some lemmas. Next, we assume that $\mathscr{G}$ is a 2-torsion generalized matrix algebra, $\mathscr{M}$ is a faithful $(\mathscr{A}, \mathscr{B})$-bimodule and $\mathscr{N}$ is a $(\mathscr{B}, \mathscr{A})$-bimodule (i.e., $\mathscr{G}_{12}$ is a faithful $(\mathscr{G}_{11}, \mathscr{G}_{22})$-bimodule and $\mathscr{G}_{21}$ is a $(\mathscr{G}_{22}, \mathscr{G}_{11})$-bimodule), $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ (i.e., $A_{ij} B_{ji} = 0, \forall A_{ij} \in \mathscr{G}_{ij}$, and $B_{ji} \in \mathscr{G}_{ji} (1 \leq i \neq j \leq 2)$),

$\Omega = \{A \in \mathcal{G} : A^2 = 0\}$, and Δ is a nonlinear Jordan derivable mapping of $\mathcal{G}$ by Lie product square-zero elements.

Lemma 1. *For any* $A_{ii} \in \mathcal{G}_{ii}, A_{ij}, B_{ij} \in \mathcal{G}_{ij}, \quad C_{ji} \in \mathcal{G}_{ji}$ $(1 \leq i \neq j \leq 2)$, *then*

(i) $\Delta(0) = 0$

(ii) $\Delta(P_i) = P_i \Delta(P_i) P_j + P_j \Delta(P_i) P_i, \quad \Delta(P_1) = -\Delta(P_2)$, *and* $A_{ij} \circ \Delta(P_1) = A_{ij} \circ \Delta(P_2) = 0$

(iii) $\Delta(A_{ij}) = P_i \Delta(A_{ij}) P_j + P_j \Delta(A_{ij}) P_i$ *and* $A_{ij} \circ \Delta(B_{ij}) = A_{ij} \circ \Delta(C_{ji}) = 0$

(iv) $P_j \Delta(A_{ii}) P_j = 0, P_i \Delta(A_{ii}) P_j = A_{ii} \Delta(P_i)$ *and* $P_j \Delta(A_{ii}) P_i = \Delta(P_i) A_{ii}$

Proof

(i) Since Δ is a nonlinear Jordan derivable mapping of $\mathcal{G}$ by Lie product square-zero elements, then for any $A, B \in \mathcal{G}$ with $[A, B] \in \Omega$, we have

$$\Delta(A \circ B) = \Delta(A) \circ B + A \circ \Delta(B). \tag{8}$$

Therefore, taking $A = B = 0$ in equation (8), we have $\Delta(0) = \Delta(0) \circ 0 + 0 \circ \Delta(0) = 0$.

(ii) Since $[P_i, P_j] = 0 \in \Omega (1 \leq i \neq j \leq 2)$ and taking $A = P_i$ and $B = P_j$ in equation (8), we have

$$0 = \Delta(P_i \circ P_j) = \Delta(P_i) P_j + P_j \Delta(P_i) + P_i \Delta(P_j) + \Delta(P_j) P_i. \tag{9}$$

Multiplying equation (9) from both the sides by P_j and then by the property of 2-torsion freeness of $\mathcal{G}$, we have

$$P_j \Delta(P_i) P_j = 0. \tag{10}$$

Multiplying equation (9) from the left by P_i and from the right by P_j, we get

$$P_i \Delta(P_i) P_j = -P_i \Delta(P_j) P_j. \tag{11}$$

Next, we show that $P_i \Delta(P_i) P_i = 0 \ (1 \leq i \leq 2)$. Indeed, for any $X_{12} \in \mathcal{G}_{12}$, since $[P_1, X_{12}] = X_{12} \in \Omega$ and taking $A = P_1$ and $B = X_{12}$ in equation (8), we obtain

$$\begin{aligned}
\Delta(X_{12}) &= \Delta(P_1 \circ X_{12}) \\
&= \Delta(P_1) \circ X_{12} + P_1 \circ \Delta(X_{12}) \\
&= \Delta(P_1) X_{12} + X_{12} \Delta(P_1) + P_1 \Delta(X_{12}) \\
&\quad + \Delta(X_{12}) P_1.
\end{aligned} \tag{12}$$

Multiplying equation (12) from the left by P_1 and from the right by P_2 and then by equation (10), we have $P_1 \Delta(P_1) X_{12} = P_1 \Delta(P_1) P_1 X_{12} = 0$; similarly, we can get that $X_{12} \Delta(P_2) P_2 = X_{12} P_2 \Delta(P_2) P_2 = 0$; therefore, we obtain from the faithfulness of $\mathcal{G}_{12}$ that

$$\begin{aligned}
P_1 \Delta(P_1) P_1 &= 0, \\
P_2 \Delta(P_2) P_2 &= 0.
\end{aligned} \tag{13}$$

Thus, we obtain from equations (10)–(12) that $\Delta(P_i) = P_i \Delta(P_i) P_j + P_j \Delta(P_i) P_i \ (1 \leq i \neq j \leq 2)$ and $\Delta(P_1) = -\Delta(P_2)$, and then, it follows from $\mathcal{MN} = \mathcal{NM} = 0$ that $A_{ij} \circ \Delta(P_1) = A_{ij} \circ \Delta(P_2) = 0$ for all $A_{ij} \in \mathcal{G}_{ij}(1 \leq i \neq j \leq 2)$.

(iii) For any $A_{ij} \in \mathcal{G}_{ij}$ and $C_{ji} \in \mathcal{G}_{ji}(1 \leq i \neq j \leq 2)$, since $[P_i, A_{ij}] = A_{ij} \in \Omega$ and taking $A = P_i$ and $B = A_{ij}$ in equation (8), it follows from Lemma 1 (ii) that

$$\Delta(A_{ij}) = \Delta(P_i \circ A_{ij}) = \Delta(P_i) \circ A_{ij} + P_i \circ \Delta(A_{ij}) = P_i \Delta(A_{ij}) + \Delta(A_{ij}) P_i. \tag{14}$$

Multiplying (14) by P_i and P_j from both sides, respectively, we get that

$$P_j \Delta(A_{ij}) P_j = P_i \Delta(A_{ij}) P_i = 0. \tag{15}$$

This implies that $\Delta(A_{ij}) = P_i \Delta(A_{ij}) P_j + P_j \Delta(A_{ij}) P_i$ $(1 \leq i \neq j \leq 2)$, and then by $\mathcal{MN} = \mathcal{NM} = 0$, we get $A_{ij} \circ \Delta(B_{ij}) = A_{ij} \circ \Delta(C_{ji}) = 0$.

(iv) For any $A_{ii} \in \mathcal{G}_{ii} \ (1 \leq i \leq 2)$, since $[P_j, A_{ii}] = 0 \in \Omega$ $(1 \leq i \neq j \leq 2)$ and taking $A = P_j$ and $B = A_{ii}$ in equation (8), we get

$$0 = \Delta(P_j \circ A_{ii}) = \Delta(P_j) A_{ii} + A_{ii} \Delta(P_j) + P_j \Delta(A_{ii}) + \Delta(A_{ii}) P_j. \tag{16}$$

Multiplying equation (16) by P_j from both sides and then by the property of 2-torsion freeness of $\mathcal{G}$, we have $P_j \Delta(A_{ii}) P_j = 0$.

Multiplying equation (16) from the left by P_i and from the right by P_j, it follows from Lemma 1 (ii) that

$$P_i \Delta(A_{ii}) P_j = -A_{ii} \Delta(P_j) P_j = A_{ii} \Delta(P_i) P_j = A_{ii} \Delta(P_i). \tag{17}$$

Similarly, we can show that $P_j \Delta(A_{ii}) P_i = \Delta(P_i) A_{ii}$ holds. The proof is completed. □

Lemma 2. *For any* $A_{ii}, B_{ii} \in \mathscr{G}_{ii}, A_{ij}, B_{ij} \in \mathscr{G}_{ij}, A_{ji} \in \mathscr{G}_{ji}$, *and* $A_{jj} \in \mathscr{G}_{jj} (1 \leq i \neq j \leq 2)$, *then*

(i) $\Delta(A_{ii} + A_{ij}) = \Delta(A_{ii}) + \Delta(A_{ij})$

(ii) $\Delta(A_{ij} + A_{jj}) = \Delta(A_{ij}) + \Delta(A_{jj})$

(iii) $\Delta(A_{ij} + B_{ij}) = \Delta(A_{ij}) + \Delta(B_{ij})$

(iv) $\Delta(A_{ij} + A_{ji}) = \Delta(A_{ij}) + \Delta(A_{ji})$

(v) $\Delta(A_{ii} + B_{ii}) = \Delta(A_{ii}) + \Delta(B_{ii})$

(vi) $\Delta(A_{ii} + A_{ij} + A_{ji}) = \Delta(A_{ii}) + \Delta(A_{ij}) + \Delta(A_{ji})$

(vii) $\Delta(A_{ii} + A_{ij} + A_{ji} + A_{jj}) = \Delta(A_{ii}) + \Delta(A_{ij}) + \Delta(A_{ji}) + \Delta(A_{jj})$

Proof

(i) For any $A_{ii} \in \mathscr{G}_{ii}$ and $A_{ij} \in \mathscr{G}_{ij} (1 \leq i \neq j \leq 2)$, since $[A_{ii} + A_{ij}, P_j] = A_{ij} \in \Omega$, taking $A = A_{ii} + A_{ij}$ and $B = P_j$ in equation (8), and so by Lemma 1 (ii)–(iv), we have

$$\begin{aligned}
\Delta(A_{ij}) &= \Delta((A_{ii} + A_{ij}) \circ P_j) \\
&= \Delta(A_{ii} + A_{ij}) \circ P_j + (A_{ii} + A_{ij}) \circ \Delta(P_j) \\
&= \Delta(A_{ii} + A_{ij})P_j + P_j\Delta(A_{ii} + A_{ij}) + A_{ii}\Delta(P_j) + \Delta(P_j)A_{ii} \\
&= \Delta(A_{ii} + A_{ij})P_j + P_j\Delta(A_{ii} + A_{ij}) - P_i\Delta(A_{ii})P_j - P_j\Delta(A_{ii})P_i.
\end{aligned} \tag{18}$$

This yields that

$$\Delta(A_{ij}) + P_i\Delta(A_{ii})P_j + P_j\Delta(A_{ii})P_i = \Delta(A_{ii} + A_{ij})P_j + P_j\Delta(A_{ii} + A_{ij}). \tag{19}$$

Multiplying equation (19) by P_j from both sides and then by the property of 2-torsion freeness of $\mathscr{G}$ and Lemma 1 (iii), we have

$$P_j\Delta(A_{ii} + A_{ij})P_j = 0. \tag{20}$$

Multiplying equation (19) from the left by P_i and from the right by P_j, we get

$$P_i\Delta(A_{ii} + A_{ij})P_j = P_i\Delta(A_{ii})P_j + P_i\Delta(A_{ij})P_j. \tag{21}$$

Multiplying equation (19) from the left by P_j and from the right by P_i, we get

$$P_j\Delta(A_{ii} + A_{ij})P_i = P_j\Delta(A_{ii})P_i + P_j\Delta(A_{ij})P_i. \tag{22}$$

Next, we show that $P_i\Delta(A_{ii} + A_{ij})P_i = P_i\Delta(A_{ii})P_i$. Indeed, for any $A_{11} \in \mathscr{G}_{11}, A_{12}, X_{12} \in \mathscr{G}_{12}$, since $[A_{11} + A_{12}, X_{12}] = [A_{11}, X_{12}] = A_{11}X_{12} \in \Omega$, on the one hand, taking $A = A_{11} + A_{12}, B = X_{12}$ in equation (8), and then by Lemma 1 (iii), we get

$$\begin{aligned}
\Delta(A_{11}X_{12}) &= \Delta((A_{11} + A_{12}) \circ X_{12}) \\
&= \Delta(A_{11} + A_{12}) \circ X_{12} + (A_{11} + A_{12}) \circ \Delta(X_{12}) \\
&= \Delta(A_{11} + A_{12})X_{12} + X_{12}\Delta(A_{11} + A_{12}) + A_{11}\Delta(X_{12}) + \Delta(X_{12})A_{11}.
\end{aligned} \tag{23}$$

On the other hand, taking $A = A_{11}$ and $B = X_{12}$ in equation (8), we get that

$$\begin{aligned}
\Delta(A_{11}X_{12}) &= \Delta(A_{11} \circ X_{12}) \\
&= \Delta(A_{11}) \circ X_{12} + A_{11} \circ \Delta(X_{12}) \\
&= \Delta(A_{11})X_{12} + X_{12}\Delta(A_{11}) + A_{11}\Delta(X_{12}) + \Delta(X_{12})A_{11}.
\end{aligned} \tag{24}$$

Comparing equations (23) and (24), we get

$$(\Delta(A_{11} + A_{12}) - \Delta(A_{11}))X_{12} + X_{12}(\Delta(A_{11} + A_{12}) - \Delta(A_{11})) = 0. \tag{25}$$

Multiplying equation (25) from the left by P_1 and from the right by P_2, then by equation (20) and Lemma 1 (iv), we get

$$P_1(\Delta(A_{11} + A_{12}) - \Delta(A_{11}))P_1 X_{12} = 0. \tag{26}$$

Similarly, for any $X_{12} \in \mathscr{G}_{12}, A_{21} \in \mathscr{G}_{21}$, and $A_{22} \in \mathscr{G}_{22}$, we can get that

$$X_{12} P_2 (\Delta(A_{22} + A_{21}) - \Delta(A_{22}))P_2 = 0. \tag{27}$$

This yields from the faithfulness of $\mathscr{G}_{12}$ that

$$P_1\Delta(A_{11} + A_{12})P_1 = P_1\Delta(A_{11})P_1 \text{ and } P_2\Delta(A_{22} + A_{21})P_2 = P_2\Delta(A_{22})P_2. \tag{28}$$

Therefore, it follows from equations (20)–(28) and Lemma 1 (iii)-(iv) that $\Delta(A_{ii} + A_{ij}) = \Delta(A_{ii}) + \Delta(A_{ij})$.

(ii) Similar to (i), we can show that (ii) holds.

(iii) For any $A_{ij}, B_{ij} \in \mathscr{G}_{ij}(1 \le i \ne j \le 2)$, since $[P_i + A_{ij}, B_{ij} + P_j] = A_{ij} + B_{ij} \in \Omega$, taking $A = P_i + A_{ij}, B = B_{ij} + P_j$ in equation (8), and so by Lemma 2 (i)-(ii) and Lemma 1 (ii)–(iv), we have

$$\begin{aligned}
\Delta(A_{ij} + B_{ij}) &= \Delta((P_i + A_{ij}) \circ (B_{ij} + P_j)) \\
&= \Delta(P_i + A_{ij}) \circ (B_{ij} + P_j) + (P_i + A_{ij}) \circ \Delta(B_{ij} + P_j) \\
&= (\Delta(P_i) + \Delta(A_{ij})) \circ (B_{ij} + P_j) + (P_i + A_{ij}) \circ (\Delta(B_{ij}) + \Delta(P_j)) \\
&= \Delta(P_i) \circ P_j + \Delta(A_{ij}) \circ P_j + P_i \circ \Delta(B_{ij}) + P_i \circ \Delta(P_j) \\
&= \Delta(P_i) + \Delta(A_{ij}) + \Delta(B_{ij}) + \Delta(P_j) \\
&= \Delta(A_{ij}) + \Delta(B_{ij}).
\end{aligned} \tag{29}$$

(iv) For any $A_{ij} \in \mathscr{G}_{ij}$ and $A_{ji} \in \mathscr{G}_{ji}(1 \le i \ne j \le 2)$, it follows from $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M} = 0$ that $[P_j + A_{ij}, A_{ji} + P_i] = -A_{ij} + A_{ij}A_{ji} - A_{ji}A_{ij} + A_{ji} = -A_{ij} + A_{ij} \in \Omega$;

taking $A = P_j + A_{ij}$ and $B = A_{ji} + P_i$ in equation (8) and we get from $\mathcal{M}\mathcal{N} = \mathcal{N}\mathcal{M} = 0$, Lemma 2 (i)-(ii), and Lemma 1 (ii)–(iv) that

$$\begin{aligned}
\Delta(A_{ij} + A_{ji}) &= \Delta((P_j + A_{ij}) \circ (A_{ji} + P_i)) \\
&= \Delta(P_j + A_{ij}) \circ (A_{ji} + P_i) + (P_j + A_{ij}) \circ \Delta(A_{ji} + P_i) \\
&= (\Delta(P_j) + \Delta(A_{ij})) \circ (A_{ji} + P_i) + (P_j + A_{ij}) \circ (\Delta(A_{ji}) + \Delta(P_i)) \\
&= \Delta(P_j) \circ P_i + \Delta(A_{ij}) \circ P_i + P_j \circ \Delta(A_{ji}) + P_j \circ \Delta(P_i) \\
&= \Delta(P_j) + \Delta(A_{ij}) + \Delta(A_{ji}) + \Delta(P_i) \\
&= \Delta(A_{ij}) + \Delta(A_{ji}).
\end{aligned} \tag{30}$$

(v) For any $A_{11}, B_{11} \in \mathscr{G}_{11}$, it follows from Lemma 1 (iv) that

$$\Delta(A_{11}) = P_1 \Delta(A_{11})P_1 + A_{11}\Delta(P_1) + \Delta(P_1)A_{11}, \tag{31}$$

$$\Delta(B_{11}) = P_1 \Delta(B_{11})P_1 + B_{11}\Delta(P_1) + \Delta(P_1)B_{11}, \tag{32}$$

$$\Delta(A_{11} + B_{11}) = P_1 \Delta(A_{11} + B_{11})P_1 + (A_{11} + B_{11})\Delta(P_1) + \Delta(P_1)(A_{11} + B_{11}). \tag{33}$$

In the following, we show that $P_1 \Delta(A_{11} + B_{11})P_1 = P_1 \Delta(A_{11})P_1 + P_1 \Delta(B_{11})P_1$. Indeed, for any $A_{11}, B_{11} \in \mathscr{G}_{11}, X_{12} \in \mathscr{G}_{12}$, on the one hand, since $[A_{11} + B_{11}, X_{12}] = (A_{11} + B_{11})X_{12} \in \Omega$ and taking $A = A_{11} + B_{11}$ and $B = X_{12}$ in equation (8), then we get

$$\Delta((A_{11} + B_{11})X_{12}) = \Delta(A_{11} + B_{11}) \circ X_{12} + (A_{11} + B_{11}) \circ \Delta(X_{12}). \tag{34}$$

On the other hand, since $[A_{11}, X_{12}] = A_{11}X_{12} \in \Omega$ and $[B_{11}, X_{12}] = B_{11}X_{12} \in \Omega$, we get, respectively,

$$\Delta(A_{11}X_{12}) = \Delta(A_{11} \circ X_{12}) = \Delta(A_{11}) \circ X_{12} + A_{11} \circ \Delta(X_{12}), \tag{35}$$

$$\Delta(B_{11}X_{12}) = \Delta(B_{11} \circ X_{12}) = \Delta(B_{11}) \circ X_{12} + B_{11} \circ \Delta(X_{12}). \tag{36}$$

Therefore, it follows from Lemma 2 (iii) and equations (34)–(36) that

$$(\Delta(A_{11} + B_{11}) - \Delta(A_{11}) - \Delta(B_{11}))X_{12} + X_{12}(\Delta(A_{11} + B_{11}) - \Delta(A_{11}) - \Delta(B_{11})) = 0. \tag{37}$$

Multiplying equation (37) from the left by P_1 and from the right by P_2 and then by Lemma 1 (iv), we get

$$P_1(\Delta(A_{11} + B_{11}) - \Delta(A_{11}) - \Delta(B_{11}))P_1 X_{12} = 0. \tag{38}$$

Therefore, by the faithfulness of $\mathscr{G}_{12}$, we get that

$$P_1 \Delta(A_{11} + B_{11})P_1 = P_1 \Delta(A_{11})P_1 + P_1 \Delta(B_{11})P_1. \tag{39}$$

Hence, we get from equations (31)–(33) and (39) and Lemma 1 (iv) that $\Delta(A_{11} + B_{11}) = \Delta(A_{11}) + \Delta(B_{11})$. Similarly, we can show that $\Delta(A_{22} + B_{22}) = \Delta(A_{22}) + \Delta(B_{22})$ holds for all $A_{22}, B_{22} \in \mathscr{G}_{22}$.

(vi) For any $A_{ii} \in \mathscr{G}_{ii}, A_{ij} \in \mathscr{G}_{ij}$, and $A_{ji} \in \mathscr{G}_{ji}$ $(1 \leq i \neq j \leq 2)$, it follows from $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ that $[A_{ii} + A_{ij} + A_{ji}, P_j] = A_{ij} - A_{ji} \in \Omega$; taking $A = A_{ii} + A_{ij} + A_{ji}, B = P_j$ in equation (8) and then by Lemma 1 (ii)–(iv) and Lemma 2 (iv), we obtain

$$\Delta\left(A_{ij}\right) + \Delta\left(A_{ji}\right) = \Delta\left(A_{ij} + A_{ji}\right)$$
$$= \Delta\left(\left(A_{ii} + A_{ij} + A_{ji}\right) \circ P_j\right)$$
$$= \Delta\left(A_{ii} + A_{ij} + A_{ji}\right) \circ P_j + \left(A_{ii} + A_{ij} + A_{ji}\right) \circ \Delta\left(P_j\right) \tag{40}$$
$$= \Delta\left(A_{ii} + A_{ij} + A_{ji}\right)P_j + P_j\Delta\left(A_{ii} + A_{ij} + A_{ji}\right) + A_{ii}\Delta\left(P_j\right) + \Delta\left(P_j\right)A_{ii}$$
$$= \Delta\left(A_{ii} + A_{ij} + A_{ji}\right)P_j + P_j\Delta\left(A_{ii} + A_{ij} + A_{ji}\right) - P_i\Delta(A_{ii})P_j - P_j\Delta(A_{ii})P_i.$$

This implies that

$$P_i\Delta\left(A_{ii} + A_{ij} + A_{ji}\right)P_j = P_i\Delta(A_{ii})P_j + P_i\Delta\left(A_{ij}\right)P_j + P_i\Delta\left(A_{ji}\right)P_j, \tag{41}$$

$$P_j\Delta\left(A_{ii} + A_{ij} + A_{ji}\right)P_i = P_j\Delta(A_{ii})P_i + P_j\Delta\left(A_{ij}\right)P_i + P_j\Delta\left(A_{ji}\right)P_i, \tag{42}$$

$$P_j\Delta\left(A_{ii} + A_{ij} + A_{ji}\right)P_j = 0. \tag{43}$$

In the following, we show that $P_i\Delta(A_{ii} + A_{ij} + A_{ji})P_i = P_i\Delta(A_{ii})P_i$. Indeed, for any $A_{11} \in \mathscr{G}_{11}$, $A_{12}, X_{12} \in \mathscr{G}_{12}, A_{21} \in \mathscr{G}_{21}$, on the one hand, it follows from $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ that $[A_{11} + A_{12} + A_{21}, X_{12}] = A_{11}X_{12} - X_{12}A_{21} + A_{21}X_{12} = A_{11}X_{12} \in \Omega$; taking $A = A_{11} + A_{12} + A_{21}, B = X_{12}$ in equation (8) and so by $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ and Lemma 1 (iii), we have

$$\Delta(A_{11}X_{12}) = \Delta\left(\left(A_{11} + A_{12} + A_{21}\right) \circ X_{12}\right)$$
$$= \Delta\left(A_{11} + A_{12} + A_{21}\right) \circ X_{12} + \left(A_{11} + A_{12} + A_{21}\right)$$
$$\circ \left(X_{12}\right)$$
$$= \Delta\left(A_{11} + A_{12} + A_{21}\right) \circ X_{12} + A_{11} \circ \left(X_{12}\right). \tag{44}$$

On the other hand, since $[A_{11}, X_{12}] = A_{11}X_{12} \in \Omega$, we get

$$\Delta(A_{11}X_{12}) = \Delta\left(A_{11} \circ X_{12}\right)$$
$$= \Delta\left(A_{11}\right) \circ X_{12} + A_{11} \circ \left(X_{12}\right). \tag{45}$$

Comparing equations (44) and (45), we get

$$\left(\Delta\left(A_{11} + A_{12} + A_{21}\right) - \Delta\left(A_{11}\right)\right)X_{12} + X_{12}\left(\Delta\left(A_{11} + A_{12} + A_{21}\right) - \Delta\left(A_{11}\right)\right) = 0. \tag{46}$$

Multiplying equation (46) from the left by P_1 and from the right by P_2 and then by Lemma 1 (iv), equation (43), and the faithfulness of $\mathscr{G}_{12}$, we get

$$P_1\Delta\left(A_{11} + A_{12} + A_{21}\right)P_1 = P_1\Delta\left(A_{11}\right)P_1. \tag{47}$$

Similarly, for any $A_{22} \in \mathscr{G}_{22}, A_{12} \in \mathscr{G}_{12}$, and $A_{21} \in \mathscr{G}_{21}$, we can get that

$$P_2\Delta\left(A_{22} + A_{12} + A_{21}\right)P_2 = P_2\Delta\left(A_{22}\right)P_2. \tag{48}$$

Hence, for any $A_{ii} \in \mathscr{G}_{ii}, A_{ij} \in \mathscr{G}_{ij}$, and $A_{ji} \in \mathscr{G}_{ji}$ $(1 \le i \ne j \le 2)$, we get that
$$P_i\Delta\left(A_{ii} + A_{ij} + A_{ij}\right)P_i = P_i\Delta\left(A_{ii}\right)P_i. \tag{49}$$

Therefore, it follows from equations (41)–(48) and Lemma 1 (iii)-(iv) that $\Delta\left(A_{ii} + A_{ij} + A_{ji}\right) = \Delta\left(A_{ii}\right) + \Delta\left(A_{ij}\right) + \Delta\left(A_{ji}\right)$.

(vii) For any $A_{ii} \in \mathscr{G}_{ii}, A_{ij} \in \mathscr{G}_{ij}, A_{ji} \in \mathscr{G}_{ji}$, and $A_{jj} \in \mathscr{G}_{jj}$ $(1 \le i \ne j \le 2)$, it follows from $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ that $[A_{ii} + A_{ij} + A_{ji} + A_{jj}, P_j] = A_{ij} - A_{ji} \in \Omega$;

taking $A = A_{ii} + A_{ij} + A_{ji} + A_{jj}, B = P_j$ in equation (8) and so by Lemma 2 (vi) and Lemma 1 (iv), we have

$$
\begin{aligned}
\Delta(A_{ij}) + \Delta(A_{ji}) + 2\Delta(A_{jj}) &= \Delta(A_{ij} + A_{ji} + 2A_{jj}) \\
&= \Delta((A_{ii} + A_{ij} + A_{ji} + A_{jj}) \circ P_j) \\
&= \Delta(A_{ii} + A_{ij} + A_{ji} + A_{jj}) \circ P_j + (A_{ii} + A_{ij} + A_{ji} + A_{jj}) \circ \Delta(P_j) \\
&= \Delta(A_{ii} + A_{ij} + A_{ji} + A_{jj}) \circ P_j + (A_{ii} + A_{jj}) \circ \Delta(P_j).
\end{aligned}
\tag{50}
$$

Similarly, we obtain

$$
2\Delta(A_{ii}) + \Delta(A_{ij}) + \Delta(A_{ji}) = \Delta(A_{ii} + A_{ij} + A_{ji} + A_{jj}) \circ P_i + (A_{ii} + A_{jj}) \circ \Delta(P_i).
\tag{51}
$$

Therefore, it follows from (50) and (51), Lemma 1 (ii), and the property of 2-torsion freeness of $\mathscr{G}$ that $\Delta(A_{ii} + A_{ij} + A_{ji} + A_{jj}) = \Delta(A_{ii}) + \Delta(A_{ij}) + \Delta(A_{ji}) + \Delta(A_{jj})$. The proof is completed. $\square$

Lemma 3. Δ *is an additive mapping.*

$$
\begin{aligned}
\Delta(A + B) &= \Delta((A_{11} + A_{12} + A_{21} + A_{22}) + (B_{11} + B_{12} + B_{21} + B_{22})) \\
&= \Delta((A_{11} + B_{11}) + (A_{12} + B_{12}) + (A_{21} + B_{21}) + (A_{22} + B_{22})) \\
&= \Delta(A_{11} + B_{11}) + \Delta(A_{12} + B_{12}) + \Delta(A_{21} + B_{21}) + \Delta(A_{22} + B_{22}) \\
&= \Delta(A_{11}) + \Delta(B_{11}) + \Delta(A_{12}) + \Delta(B_{12}) + \Delta(A_{21}) + \Delta(B_{21}) + \Delta(A_{22}) + \Delta(B_{22}) \\
&= \Delta(A_{11} + A_{12} + A_{21} + A_{22}) + \Delta(B_{11} + B_{12} + B_{21} + B_{22}) \\
&= \Delta(A) + \Delta(B).
\end{aligned}
\tag{52}
$$

Therefore, Δ is an additive mapping on $\mathscr{G}$. The proof is completed. $\square$

Remark 1. For any $A \in \mathscr{G}$, we define a mapping $f: \mathscr{G} \longmapsto \mathscr{G}$ as

$$
f(A) = P_2\Delta(P_1 A P_2)P_1 + P_1\Delta(P_2 A P_1)P_2.
\tag{53}
$$

Then, by the definition of f, we can easily obtain that

$$
f(A_{ii}) = 0 \text{ and } f(A_{ij}) = P_j\Delta(A_{ij})P_i,
\tag{54}
$$

for all $A_{ii} \in \mathscr{G}_{ii}, A_{ij} \in \mathscr{G}_{ij} (1 \le i \ne j \le 2)$.

Proof. For any $A, B \in \mathscr{G}$, let $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$, where $A_{ij}, B_{ij} \in \mathscr{G}_{ij} (1 \le i, j \le 2)$, and by Lemma 2, we have

Next, we will show that f is an additive antiderivation on $\mathscr{G}$. First, we introduce Lemma 4 and get that f is an additive mapping, and then, we introduce Lemmas 5 and 6 and show that f is an additive antiderivation on $\mathscr{G}$.

Lemma 4. *Let* $f: \mathscr{G} \longmapsto \mathscr{G}$ *be as in Remark 1. Then,* f *is an additive mapping.*

Proof. For any $A, B \in \mathscr{G}$, since we have shown that Δ is an additive mapping in Lemma 3 and then by the definition of f, we obtain that

$$
\begin{aligned}
f(A+B) &= P_2\Delta\left(P_1(A+B)P_2\right)P_1 + P_1\Delta\left(P_2(A+B)P_1\right)P_2 \\
&= P_2\Delta\left(P_1AP_2 + P_1BP_2\right)P_1 + P_1\Delta\left(P_2AP_1 + P_2BP_1\right)P_2 \\
&= P_2\Delta\left(P_1AP_2\right)P_1 + P_2\Delta\left(P_1BP_2\right)P_1 + P_1\Delta\left(P_2AP_1\right)P_2 + P_1\Delta\left(P_2BP_1\right)P_2 \\
&= \left(P_2\Delta\left(P_1AP_2\right)P_1 + P_1\Delta\left(P_2AP_1\right)P_2\right) + \left(P_2\Delta\left(P_1BP_2\right)P_1 + P_1\Delta\left(P_2BP_1\right)P_2\right) \\
&= f(A) + f(B).
\end{aligned}
\tag{55}
$$

Therefore, f is an additive mapping on $\mathscr{G}$. The proof is completed. □

Lemma 5. *Let* $f: \mathscr{G} \longmapsto \mathscr{G}$ *be as in Remark 1. Then, for any* $A_{ii}, B_{ii} \in \mathscr{G}_{ii}, A_{ij}, B_{ij} \in \mathscr{G}_{ij}, B_{ji} \in \mathscr{G}_{ji}$, *and* $B_{jj} \in \mathscr{G}_{jj} (1 \le i \ne j \le 2)$,

(i) $f(A_{ii}B_{ii}) = f(B_{ii})A_{ii} + B_{ii}f(A_{ii})$

(ii) $f(A_{ii}B_{jj}) = f(B_{jj})A_{ii} + B_{jj}f(A_{ii})$

(iii) $f(A_{ij}B_{ji}) = f(B_{ji})A_{ij} + B_{ji}f(A_{ij})$

(iv) $f(A_{ij}B_{ij}) = f(B_{ij})A_{ij} + B_{ij}f(A_{ij})$

(v) $f(A_{ii}B_{ji}) = f(B_{ji})A_{ii} + B_{ji}f(A_{ii})$

(vi) $f(A_{ji}B_{jj}) = f(B_{jj})A_{ji} + B_{jj}f(A_{ji})$

(vii) $f(A_{ii}B_{ij}) = f(B_{ij})A_{ii} + B_{ij}f(A_{ii})$

(viii) $f(A_{ji}B_{ii}) = f(B_{ii})A_{ji} + B_{ii}f(A_{ji})$

Proof

(i) For any $A_{ii}, B_{ii} \in \mathscr{G}_{ii} (1 \le i \le 2)$, it follows from $f(A_{ii}B_{ii}) = 0 = f(A_{ii}) = f(B_{ii})$ that $f(A_{ii}B_{ii}) = f(B_{ii})A_{ii} + B_{ii}f(A_{ii})$.

(ii) Similar to (i), we can show holds.

(iii) For any $A_{ij} \in \mathscr{G}_{ij}$ and $B_{ji} \in \mathscr{G}_{ji} (1 \le i \ne j \le 2)$, on the one hand, it follows from $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ that $f(A_{ij}B_{ji}) = f(0) = 0$. On the other hand, by the definition of f, we have $f(B_{ji}) = P_i\Delta(B_{ji})P_j$ and $f(A_{ij}) = P_j\Delta(A_{ij})P_i$ and so $f(B_{ji})A_{ij} = 0 = B_{ji}f(A_{ij})$; therefore, we have $f(A_{ij}B_{ji}) = f(B_{ji})A_{ij} + B_{ji}f(A_{ij})$.

(iv) For any $A_{ij}, B_{ij} \in \mathscr{G}_{ij} (1 \le i \ne j \le 2)$, on the one hand, we have $f(A_{ij}B_{ij}) = f(0) = 0$. On the other hand, by the definition of f, we have $f(A_{ij}) = P_j\Delta(A_{ij})P_i$ and $f(B_{ij}) = P_j\Delta(B_{ij})P_i$, and then, it follows from $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ that $f(B_{ij})A_{ij} = (P_j\Delta(B_{ij})P_i)A_{ij} = 0$ and $B_{ij}f(A_{ij}) = B_{ij}(P_j\Delta(A_{ij})P_i) = 0$ and so $f(A_{ij}B_{ij}) = f(B_{ij})A_{ij} + B_{ij}f(A_{ij})$.

(v) Similar to (iv), we can show that (v) holds.

(vi) Similar to (iv), we can show that (vi) holds.

(vii) For any $A_{ii} \in \mathscr{G}_{ii}$ and $B_{ij} \in \mathscr{G}_{ij} (1 \le i \ne j \le 2)$, since $[A_{ii}, B_{ij}] = A_{ii}B_{ij} \in \Omega$, we have

$$
\begin{aligned}
\Delta\left(A_{ii}B_{ij}\right) &= \Delta\left(A_{ii} \circ B_{ij}\right) \\
&= \Delta(A_{ii}) \circ B_{ij} + A_{ii} \circ \Delta\left(B_{ij}\right) \\
&= \Delta(A_{ii})B_{ij} + B_{ij}\Delta(A_{ii}) + A_{ii}\Delta\left(B_{ij}\right) + \Delta\left(B_{ij}\right)A_{ii}.
\end{aligned}
\tag{56}
$$

Hence, by the definition of f, we have

$$
f\left(A_{ii}B_{ij}\right) = P_j\Delta\left(A_{ii}B_{ij}\right)P_i = P_j\Delta\left(B_{ij}\right)A_{ii} = f\left(B_{ij}\right)A_{ii}.
\tag{57}
$$

Therefore, we obtain from $f(A_{ii}) = 0$ that $f(A_{ii}B_{ij}) = f(B_{ij})A_{ii} + B_{ij}f(A_{ii})$. Similarly, we can show (viii) holds. The proof is completed. □

Lemma 6. *Let* $f: \mathscr{G} \longmapsto \mathscr{G}$ *be as in Remark 3.1. Then, f is an additive antiderivation.*

Proof. For any $A, B \in \mathscr{G}$, let $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$, where $A_{ij}, B_{ij} \in \mathscr{G}_{ij} (1 \le i, j \le 2)$, and we obtain from Lemmas 4 and 5 that

$$
\begin{aligned}
f(AB) &= f\left((A_{11} + A_{12} + A_{21} + A_{22})(B_{11} + B_{12} + B_{21} + B_{22})\right) \\
&= f(A_{11}B_{11}) + f(A_{11}B_{12}) + f(A_{11}B_{21}) + f(A_{11}B_{22}) + f(A_{12}B_{11}) + f(A_{12}B_{12}) + f(A_{12}B_{21}) + f(A_{12}B_{22}) + f(A_{21}B_{11}) \\
&\quad + f(A_{21}B_{12}) + f(A_{21}B_{21}) + f(A_{21}B_{22}) + f(A_{22}B_{11}) + f(A_{22}B_{12}) + f(A_{22}B_{21}) + f(A_{22}B_{22}) = f(B_{11})(A_{11} + A_{12} + A_{21} + A_{22}) \\
&\quad + f(B_{12})(A_{11} + A_{12} + A_{21} + A_{22}) + f(B_{21})(A_{11} + A_{12} + A_{21} + A_{22}) + f(B_{22})(A_{11} + A_{12} + A_{21} + A_{22}) \\
&\quad + (B_{11} + B_{12} + B_{21} + B_{22})f(A_{11}) \\
&\quad + (B_{11} + B_{12} + B_{21} + B_{22})f(A_{12}) + (B_{11} + B_{12} + B_{21} + B_{22})f(A_{21}) + (B_{11} + B_{12} + B_{21} + B_{22})f(A_{22}) = f(B)A + Bf(A).
\end{aligned}
\tag{58}
$$

Therefore, f is an additive antiderivation on $\mathscr{G}$. The proof is completed. □

Remark 2. For any $A \in \mathscr{G}$, we define a mapping $\delta \colon \mathscr{G} \longmapsto \mathscr{G}$ as

$$\delta(A) = \Delta(A) - f(A). \tag{59}$$

Then, we obtain from Lemmas 3 and 4 that δ is an additive mapping on $\mathscr{G}$.

In the following, we will introduce Lemmas 7–9 and show that δ is an additive derivation on $\mathscr{G}$.

Lemma 7. *Let $\delta \colon \mathscr{G} \longmapsto \mathscr{G}$ be as in Remark 2. Then, for any $A_{ii} \in \mathscr{G}_{ii}$ and $A_{ij} \in \mathscr{G}_{ij}$ ($1 \le i \ne j \le 2$),*

(i) $\delta(P_1) = -\delta(P_2)$ and $\delta(P_i) = P_i\delta(P_i)P_j + P_j\delta(P_i)P_i$

(ii) $\delta(A_{ij}) = P_i\delta(A_{ij})P_j$

(iii) $P_j\delta(A_{ii})P_j = 0, P_i\delta(A_{ii})P_j = A_{ii}\delta(P_i)$ and $P_j\delta(A_{ii})P_i = \delta(P_i)A_{ii}$

Proof. By Lemma 1 and Remarks 1 and 2, we can easily check that Lemma 7 holds. The proof is completed. □

Lemma 8. *Let $\delta \colon \mathscr{G} \longmapsto \mathscr{G}$ be as in Remark 2. Then, for any $A_{ii}, B_{ii} \in \mathscr{G}_{ii}, A_{ij}, B_{ij} \in \mathscr{G}_{ij}, A_{ji}, B_{ji} \in \mathscr{G}_{ji}$, and $B_{jj} \in \mathscr{G}_{jj}$ ($1 \le i \ne j \le 2$),*

(i) $\delta(A_{ii}B_{ij}) = \delta(A_{ii})B_{ij} + A_{ii}\delta(B_{ij})$

(ii) $\delta(A_{ij}B_{jj}) = \delta(A_{ij})B_{jj} + A_{ij}\delta(B_{jj})$

(iii) $\delta(A_{ii}B_{ii}) = \delta(A_{ii})B_{ii} + A_{ii}\delta(B_{ii})$

(iv) $\delta(A_{ii}B_{jj}) = \delta(A_{ii})B_{jj} + A_{ii}\delta(B_{jj})$

(v) $\delta(A_{ij}B_{ji}) = \delta(A_{ij})B_{ji} + A_{ij}\delta(B_{ji})$

(vi) $\delta(A_{ij}B_{ij}) = \delta(A_{ij})B_{ij} + A_{ij}\delta(B_{ij})$

(vii) $\delta(A_{ii}B_{ji}) = \delta(A_{ii})B_{ji} + A_{ii}\delta(B_{ji})$

(viii) $\delta(A_{ji}B_{jj}) = \delta(A_{ji})B_{jj} + A_{ji}\delta(B_{jj})$

Proof

(i) For any $A_{ii}, B_{ij} \in \mathscr{G}_{ij}$ ($1 \le i \ne j \le 2$), on the one hand, since $[A_{ii}, B_{ij}] = A_{ii}B_{ij} \in \Omega$, we get that

$$\Delta(A_{ii}B_{ij}) = \Delta(A_{ii})B_{ij} + B_{ij}\Delta(A_{ii}) + A_{ii}\Delta(B_{ij}) + \Delta(B_{ij})A_{ii}. \tag{60}$$

And then, it follows from $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$, Lemma 1, and Remark 1 that

$$\Delta(A_{ii}B_{ij}) = \Delta(A_{ii})B_{ij} + A_{ii}\Delta(B_{ij}) + f(B_{ij})A_{ii}. \tag{61}$$

On the other hand, we obtain from Lemma 6 and $f(A_{ii}) = 0$ that

$$f(A_{ii}B_{ij}) = f(B_{ij})A_{ii}. \tag{62}$$

Therefore, by Remark 2, equations (61) and (62), $A_{ii}f(B_{ij}) = A_{ii}(P_j\Delta(B_{ij})P_i) = 0$, and $f(A_{ii}) = 0$, we get

$$\begin{aligned}
\delta(A_{ii}B_{ij}) &= \Delta(A_{ii}B_{ij}) - f(A_{ii}B_{ij}) \\
&= \Delta(A_{ii})B_{ij} + A_{ii}\Delta(B_{ij}) \\
&= (\Delta(A_{ii}) - f(A_{ii}))B_{ij} + A_{ii}(\Delta(B_{ij}) - f(B_{ij})) \\
&= \delta(A_{ii})B_{ij} + A_{ii}\delta(B_{ij}).
\end{aligned} \tag{63}$$

(ii) Similarly, we can show that (ii) holds.

(iii) For any $A_{11}, B_{11} \in \mathscr{G}_{11}$ and $Y_{12} \in \mathscr{G}_{12}$, by Lemma 8 (i), on the one hand, we get

$$\delta(A_{11}B_{11}Y_{12}) = \delta((A_{11}B_{11})Y_{12}) = \delta(A_{11}B_{11})Y_{12} + A_{11}B_{11}\delta(Y_{12}). \tag{64}$$

On the other hand, we have

$$\delta(A_{11}B_{11}Y_{12}) = \delta(A_{11}(B_{11}Y_{12})) = \delta(A_{11})B_{11}Y_{12} + A_{11}\delta(B_{11})Y_{12} + A_{11}B_{11}\delta(Y_{12}). \tag{65}$$

Comparing equations (64) and (65), we get

$$(\delta(A_{11}B_{11}) - \delta(A_{11})B_{11} - A_{11}\delta(B_{11}))Y_{12} = 0. \tag{66}$$

This yields from the faithfulness of $\mathscr{G}_{12}$ that

$$P_1\delta(A_{11}B_{11})P_1 = P_1\delta(A_{11})B_{11} + A_{11}\delta(B_{11})P_1. \tag{67}$$

Furthermore, by Lemma 7 (i) and (iii), we have

$$\begin{aligned}
P_1\delta(A_{11}B_{11})P_2 + P_2\delta(A_{11}B_{11})P_1 &= A_{11}B_{11}\delta(P_1) + \delta(P_1)A_{11}B_{11} \\
&= A_{11}\delta(B_{11})P_2 + P_2\delta(A_{11})B_{11}.
\end{aligned} \tag{68}$$

Therefore, we obtain from Lemma 7 (iii) and equations (67) and (68) that

$$
\begin{aligned}
\delta(A_{11}B_{11}) &= P_1\delta(A_{11}B_{11})P_1 + P_1\delta(A_{11}B_{11})P_2 + P_2\delta(A_{11}B_{11})P_1 \\
&= P_1\delta(A_{11})B_{11} + A_{11}\delta(B_{11})P_1 + A_{11}\delta(B_{11})P_2 + P_2\delta(A_{11})B_{11} \\
&= \text{delta}(A_{11})B_{11} + A_{11}\delta(B_{11}).
\end{aligned}
\tag{69}
$$

Similarly, we can show $\delta(A_{22}B_{22}) = \delta(A_{22})B_{22} + A_{22}\delta(B_{22})$ holds for all $A_{22}, B_{22} \in \mathscr{G}_{22}$.

(iv) For any $A_{ii} \in \mathscr{G}_{ii}$ and $B_{jj} \in \mathscr{G}_{jj}$ $(1 \le i \ne j \le 2)$, by Lemma 1 (iv), we have

$$
\Delta(A_{ii})B_{jj} + A_{ii}\Delta(B_{jj}) \in \mathscr{G}_{ij} \text{ and } B_{jj}\Delta(A_{ii}) + \Delta(B_{jj})A_{ii} \in \mathscr{G}_{ji}.
\tag{70}
$$

Furthermore, since $[A_{ii}, B_{jj}] = 0 \in \Omega$, we get

$$
\begin{aligned}
0 &= \Delta(A_{ii} \circ B_{jj}) \\
&= \Delta(A_{ii}) \circ B_{jj} + A_{ii} \circ \Delta(B_{jj}) \\
&= \Delta(A_{ii})B_{jj} + B_{jj}\Delta(A_{ii}) + A_{ii}\Delta(B_{jj}) + \Delta(B_{jj})A_{ii} \\
&= (\Delta(A_{ii})B_{jj} + A_{ii}\Delta(B_{jj})) + (B_{jj}\Delta(A_{ii}) + \Delta(B_{jj})A_{ii}).
\end{aligned}
\tag{71}
$$

And so, this yields from equation (70) that

$$
\Delta(A_{ii})B_{jj} + A_{ii}\Delta(B_{jj}) = 0.
\tag{72}
$$

Therefore, it follows from equation (72) and $\delta(A_{ii}B_{jj}) = \delta(0) = 0 = f(A_{ii}) = f(B_{jj})$ that

$$
\delta(A_{ii}B_{jj}) = (\Delta(A_{ii}) - f(A_{ii}))B_{jj} + A_{ii}(\Delta(B_{jj}) - f(B_{jj})) = \delta(A_{ii})B_{jj} + A_{ii}\delta(B_{jj}).
\tag{73}
$$

(v) For any $A_{ij} \in \mathscr{G}_{ij}$ and $B_{ji} \in \mathscr{G}_{ji}$ $(1 \le i \ne j \le 2)$, it follows from $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ and Lemma 7 (ii) that

$$
\delta(A_{ij}B_{ji}) = \delta(0) = 0.
\tag{74}
$$

And,

$$
\delta(A_{ij})B_{ji} = (P_i\delta(A_{ij})P_j)B_{ji} = 0 \text{ and } A_{ij}\delta(B_{ji}) = A_{ij}(P_j\delta(B_{ji})P_i) = 0.
\tag{75}
$$

Thus, we obtain from (74) and (75) that

$$
\delta(A_{ij}B_{ji}) = \delta(A_{ij})B_{ji} + A_{ij}\delta(B_{ji}).
\tag{76}
$$

(vi) Similar to (v), we can show that (vi) holds.

(vii) For any $A_{ii} \in \mathscr{G}_{ii}$, $A_{ij} \in \mathscr{G}_{ij}$, and $B_{ji} \in \mathscr{G}_{ji}$ $(1 \le i \ne j \le 2)$, on the one hand, we get

$$
\delta(A_{ii}B_{ji}) = \delta(0) = 0.
\tag{77}
$$

On the other hand, it follows from $\mathscr{M}\mathscr{N} = \mathscr{N}\mathscr{M} = 0$ and Lemma 7 (ii)-(iii) that

$$
\delta(A_{ii})B_{ji} = (P_i\delta(A_{ii})P_j)B_{ji} = 0 \text{ and } A_{ii}\delta(B_{ji}) = A_{ii}(P_j\delta(B_{ji})P_i) = 0.
\tag{78}
$$

Comparing equations (77) and (78), we get

$$
\delta(A_{ii}B_{ji}) = \delta(A_{ii})B_{ji} + A_{ii}\delta(B_{ji}).
\tag{79}
$$

(viii) Similar to (vii), we can show that (viii) holds. The proof is completed.   □

Lemma 9. *Let* $\delta: \mathscr{G} \longmapsto \mathscr{G}$ *be as in Remark 2. Then,* δ *is an additive derivation.*

$$
\begin{aligned}
\delta(\mathrm{AB}) &= \delta\left((A_{11} + A_{12} + A_{21} + A_{22})(B_{11} + B_{12} + B_{21} + B_{22})\right) \\
&= \delta(A_{11}B_{11}) + \delta(A_{11}B_{12}) + \delta(A_{11}B_{21}) + \delta(A_{11}B_{22}) + \delta(A_{12}B_{11}) + \delta(A_{12}B_{12}) + \delta(A_{12}B_{21}) + \delta(A_{12}B_{22}) + \delta(A_{21}B_{11}) \\
&\quad + \delta(A_{21}B_{12}) + \delta(A_{21}B_{21}) + \delta(A_{21}B_{22}) + \delta(A_{22}B_{11}) \\
&\quad + \delta(A_{22}B_{12}) + \delta(A_{22}B_{21}) + \delta(A_{22}B_{22}) = \delta(A_{11})(B_{11} + B_{12} + B_{21} + B_{22}) + \delta(A_{12})(B_{11} + B_{12} + B_{21} + B_{22}) \\
&\quad + \delta(A_{21})(B_{11} + B_{12} + B_{21} + B_{22}) + \delta(A_{22})(B_{11} + B_{12} + B_{21} + B_{22}) \\
&\quad + (A_{11} + A_{12} + A_{21} + A_{22})\delta(B_{11}) + (A_{11} + A_{12} + A_{21} + A_{22})\delta(B_{12}) + (A_{11} + A_{12} + A_{21} + A_{22})\delta(B_{21}) \\
&\quad + (A_{11} + A_{12} + A_{21} + A_{22})\delta(B_{22}) = \delta(A)B + A\delta(B).
\end{aligned}
$$
(80)

Therefore, δ is an additive derivation on $\mathscr{G}$. The proof is completed.   □

Proof. of Theorem 1. For any $A \in \mathscr{G}$, let $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathscr{G}_{ij}$ $(1 \le i, j \le 2)$; by Remarks 1 and 2 and Lemmas 6 and 9, we obtain that

$$\Delta(A) = \delta(A) + f(A),$$
(81)

where δ is an additive derivation and f is an additive antiderivation, respectively. Furthermore, $f(A_{ii}) = 0$ for all $A_{ii} \in \mathscr{G}_{ii} (1 \le i \le 2)$.

In the following, we check that δ and f are unique. Let $d: \mathscr{G} \longmapsto \mathscr{G}$ be an additive derivation and $h: \mathscr{G} \longmapsto \mathscr{G}$ be an additive antiderivation such that $h(A_{ii}) = 0$ for all $A_{ii} \in \mathscr{G}_{ii}$ $(1 \le i \le 2)$. Suppose that

$$\Delta(A) = d(A) + h(A).$$
(82)

Then, for any $A \in \mathscr{G}$, by Theorem 1, we get that $\delta(A) + f(A) = \Delta(A) = d(A) + h(A)$ and so $\delta(A) - d(A) = h(A) - f(A)$ for all $A \in \mathscr{G}$; this yields from δ and d two additive derivations that $h - f$ is an additive derivation; moreover, by Remark 1, we get $(h - f)(A_{ii}) = 0$ for all $A_{ii} \in \mathscr{G}_{ii}$ $(1 \le i \le 2)$.

Next, we show that $(h - f)(A_{ij}) = 0$ for all $A_{ij} \in \mathscr{G}_{ij}$ $(1 \le i \ne j \le 2)$. Indeed, for any $A_{ij} \in \mathscr{G}_{ij}$ $(1 \le i \ne j \le 2)$, on the one hand, since $h - f$ is an additive derivation and $h(P_i) = f(P_i) = 0 (1 \le i \le 2)$ and so we get that

$$
\begin{aligned}
(h - f)(A_{ij}) &= (h - f)(P_i A_{ij}) \\
&= (h - f)(P_i)A_{ij} + P_i(h - f)(A_{ij}) \\
&= P_i(h - f)(A_{ij}).
\end{aligned}
$$
(83)

Proof. For any $A, B \in \mathscr{G}$, let $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$, where $A_{ij}, B_{ij} \in \mathscr{G}_{ij} (1 \le i, j \le 2)$; since δ is an additive mapping and then by Lemma 8, we have

On the other hand, since h and f are two additive antiderivations and $h(P_j) = f(P_j) = 0 (1 \le j \le 2)$, we get that

$$
\begin{aligned}
(h - f)(A_{ij}) &= (h - f)(A_{ij}P_j) \\
&= h(A_{ij}P_j) - f(A_{ij}P_j) \\
&= h(P_j)A_{ij} + P_j h(A_{ij}) - f(P_j)A_{ij} - P_j f(A_{ij}) \\
&= P_j h(A_{ij}) - P_j f(A_{ij}) \\
&= P_j(h - f)(A_{ij}).
\end{aligned}
$$
(84)

Thus, this yields from equations (83) and (84) that $(h - f)(A_{ij}) = 0$ for all $A_{ij} \in \mathscr{G}_{ij}$ $(1 \le i \ne j \le 2)$. Therefore, $(h - f)(A) = (h - f)(A_{11}) + (h - f)(A_{12}) + (h - f)(A_{21}) + (h - f)(A_{22}) = 0$ and so $h = f$ and $\delta = d$. The proof is completed.   □

Remark 3. By the above lemmas, Remarks 1 and 2, and the proof of Theorem 1, we can easily obtain that if Δ is a nonlinear Jordan derivable mapping of $\mathscr{G}$ by Lie product square-zero elements, then the following statements are equivalent:

(i) Δ is an additive derivation

(ii) $P_j\Delta(A_{ij})P_i = 0, \forall A_{ij} \in \mathscr{G}_{ij}$ $(1 \le i \ne j \le 2)$

(iii) $\Delta(A_{ij}) \in \mathscr{G}_{ij}, \forall A_{ij} \in \mathscr{G}_{ij}$ $(1 \le i \ne j \le 2)$

Next, we give an application of Remark 3 to triangular algebras and we obtain that every nonlinear Jordan derivable mapping of a triangular algebra by Lie product square-zero elements is an additive derivation.

Corollary 1. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital algebras, $\mathscr{M}$ be a unital $(\mathscr{A}, \mathscr{B})$-bimodule, which is faithful as both a left $\mathscr{A}$-module and a right $\mathscr{B}$-module, $\mathscr{U} = Tri(\mathscr{A}, \mathscr{M}, \mathscr{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathscr{A}, m \in \mathscr{M}, b \in \mathscr{B} \right\}$ be a 2-torsion free triangular algebra, and $\Delta: \mathscr{U} \longmapsto \mathscr{U}$ be a nonlinear Jordan derivable mapping of $\mathscr{U}$ by Lie product square-zero elements; then, Δ is an additive derivation.*

Proof. of Corollary 1. Let $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ be the identities of the algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, and let 1 be the identity of the triangular algebra $\mathscr{U}$. We denote

$$P_1 = \begin{pmatrix} 1_{\mathscr{A}} & 0 \\ 0 & 0 \end{pmatrix} \text{ by the standard idempotent of } \mathscr{U}, \quad P_2 = 1 - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathscr{B}} \end{pmatrix}, \tag{85}$$

$$\mathscr{U}_{ij} = P_i \mathscr{U} P_j \quad \text{for } 1 \leq i, j \leq 2. \tag{86}$$

It is clear that the triangular algebra $\mathscr{U}$ may be represented as

$$\mathscr{U} = P_1 \mathscr{U} P_1 + P_1 \mathscr{U} P_2 + P_2 \mathscr{U} P_2 = \mathscr{A} + \mathscr{M} + \mathscr{B}. \tag{87}$$

For any $A \in \mathscr{U}$, let $A = A_{11} + A_{12} + A_{22}$, where $A_{ij} \in \mathscr{U}_{ij}$ $(1 \leq i, j \leq 2)$; then for any $A_{12} \in \mathscr{U}_{12}$, since $\mathscr{U}_{21} = \{0\}$ and by Lemma 1 (iii), we get

$$\Delta(A_{12}) = P_1 \Delta(A_{12}) P_2 + P_2 \Delta(A_{12}) P_1 = P_1 \Delta(A_{12}) P_2. \tag{88}$$

Therefore, we obtain from Remark 3 (iii) that Δ is an additive derivation. The proof is completed. □

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

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