Research Article

On the Norms of RFMLR-Circulant Matrices with the Exponential and Trigonometric Functions

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In this paper, based on combinatorial methods and the structure of RFMLR-circulant matrices, we study the spectral norms of RFMLR-circulant matrices involving exponential forms and trigonometric functions. Firstly, we give some properties of exponential forms and trigonometric functions. Furthermore, we study Frobenius norms, the lower and upper bounds for the spectral norms of RFMLR-circulant matrices involving exponential forms and trigonometric functions by some ingenious algebra methods, and then we obtain new refined results.

1. Introduction

Matrix analysis theory is a powerful tool to study modern communication systems; especially, matrix norm is very important for neural network-based adaptive tracking control for switched nonlinear systems [1]. Recently, studying the norms of special matrices has been a hot topic in matrix theory. Especially, some scholars studied the norms of r -circulant matrices, geometric circulant matrices, and r -Hankel and r -Toeplitz matrices with some famous numbers and polynomials. For example, on the spectral norms of circulant matrices, r -circulant matrices, geometric circulant matrices, and r -Hankel and r -Toeplitz matrices with Fibonacci number, Lucas number, generalized Fibonacci and Lucas numbers, and generalized k -Horadam numbers have been studied [2–8]. We have obtained several results [9, 10] of the norms of matrices mentioned above with exponential forms $e(k/n)$ and trigonometric functions $\cos(k\pi/n)$ and $\sin(k\pi/n)$. There is a RFMLR-circulant matrix applied first by Jiang [11]. As far as we know, it seems that no one has studied the upper and lower estimate problems for the spectral norms of RFMLR-circulant matrices involving exponential forms $e(k/n)$ and trigonometric functions $\cos(k\pi/n)$ and $\sin(k\pi/n)$ yet. Although some scholars at home and abroad have given algorithms about norms of circulant matrices, the

computational complexity of these algorithms is amazing with the increase of matrix order. To overcome this defect, we have constructed matrix factorization and then we study the norms of RFMLR-circulant matrices involving exponential forms and trigonometric functions. Compared with the existing methods, the algorithm model of this study is easy to implement and then we use some ingenious methods to obtain new refined results. All results can be well applied in adaptive feedback control, and these have potential applications in neural network nonlinear system-based norms.

For $e(x)$, $e(x) = e^{2\pi ix}$ and then $|e(x)| = 1$; by $e^{i\theta} = \cos \theta + i \sin \theta$, note that $e(0) = e(1) = e(-1) = e(n) = 1$, and by the trigonometric sums, we have

$$\sum_{k=1}^n e\left(\frac{km}{n}\right) = \begin{cases} n, & n|m, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Particularly, $\sum_{k=0}^{n-1} e(k/n) = 0$, $\cos \theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin \theta = (e^{i\theta} - e^{-i\theta})/2$. By the relationship between exponential forms $e(k/n)$ and trigonometric functions $\cos(k\pi/n)$ and $\sin(k\pi/n)$, we can obtain some power sums of these functions.

A $n \times n$ row first-minus-last right- (RFMLR-) circulant matrix with the first row $(a_0, a_1, \dots, a_{n-1})$, denoted by $A = \text{RFMLRcircfr}(a_0, a_1, \dots, a_{n-1})$, is defined by [11]

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 - a_{n-1} & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} - a_{n-2} & a_0 - a_{n-1} & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 - a_1 & a_3 - a_2 & \cdots & a_{n-1} - a_{n-2} & a_0 - a_{n-1} \end{pmatrix}_{n \times n}. \quad (2)$$

Obviously, the RFMLR-circulant matrix is determined by its first row, and we define $\Theta_{(1,-1)}$ as the basic RFMLR-circulant matrix with the first row $(0, 1, 0, \dots, 0)$, namely,

$$\Theta_{(1,-1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix}_{n \times n}. \quad (3)$$

We can obtain $\Theta_{(1,-1)}^n = I_n - \Theta_{(1,-1)}$. According to the structure of the power of the basic RFMLR-circulant matrix $\Theta_{(1,-1)}$, it is clear that

$$A = \text{RFMLRcircfr}(a_0, a_1, \dots, a_{n-1}) = \sum_{i=0}^{n-1} a_i \Theta_{(1,-1)}^i. \quad (4)$$

Inspired by reference [11], based on preliminary work, in this paper, we shall use identities of exponential forms $e(k/n)$ and trigonometric functions $\cos(k\pi/n)$ and $\sin(k\pi/n)$ and power sums of $e(k/n)$, $\cos(k\pi/n)$, and $\sin(k\pi/n)$ to study the norms of RFMLR-circulant matrices:

$$\begin{aligned} A &= \text{RFMLRcircfr}\left(e\left(\frac{0}{n}\right), e\left(\frac{1}{n}\right), e\left(\frac{3}{n}\right), \dots, e\left(\frac{n-1}{n}\right)\right), \\ B &= \text{RFMLRcircfr}\left(\cos\left(\frac{0 \cdot \pi}{n}\right), \cos\left(\frac{1 \cdot \pi}{n}\right), \cos\left(\frac{2 \cdot \pi}{n}\right), \dots, \cos\left(\frac{(n-1) \cdot \pi}{n}\right)\right), \\ C &= \text{RFMLRcircfr}\left(\sin\left(\frac{0 \cdot \pi}{n}\right), \sin\left(\frac{1 \cdot \pi}{n}\right), \sin\left(\frac{2 \cdot \pi}{n}\right), \dots, \sin\left(\frac{(n-1) \cdot \pi}{n}\right)\right). \end{aligned} \quad (5)$$

We give Frobenius norms, the lower and upper bounds for the spectral norms of these matrices. Some interesting and concise results are stated by the following theorems.

Theorem 1. Let

$$A = \text{RFMLRcircfr}\left(e\left(\frac{0}{n}\right), e\left(\frac{1}{n}\right), e\left(\frac{3}{n}\right), \dots, e\left(\frac{n-1}{n}\right)\right), \quad (6)$$

be a $n \times n$ RFMLR-circulant matrix; then, we have

$$\sqrt{\frac{n+1}{2} + 2(n-1)\sin^2\left(\frac{\pi}{n}\right)} \leq \|A\|_2 \leq 2n-1. \quad (7)$$

Theorem 2. Let

$$B = \text{RFMLRcircfr}\left(\cos\left(\frac{0 \cdot \pi}{n}\right), \cos\left(\frac{1 \cdot \pi}{n}\right), \cos\left(\frac{2 \cdot \pi}{n}\right), \dots, \cos\left(\frac{(n-1) \cdot \pi}{n}\right)\right), \quad (8)$$

be a $n \times n$ RFMLR-circulant matrix; then, we have the following:

If n is even,

$$\sqrt{\frac{\alpha}{n}} \leq \|B\|_2 \leq 2 \cot\left(\frac{\pi}{2n}\right) - 1. \quad (9)$$

If n is odd,

$$\sqrt{\frac{\alpha}{n}} \leq \|B\|_2 \leq 2 \csc\left(\frac{\pi}{2n}\right) - 1, \quad (10)$$

where

$$\alpha = \frac{5n^2 - 8n + 4}{4} + \frac{1}{4} \sec\left(\frac{\pi}{n}\right) - (n^2 - 6n + 6) \cos\left(\frac{\pi}{n}\right). \quad (11)$$

Theorem 3. Let

$$C = \text{RFMLRcircfr}\left(\sin\left(\frac{0 \cdot \pi}{n}\right), \sin\left(\frac{1 \cdot \pi}{n}\right), \sin\left(\frac{2 \cdot \pi}{n}\right), \dots, \sin\left(\frac{(n-1) \cdot \pi}{n}\right)\right), \quad (12)$$

be a $n \times n$ RFMLR-circulant matrix, we have

$$\sqrt{\frac{\beta}{n}} \|C\|_2 \leq 2 \cot\left(\frac{\pi}{2n}\right), \quad (13)$$

where

$$\beta = \frac{2n^2 + 3n - 2}{4} - \frac{1}{8} \sec\left(\frac{\pi}{n}\right). \quad (14)$$

2. Preliminaries

Definition 1 (see [9]). Let any matrix $A = (a_{ij}) \in M_{m \times n}(C)$; the spectral norm and the Euclidean norm of matrix A are defined by

$$\begin{aligned} \|A\|_2 &= \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^H A)}, \\ \|A\|_F &= \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{(1/2)}, \end{aligned} \quad (15)$$

respectively, where $\lambda_i(A^H A)$ is the eigenvalues of matrices $A^H A$ and A^H is the conjugate transpose of A .

The following important inequalities hold between the Frobenius norm and spectral norm:

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F. \quad (16)$$

Lemma 1. For exponential forms $e(k/n)$,

$$\begin{aligned} \sum_{k=0}^{n-1} \left| e\left(\frac{k}{n}\right) \right| &= n, \\ \sum_{k=0}^{n-1} e\left(\frac{k}{n}\right) &= \sum_{k=0}^{n-1} e\left(\frac{-k}{n}\right) = \sum_{k=0}^{n-1} e^2\left(\frac{k}{n}\right) = 0. \end{aligned} \quad (17)$$

Proof. Using the definition of $e(x)$, $e(x) = e^{2\pi i x}$ and $e^{i\theta} = \cos \theta + i \sin \theta$, we have $|e(x)| = 1$, namely, $\sum_{k=0}^{n-1} |e(k/n)| = n$.

$e(k/n)$ is a geometric sequence, the common ratio is $e(1/n)$, and $e(x)e(y) = e(x+y)$, so we have

$$\sum_{k=0}^{n-1} e\left(\frac{k}{n}\right) = \sum_{k=0}^{n-1} e\left(\frac{-k}{n}\right) = \sum_{k=0}^{n-1} e^2\left(\frac{k}{n}\right) = 0. \quad (18)$$

□

Lemma 2. For any positive integer $n \geq 2$, we have

$$\begin{aligned} \sum_{k=0}^{n-1} \cos\left(\frac{k\pi}{n}\right) &= 1, \\ \sum_{k=0}^{n-1} \sin\left(\frac{k\pi}{n}\right) &= \cot\left(\frac{\pi}{2n}\right), \\ \sum_{k=0}^{n-1} \cos^2\left(\frac{k\pi}{n}\right) &= \sum_{k=0}^{n-1} \sin^2\left(\frac{k\pi}{n}\right) = \frac{n}{2}. \end{aligned} \quad (19)$$

Proof. By the relationship between exponential forms and trigonometric functions, $\cos \theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin \theta = (e^{i\theta} - e^{-i\theta})/2$, and using Lemma 1, we can obtain sums mentioned above. □

Lemma 3. If n is even,

$$\sum_{j=1}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| = \cot\left(\frac{\pi}{2n}\right) - 1. \quad (20)$$

If n is odd,

$$\sum_{j=1}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| = \csc\left(\frac{\pi}{2n}\right) - 1. \quad (21)$$

Proof. If n is even,

$$\begin{aligned} \sum_{j=0}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| &= 2 \sum_{j=1}^{(n/2)-1} \cos\left(\frac{j\pi}{n}\right) + 1 = 2 \sum_{j=0}^{(n/2)-1} \cos\left(\frac{j\pi}{n}\right) - 1, \\ \sum_{j=0}^{(n/2)-1} \cos\left(\frac{j\pi}{n}\right) &= \sum_{j=0}^{(n/2)-1} \frac{e(j/2n) + e(-j/2n)}{2} = \frac{1}{2} \left(\frac{1 - e(1/4)}{1 - e(1/2n)} + \frac{1 - e(-1/4)}{1 - e(-1/2n)} \right) \\ &= \frac{1}{2} \frac{2 - e(1/4) - e(-1/4) - e(1/2n) - e(-1/2n) + e((1/2n) - (1/4)) + e((1/4) - (1/2n))}{2 - e(1/2n) - e(-1/2n)} \\ &= \frac{1}{2} \frac{1 - \cos(\pi/n) + \sin(\pi/n)}{1 - \cos(\pi/n)} \\ &= \frac{1}{2} \left(1 + \cot\left(\frac{\pi}{2n}\right) \right). \end{aligned} \quad (22)$$

If n is odd,

$$\begin{aligned}
 \sum_{j=0}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| &= \sum_{j=1}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| + 1 = 2 \sum_{j=1}^{(n-1)/2} \cos\left(\frac{j\pi}{n}\right) + 1 \\
 &= 2 \sum_{j=0}^{(n-1)/2} \cos\left(\frac{j\pi}{n}\right) - 1 \\
 &= \sum_{j=0}^{(n-1)/2} \left(e\left(\frac{j}{2n}\right) + e\left(\frac{-j}{2n}\right) \right) - 1 \\
 &= \frac{1 - \cos(\pi/n) - \cos((n+1)\pi/2n) + \cos((n-1)\pi/2n)}{1 - \cos(\pi/n)} - 1 \\
 &= \frac{\cos((n-1)\pi/2n) - \cos((n-1)\pi/2n)}{1 - \cos(\pi/n)} \\
 &= \csc\left(\frac{\pi}{2n}\right).
 \end{aligned} \tag{23}$$

3. Proofs of Theorems

The following is the proof of Theorem 1.

Proof. The matrix

$$A = \text{RFMLRcircfr}\left(e\left(\frac{0}{n}\right), e\left(\frac{1}{n}\right), e\left(\frac{3}{n}\right), \dots, e\left(\frac{n-1}{n}\right)\right), \quad \square \tag{24}$$

is of the following form:

$$A = \begin{pmatrix} e\left(\frac{0}{n}\right) & e\left(\frac{1}{n}\right) & e\left(\frac{2}{n}\right) & \cdots & e\left(\frac{n-2}{n}\right) & e\left(\frac{n-1}{n}\right) \\ e\left(\frac{n-1}{n}\right) & e_1 & e\left(\frac{1}{n}\right) & \cdots & e\left(\frac{n-3}{n}\right) & e\left(\frac{n-2}{n}\right) \\ e\left(\frac{n-2}{n}\right) & e_2 & e_1 & \cdots & e\left(\frac{n-4}{n}\right) & e\left(\frac{n-3}{n}\right) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ e\left(\frac{1}{n}\right) & e\left(\frac{2}{n}\right) - e\left(\frac{1}{n}\right) & e\left(\frac{3}{n}\right) - e\left(\frac{2}{n}\right) & \cdots & e_2 & e_1 \end{pmatrix}_{n \times n}, \tag{25}$$

where

$$\begin{aligned}
 e_1 &= e\left(\frac{0}{n}\right) - e\left(\frac{n-1}{n}\right), \\
 e_2 &= e\left(\frac{n-1}{n}\right) - e\left(\frac{n-2}{n}\right).
 \end{aligned} \tag{26}$$

Using the definition of Frobenius norm and Lemma 1, $a_k = e(k/n)$, we have

$$\begin{aligned}
 \|A\|_F^2 &= \sum_{k=0}^{n-1} |a_k|^2 + (n-1)|a_1|^2 + (n-2)|a_2|^2 + \cdots + |a_{n-1}|^2 + \\
 &\quad (n-1)|a_0 - a_{n-1}|^2 + (n-2)|a_{n-1} - a_{n-2}|^2 + \cdots + |a_2 - a_1|^2,
 \end{aligned} \tag{27}$$

and then $|a_k| = |e(k/n)| = 1$,

$$\begin{aligned} |a_0 - a_{n-1}|^2 &= \left| e\left(\frac{0}{n}\right) - e\left(\frac{n-1}{n}\right) \right|^2 = \left| e\left(\frac{0}{n}\right) - e\left(\frac{-1}{n}\right) \right|^2 = \left| e\left(\frac{1}{n}\right) - 1 \right|^2, \\ |a_{n-1} - a_{n-2}|^2 &= \left| e\left(\frac{n-1}{n}\right) - e\left(\frac{n-2}{n}\right) \right|^2 = \left| e\left(\frac{-1}{n}\right) - e\left(\frac{-2}{n}\right) \right|^2 = \left| e\left(\frac{1}{n}\right) - 1 \right|^2, \\ \left| e\left(\frac{1}{n}\right) - 1 \right|^2 &= \left| \cos\left(\frac{2\pi}{n}\right) - 1 + i \sin\left(\frac{2\pi}{n}\right) \right|^2 = 4 \sin^2\left(\frac{\pi}{n}\right), \end{aligned} \quad (28)$$

that is to say,

$$\|A\|_F^2 = \frac{n(n+1)}{2} + 2n(n-1)\sin^2\left(\frac{\pi}{n}\right). \quad (29)$$

Using $\|A\|_2 \geq (1/\sqrt{n})\|A\|_F$, we can obtain the lower bound

$$\|A\|_2 \geq \sqrt{\frac{n+1}{2} + 2(n-1)\sin^2\left(\frac{\pi}{n}\right)}. \quad (30)$$

In another case, let the matrices Q_1 , Q_2 , and Q_3 be defined by

$$\begin{aligned} Q_1 &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}, \\ Q_2 &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{n \times n}, \\ Q_3 &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{n \times n}. \end{aligned} \quad (31)$$

Then, we can obtain $A = \sum_{k=0}^{n-1} a_k Q_1^k - \sum_{k=1}^{n-2} a_{n-k-1} Q_2^k - a_{n-1} Q_3$, by identities of matrix norms,

$$\begin{aligned} \|A\|_2 &= \left\| \sum_{k=0}^{n-1} a_k Q_1^k - \sum_{k=1}^{n-2} a_{n-k-1} Q_2^k - a_{n-1} Q_3 \right\|_2 \\ &\leq \sum_{k=0}^{n-1} |a_k| \|Q_1\|_2^k + \sum_{k=1}^{n-2} |a_{n-k-1}| \|Q_2\|_2^k + |a_{n-1}| \|Q_3\|_2. \end{aligned} \quad (32)$$

Since

$$\begin{aligned} Q_1^H Q_1 &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}, \\ Q_2^H Q_2 &= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}, \\ Q_3^H Q_3 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}. \end{aligned} \quad (33)$$

Hence,

$$\begin{aligned} \|Q_1\|_2 &= \|Q_2\|_2 = \|Q_3\|_2 = 1, \\ \|A\|_2 &\leq \sum_{k=0}^{n-1} |a_k| + \sum_{k=1}^{n-2} |a_{n-k-1}| + |a_{n-1}|, \end{aligned} \quad (34)$$

and by $|a_k| = |e(k/n)| = 1$, $\|A\|_2 \leq 2n - 1$; thus,

$$\sqrt{\frac{n+1}{2} + 2(n-1)\sin^2\left(\frac{\pi}{n}\right)} \leq \|A\|_2 \leq 2n - 1. \quad (35)$$

This proves Theorem 1.

Now, we prove Theorem 2. $\square$

Proof

$$B = \begin{pmatrix} \cos\left(\frac{0\pi}{n}\right) & \cos\left(\frac{1\pi}{n}\right) & \cdots & \cos\left(\frac{(n-2)\pi}{n}\right) & \cos\left(\frac{(n-1)\pi}{n}\right) \\ \cos\left(\frac{(n-1)\pi}{n}\right) & m_1 & \cdots & \cos\left(\frac{(n-3)\pi}{n}\right) & \cos\left(\frac{(n-2)\pi}{n}\right) \\ \cos\left(\frac{(n-2)\pi}{n}\right) & m_2 & \cdots & \cos\left(\frac{(n-4)\pi}{n}\right) & \cos\left(\frac{(n-3)\pi}{n}\right) \\ \vdots & \vdots & & \vdots & \vdots \\ \cos\left(\frac{1\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) - \cos\left(\frac{1\pi}{n}\right) & \cdots & m_2 & m_1 \end{pmatrix}_{n \times n}, \quad (36)$$

where

$$\begin{aligned} m_1 &= \cos\left(\frac{0\pi}{n}\right) - \cos\left(\frac{(n-1)\pi}{n}\right), \\ m_2 &= \cos\left(\frac{(n-1)\pi}{n}\right) - \cos\left(\frac{(n-2)\pi}{n}\right). \end{aligned} \quad (37)$$

Using the Frobenius norms and Theorem 1, as well as $b_j = \cos(j\pi/n)$, we have

$$\begin{aligned} \|B\|_F^2 &= n \sum_{j=0}^{n-1} b_j^2 + \sum_{j=1}^{n-1} j b_j^2 - 2 \sum_{j=1}^{n-2} j b_j b_{j+1} - 2(n-1)b_0 b_{n-1} \\ &= n \sum_{j=0}^{n-1} b_j^2 + \sum_{k=1}^{n-1} \sum_{j=n-k}^{n-1} b_j^2 - 2 \sum_{k=1}^{n-2} \sum_{j=n-k-1}^{n-2} b_j b_{j+1} - 2(n-1)b_0 b_{n-1} \\ &= n \sum_{j=0}^{n-1} b_j^2 + \sum_{k=1}^{n-1} \left(\sum_{j=0}^{n-1} b_j^2 - \sum_{j=0}^{n-k-1} b_j^2 \right) - 2 \sum_{k=1}^{n-2} \left(\sum_{j=0}^{n-2} b_j b_{j+1} - \sum_{j=0}^{n-k-2} b_j b_{j+1} \right) - 2(n-1)b_0 b_{n-1}. \end{aligned} \quad (38)$$

By Lemma 1, $\sum_{j=0}^{n-1} \cos^2(j\pi/n) = (n/2)$. Using the identities $\cos \theta = (e^{i\theta} + e^{-i\theta})/2$,

$$\begin{aligned} e(x) &= e^{2\pi i x}, \\ \cos\left(\frac{(2j+1)\pi}{n}\right) &= \frac{e((2j+1)/2n) + e(-(2j+1)/2n)}{2}, \end{aligned} \quad (39)$$

we can obtain

$$\begin{aligned}\sum_{j=0}^{n-1} \cos\left(\frac{(2j+1)\pi}{n}\right) &= \frac{1}{2} \sum_{j=0}^{n-1} \left(e\left(\frac{2j+1}{2n}\right) + e\left(-\frac{2j+1}{2n}\right) \right) \\ &= \frac{1}{2} \left(\frac{e(1/2n)(1-e(1))}{1-e(1/n)} + \frac{e(1/2n)(1-e(-1))}{1-e(-1/n)} \right) \\ &= 0,\end{aligned}\tag{40}$$

$$\begin{aligned}\sum_{j=0}^{n-1} \cos\left(\frac{j\pi}{n}\right) \cos\left(\frac{(j+1)\pi}{n}\right) &= \frac{1}{2} \sum_{j=0}^{n-1} \cos\left(\frac{2(j+1)\pi}{n}\right) + \frac{n}{2} \cos\left(\frac{\pi}{n}\right) \\ &= \frac{n}{2} \cos\left(\frac{\pi}{n}\right).\end{aligned}$$

Hence,

$$\sum_{j=0}^{n-2} \cos\left(\frac{j\pi}{n}\right) \cos\left(\frac{(j+1)\pi}{n}\right) = \frac{n-2}{2} \cos\left(\frac{\pi}{n}\right), \tag{41}$$

and then, we can obtain

$$\sum_{k=1}^{n-2} \sum_{j=0}^{n-k-2} \cos\left(\frac{j\pi}{n}\right) \cos\left(\frac{(j+1)\pi}{n}\right) = \frac{1}{8} \sec\left(\frac{\pi}{n}\right) + \frac{(n-1)(n-2)}{4}.\tag{42}$$

Therefore,

$$\|B\|_F^2 = \frac{5n^2 - 8n + 4}{4} + \frac{1}{4} \sec\left(\frac{\pi}{n}\right) - (n^2 - 6n + 6) \cos\left(\frac{\pi}{n}\right).\tag{43}$$

We can obtain $\|B\|_2 \geq (1/\sqrt{n})\|B\|_F = \sqrt{\alpha/n}$, where

$$\alpha = \frac{5n^2 - 8n + 4}{4} + \frac{1}{4} \sec\left(\frac{\pi}{n}\right) - (n^2 - 6n + 6) \cos\left(\frac{\pi}{n}\right).\tag{44}$$

In another case, using Theorem 1, and for the matrices Q_1 , Q_2 , and Q_3 as mentioned above, we have

$$\begin{aligned}B &= \sum_{j=0}^{n-1} b_j Q_1^j - \sum_{j=1}^{n-2} b_{n-j-1} Q_2^j - b_{n-1} Q_3, \\ \|B\|_2 &= \left\| \sum_{j=0}^{n-1} b_j Q_1^j - \sum_{j=1}^{n-2} b_{n-j-1} Q_2^j - b_{n-1} Q_3 \right\|_2 \\ &\leq \sum_{j=0}^{n-1} |b_j| \|Q_1\|_2^j + \sum_{j=1}^{n-2} |b_{n-j-1}| \|Q_2\|_2^j + |b_{n-1}| \|Q_3\|_2 \\ &= \sum_{j=0}^{n-1} |b_j| + \sum_{j=1}^{n-2} |b_{n-j-1}| + |b_{n-1}| \\ &= \sum_{j=0}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| + \sum_{j=1}^{n-2} \left| \cos\left(\frac{(n-j-1)\pi}{n}\right) \right| + \left| \cos\left(\frac{(n-1)\pi}{n}\right) \right| \\ &= \sum_{j=0}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| + \sum_{j=1}^{n-2} \left| \cos\left(\frac{(j+1)\pi}{n}\right) \right| + \left| \cos\left(\frac{\pi}{n}\right) \right| \\ &= 2 \sum_{j=0}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| - 1.\end{aligned}\tag{45}$$

By Lemma 3, if n is even,

$$\sum_{j=0}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| = \cot\left(\frac{\pi}{2n}\right). \quad (46)$$

Therefore, if n is even, $\|B\|_2 \leq 2 \cot(\pi/2n) - 1$.
If n is odd,

$$\sum_{j=0}^{n-1} \left| \cos\left(\frac{j\pi}{n}\right) \right| = \csc\left(\frac{\pi}{2n}\right). \quad (47)$$

Therefore, if n is odd, $\|B\|_2 \leq 2 \csc(\pi/2n) - 1$.

This proves Theorem 2.

Now, we prove Theorem 3. $\square$

Proof

$$C = \begin{pmatrix} \sin\left(\frac{0\pi}{n}\right) & \sin\left(\frac{1\pi}{n}\right) & \cdots & \sin\left(\frac{(n-2)\pi}{n}\right) & \sin\left(\frac{(n-1)\pi}{n}\right) \\ \sin\left(\frac{(n-1)\pi}{n}\right) & s_1 & \cdots & \sin\left(\frac{(n-3)\pi}{n}\right) & \sin\left(\frac{(n-2)\pi}{n}\right) \\ \sin\left(\frac{(n-2)\pi}{n}\right) & s_2 & \cdots & \sin\left(\frac{(n-4)\pi}{n}\right) & \sin\left(\frac{(n-3)\pi}{n}\right) \\ \vdots & \vdots & & \vdots & \vdots \\ \sin\left(\frac{1\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) - \sin\left(\frac{1\pi}{n}\right) & \cdots & s_2 & s_1 \end{pmatrix}_{n \times n}, \quad (48)$$

where

$$\begin{aligned} s_1 &= \sin\left(\frac{0\pi}{n}\right) - \sin\left(\frac{(n-1)\pi}{n}\right), \\ s_2 &= \sin\left(\frac{(n-1)\pi}{n}\right) - \sin\left(\frac{(n-2)\pi}{n}\right). \end{aligned} \quad (49)$$

Using the Frobenius norms and Theorem 1, as well as $c_j = \sin(j\pi/n)$, we have

$$\begin{aligned} \|C\|_F^2 &= n \sum_{j=0}^{n-1} c_j^2 + \sum_{j=1}^{n-1} j c_j^2 - 2 \sum_{j=1}^{n-2} j c_j c_{j+1} - 2(n-1)c_0 c_{n-1} \\ &= n \sum_{j=0}^{n-1} c_j^2 + \sum_{k=1}^{n-1} \sum_{j=n-k}^{n-1} c_j^2 - 2 \sum_{k=1}^{n-2} \sum_{j=n-k-1}^{n-2} c_j c_{j+1} \\ &= n \sum_{j=0}^{n-1} c_j^2 + \sum_{k=1}^{n-1} \left(\sum_{j=0}^{n-1} c_j^2 - \sum_{j=0}^{n-k-1} c_j^2 \right) \\ &\quad - 2 \sum_{k=1}^{n-2} \left(\sum_{j=0}^{n-2} c_j c_{j+1} - \sum_{j=0}^{n-k-2} c_j c_{j+1} \right). \end{aligned} \quad (50)$$

By Lemma 1, $\sum_{j=0}^{n-1} \sin^2(j\pi/n) = (n/2)$. Using the identities

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2},$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

$$e(x) = e^{2\pi i x},$$

$$\begin{aligned} \sum_{j=0}^{n-2} \sin\left(\frac{j\pi}{n}\right) \sin\left(\frac{(j+1)\pi}{n}\right) &= \frac{1}{2} \sum_{j=0}^{n-2} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{2(j+1)\pi}{n}\right) \right) \\ &= \frac{n}{2} \cos\left(\frac{\pi}{n}\right), \end{aligned} \quad (51)$$

and by Theorem 1, we can obtain

$$\begin{aligned} &\sum_{k=1}^{n-2} \sum_{j=0}^{n-k-2} \sin\left(\frac{j\pi}{n}\right) \sin\left(\frac{(j+1)\pi}{n}\right) \\ &= \sum_{k=1}^{n-2} \frac{n-k-1}{2} \cos\left(\frac{\pi}{n}\right) - \frac{1}{2} \sum_{k=1}^{n-2} \sum_{j=0}^{n-k-2} \cos\left(\frac{2(j+1)\pi}{n}\right) \\ &= \frac{n(n-2)}{2} \cos\left(\frac{\pi}{n}\right) - \frac{1}{16} \sec\left(\frac{\pi}{n}\right) - \frac{(n-1)(n-2)}{8}. \end{aligned} \quad (52)$$

Therefore,

$$\|C\|_F^2 = \frac{2n^2 + 3n - 2}{4} - \frac{1}{8} \sec\left(\frac{\pi}{n}\right). \quad (53)$$

We can obtain $\|C\|_2 \geq (1/\sqrt{n})\|C\|_F = \sqrt{(\beta/n)}$, where

$$\beta = \frac{2n^2 + 3n - 2}{4} - \frac{1}{8} \sec\left(\frac{\pi}{n}\right). \quad (54)$$

In another case, using Theorem 1, and for the matrices Q_1 , Q_2 , and Q_3 as mentioned above, by $c_j = \sin(j\pi/n) \geq 0$, $\|Q_1\|_2 = \|Q_2\|_2 = \|Q_3\|_2 = 1$. We have

$$\begin{aligned} C &= \sum_{j=0}^{n-1} c_j Q_1^j - \sum_{j=1}^{n-2} c_{n-j-1} Q_2^j - c_{n-1} Q_3, \\ \|C\|_2 &= \left\| \sum_{j=0}^{n-1} c_j Q_1^j - \sum_{j=1}^{n-2} c_{n-j-1} Q_2^j - c_{n-1} Q_3 \right\|_2 \\ &\leq \sum_{j=0}^{n-1} |c_j| \|Q_1\|_2^j + \sum_{j=1}^{n-2} |c_{n-j-1}| \|Q_2\|_2^j + |c_{n-1}| \|Q_3\|_2 \\ &= \sum_{j=0}^{n-1} |c_j| + \sum_{j=1}^{n-2} |c_{n-j-1}| + |c_{n-1}| \\ &= \sum_{j=0}^{n-1} \sin\left(\frac{j\pi}{n}\right) + \sum_{j=1}^{n-2} \sin\left(\frac{(n-j-1)\pi}{n}\right) + \sin\left(\frac{(n-1)\pi}{n}\right) \\ &= 2 \sum_{j=0}^{n-1} \sin\left(\frac{j\pi}{n}\right) \\ &= 2 \cot\left(\frac{\pi}{2n}\right). \end{aligned} \quad (55)$$

Thus, the result is obtained as follows:

$$\sqrt{\frac{\beta}{n}} \leq \|C\|_2 \leq 2 \cot\left(\frac{\pi}{2n}\right), \quad (56)$$

where

$$\beta = \frac{2n^2 + 3n - 2}{4} - \frac{1}{8} \sec\left(\frac{\pi}{n}\right). \quad (57)$$

This completes all of the theorems. $\square$

4. Conclusion

The spectral norms of RFMLR-circulant matrices involving exponential forms and trigonometric functions are investigated in this paper. The computation complexity of this paper is lower than the previous work. By using the algorithms of this paper, we can further study the identities of $f(x)$ -circulant matrices, such as RFP r L r R-circulant matrix and RFML r R-circulant matrix. Simulation is very important for applications of our work, and these will be our further topics to study.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares conflicts of interest.

Authors' Contributions

The author contributed to each part of this work seriously and read and approved the final version of the manuscript.

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