Research Article

Existence of Limit Cycles for a Class of Quartic Polynomial Differential System Depending on Parameters

Sarah Abdullah Qadha ^{1,2}, Muneera Abdullah Qadha ^{1,2} and Haibo Chen¹

¹School of Mathematics and Statistics, Central South University, Changsha 410083, China

²Department of Mathematics, Faculty of Education at Al-Mahweet, Sana'a University, Al-Mahweet, Yemen

Correspondence should be addressed to Sarah Abdullah Qadha; sarah_qadha@csu.edu.cn

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We studied the existence of limit cycles for the quartic polynomial differential systems depending on parameters. To prove that, first, we used the formal series method based on Poincaré's ideas to determine the center-focus. Then, by the Hopf bifurcation theory, we obtained the sufficient condition for the existence of the limit cycles. Finally, we provided some numerical examples for illustration.

1. Introduction

In the twentieth century, progress in applied electronics has been rapid. Physicists invented the triode vacuum tube, making stable self-excited oscillations of constant amplitude. Thus, propagating sound and pictures through electronics became possible. However, it was not possible to describe this oscillation phenomenon by linear differential equations. In [1] 1926, van der Pol first obtained a differential equation to describe oscillations of the constant amplitude of a triode vacuum tube, as follows:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0. \quad (1)$$

This was the main reason for the appearance of the nonlinear equations and qualitative theory. One of the main issues in analysing the qualitative theory of differential equations is the limit cycle. The limit cycle was discovered by Poincaré (1881–1886). Recently, the problem of limit cycles has become increasingly critical and has attracted the attention of many pure and applied mathematicians, see for instance [1–10]. The necessary problems in the qualitative theory of differential equation include the existence of the limit cycle.

Existence, nonexistence, uniqueness, and other properties of limit cycles have been studied extensively by mathematicians and physicists. Then, since the 1950s, many mathematical models in physics, engineering, chemistry, biology, and economics have been displayed as autonomous plane systems with limit cycles. Many researchers have studied the existence of limit cycles for the quadratic and cubic systems, and numerous research achievements have been obtained, see for instance [3, 5, 7, 10–18].

For examples, consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= P(x, y), \\ \frac{dy}{dt} &= Q(x, y), \end{aligned} \quad (2)$$

where $P(x, y)$ and $Q(x, y)$ denote homogeneous polynomial of degree n and real variable x, y and some parameter.

Consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= -y + P_n(x, y), \\ \frac{dy}{dt} &= x + Q_n(x, y), \end{aligned} \quad (3)$$

where $P_n(x, y)$ and $Q_n(x, y)$ are polynomials of degree n , with real coefficients without constant terms. System (3) always has either a center or fine (weak) focus at the origin. Distinguishing between the center and a focus at the origin is the main problem.

We know that linear differential system $(dx/dt) = -y, (dy/dt) = x$ can have the center. However, the perturbation of this center inside the class of linear differential systems cannot produce a limit cycle because the linear differential system cannot have an isolated periodic solution in the set of all periodic solutions:

- (1) When $n=2$, system (3) is the quadratic polynomial differential system whose centers have been completely classified. Many papers study how many limit cycles can bifurcate from the periodic orbits of these centers. As an example, consider the following system:

$$\begin{aligned}\frac{dx}{dt} &= -y(1+x+\alpha y), \\ \frac{dy}{dt} &= x + (\lambda + \beta + \gamma)y + ax^2 \\ &\quad + (\alpha + \beta + \gamma)xy + cy^2.\end{aligned}\quad (4)$$

In [19], the authors obtained a quadratic system that has at most four limit cycles and only in the (3:1) distribution when $a = (1/2), c = -1$.

- (2) When $n=3$, system (3) is a cubic system.

In [20], Malkin obtained the necessary and sufficient conditions for system (3) for the origin to be the center. In [21], the authors proved that these conditions were satisfactory for the integrability of the system. In [22], Liu derived the formulas of the first five focus values of the origin for system (3) and proved that the origin is a center if the five focus values are all zero. This finding implies that a central symmetric cubic system with respect to the origin has at most five small amplitude limit cycles around the origin (see [23]).

Consider the following system:

$$\begin{aligned}\frac{dx}{dt} &= -y(1-ax) + (1-bx) + a_1x + a_2x^2 + a_3x^3, \\ \frac{dy}{dt} &= x(1-cx)(1-dx).\end{aligned}\quad (5)$$

In [24], Quan obtained the sufficient and necessary condition for the existence of a limit cycle in the case $a_2 = 0, a = c$, and $b = d$. The authors of [25] studied the condition for the existence of limit cycles in the system when $a = 0$ and $b = d$. One article [26] showed that the case $a = c$ and $b = d = 0$ has no limit cycle.

In this paper, we will study the existence of limit cycles for the following quartic polynomial differential systems depending on parameters:

$$\begin{aligned}\frac{dx}{dt} &= -y + kx^2 + lxy, \\ \frac{dy}{dt} &= x + mx^2y + ny^4,\end{aligned}\quad (6)$$

where k, l, m , and n are parameters, in which the linear system is the center. If the singular point for nonlinear is the center, quartic system (6) does not have a limit cycle. If the singular point for nonlinear systems is the focus or node, then quartic system (6) has the limit cycle. System (6) only has one singular point on the origin. It does not have any singular point on the neighborhood; so, the origin (the singular point) is isolated.

In the first section of this paper, we presented some necessary definitions and theorem. In the second section, we used a formal series method based on Poincaré's ideas to determine the center-focus. By using the Hopf bifurcation theory, we obtained the sufficient condition for the existence of limit cycle for quartic system (6), which bifurcated from the equilibrium point (singular point). We also obtained sufficient conditions for the existence of the limit cycle according to the change analysis of the stability of the focus when the parameters change. Finally, we provided some examples for illustration.

2. Some Preliminary Results

In this section, we introduce some definitions and notations that we will need for the existence of the quartic differential system (6).

Theorem 1 (see [1]). *Closed trajectories of different vector fields of a complete family do not intersect.*

Theorem 2 (Bendixson's criterion) (see [27]). *If P_x, Q_y is continuous in R which is simply connected and $\text{div } \vec{F} = (\partial P/\partial x) + (\partial Q/\partial y) \neq 0$ at any point of R , then the system has no closed trajectories inside R .*

Definition 1 (Hopf bifurcation of order k) (see [28], page 385). Consider the following system:

$$\begin{aligned}\frac{dx}{dt} &= P(x, y, \eta), \\ \frac{dy}{dt} &= Q(x, y, \eta),\end{aligned}\quad (7)$$

where $x, y \in \mathbb{R}^1, \eta \in \mathbb{R}^n$, and $P, Q \in C^\infty$, when two eigenvalues $\lambda_{1,2} = a(\eta) \pm ib(\eta)$ of the linear part of the system above (7) cross the imaginary axis. We say that system (7) has a *Hopf bifurcation of order k* ($k > 1$) at the origin if

- (1) $a(0) = 0$
- (2) $b(0) = b_0 \neq 0$
- (3) $\text{Re}(C_1) = \text{Re}(C_2) = \dots = \text{Re}(C_{k-1}) = 0, \quad \text{Re}(C_k) \neq 0$

where $C_1, \dots, C_k$ are the coefficients of Poincaré-Lyapunov constants.

In this case, we also say that the origin is a weak focus of order k .

Note: In two dimensions, when the focus Point switches from stable to unstable (or vice versa), then a periodic solution appears and a Hopf bifurcation occurs. [29].

Theorem 3 (see [28], page 385). If we let system (2) be a C^∞ with an equilibrium $(0, 0)$, that is, a weak focus of order k , then

- (1) If $n \geq k$, and $(5)_{\eta=0} = (2)$, then there is $\delta > 0$ and a neighborhood Γ of $(x, y) = (0, 0)$ exists. $|\eta| < \delta$, (5) has at most k limit cycles in neighborhood Γ
- (2) For any integer i , $1 \leq i \leq k$ and a neighborhood $\Gamma^* \subset \Gamma$ of $(x, y) = (0, 0)$, there exists a system of the form $(5)_{\eta=0}$ with $(5)_{\eta=0} = (2)$, and a number $\delta^* > 0$, such that $(5)_{\eta=0}$ has exactly i limit cycles in Γ^* for $\eta \in S$, where S is an open subset of $(\eta/0 < |\eta| < \delta^*)$ and $0 \in \bar{S}$

Remark 1 (see [27]). The linear part of system (6) has closed trajectories because the linear system has $\text{Tr } A = P_x(A) + Q_y(A) = 0$ and $\det A = P_x(A)Q_y(A) - P_y(A)Q_x(A) = 1 > 0$ where A is a singular point for system. λ_1 and λ_2 are two conjugate pure imaginary roots. Then, the linear system (6) can be transformed into the following:

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y}, & \text{then } \dot{r} &= 0, \\ r^2\dot{\theta} &= x\dot{y} - y\dot{x}, & \text{then } \dot{\theta} &= 1, \end{aligned} \quad (8)$$

whose solution is as follows:

$$\begin{aligned} r &= r_0 \geq 0, \\ \theta &= t + \theta_0. \end{aligned} \quad (9)$$

The orbit form has a family of closed curves around the origin A . The singular point is called a center. The simulation is presented in Figure 1.

3. Main Results

3.1. Determine the Center-Focus and It Is Stable

Theorem 4. For system (6), we have the following:

- (1) If $(2kl - m) \neq 0, kl \neq 0, ((m + kl)/4) < 0$, then $A(0, 0)$ is an unstable first-order weak focus
- (2) If $(2kl - m) \neq 0, kl \neq 0, ((m + kl)/4) > 0$, then $A(0, 0)$ is a stable first-order weak focus
- (3) If $(2kl - m) = 0, kl = 0$, then the point $A(0, 0)$ is the center point
- (4) If $kl = 0, (2kl - m) > 0$, then the point $A(0, 0)$ is an unstable second-order weak focus
- (5) If $kl = 0, (2kl - m) < 0$, then the point $A(0, 0)$ is a stable second-order weak focus
- (6) If $(2kl - m) = 0, kl < 0$, then the point $A(0, 0)$ is an unstable third-order weak focus
- (7) If $(2kl - m) = 0, kl > 0$, then the point $A(0, 0)$ is a stable third-order weak focus

Proof. By using the formal series method based on Poincaré ideas, we can easily obtain these results of this theorem. The details are omitted here. $\square$

FIGURE 1: Center.

3.2. The Nonexistence of Limit Cycle

Theorem 5. If $(2kl - m) = 0, kl = 0$, then system (6) has no limit cycle in the whole plane.

Proof. When the conditions above hold, then the point $A(0, 0)$ is the center point. And, when $kl = 0, m = 0$, then $P(-x, y) = P(x, y), Q(-x, y) = -Q(x, y)$. The vector field $(P(x, y), Q(x, y))$ is symmetrical with respect to y -axis. So, there is no limit cycle in the neighborhood of point $A(0, 0)$ when the conditions of Theorem 5 hold. The simulation is presented in Figure 2. $\square$

3.3. Existence of Limit Cycles

Theorem 6. When one of the following conditions holds, system (6) has at least one limit cycle around $A(0, 0)$:

- (1) $((kl + m)/4) < 0, kl \neq 0, (2kl - m) \neq 0, -1 \ll m < 0$
- (2) $((kl + m)/4) > 0, kl \neq 0, (2kl - m) \neq 0, 0 < m \ll 1$
- (3) $(2kl - m) > 0, kl = 0, -1 \ll m < 0$
- (4) $(2kl - m) < 0, kl = 0, 0 < m \ll 1$
- (5) $(m - 2kl) = 0, kl < 0, -1 \ll m < 0$
- (6) $(m - 2kl) = 0, kl > 0, 0 < m \ll 1$

Proof. For conditions (1), (3), and (5), the singular point $A(0, 0)$ is an unstable weak focus point of system (6). When m decreases from zero, the singular point $A(0, 0)$ of system (6) changes from an unstable weak focus to a stable weak focus. According to the Hopf bifurcation theory, these parameters change because when the stability changes, system (6) generates at least one unstable limit cycle in the neighborhood of $A(0, 0)$.

For condition (2), (4), and (6), the singular point $A(0, 0)$ is the stable weak focus point of system (6). When m increases from zero, the singular point $A(0, 0)$ of system (6) changes from a stable weak focus to an unstable weak focus, and according to the Hopf bifurcation theory, under these parameters, system (6) generates at least one stable limit cycle in the neighborhood of $A(0, 0)$. $\square$

Theorem 7. System (6) has at least three limit cycles bifurcating from the origin $A(0, 0)$

FIGURE 2: Nonexistence of limit cycle for system (6).

FIGURE 4: Existence of limit cycle of system (6), when $k = 0.2$, $l = -0.2$, $m = -0.06$, and $n = 1$. FIGURE 3: Existence of limit cycle for system (6), when $k = 1$, $l = 1$, $m = 1$, $n = 1$.

Proof. From Theorem 4, we determine that system (6) has three weak focuses. We denote the number of weak focus as k , and $k = 3$. System (6) is a class quartic; thus, we denote n as $n = 4$. According to Theorem 3, we determine that system (6) has three limit cycles around $A(0, 0)$. $\square$

3.4. Numerical Solution

Example 1. In system (6), if we take the values of the parameters as $k = 1$, $l = 1$, $m = 1$, and $n = 1$, then a limit cycle exists. The simulation is presented in Figure 3.

Example 2 (the existence). In system (6), if we take values of the parameters as $k = 0.2$, $l = -0.2$, $m = -0.06$, and $n = 1$, then by Theorem 6 Case 1, there exists a limit cycle around $A(0, 0)$. The simulation is presented in Figure 4.

In system (6), if we take values of the parameters as $k = -1$, $l = 1$, $m = 1$, and $n = 1$, then by Theorem 6 Case 2, there exists a limit cycle around $A(0, 0)$. The simulation is presented in Figure 5.

In system (6), if we take values of the parameters as $k = 1$, $l = 0$, $m = -0.2$, and $n = 1$, then by Theorem 6 Case 3, there exists a limit cycle around $A(0, 0)$. The simulation is presented in Figure 6.

In system (6), if we take values of the parameters as $k = -1$, $l = 0$, $m = 1$, and $n = 1$, then by Theorem 6 Case 4,

FIGURE 5: Existence of limit cycle of system (6), when $k = -1$, $l = 1$, $m = 1$, and $n = 1$. FIGURE 6: Existence of limit cycle of system (6), when $k = 1$, $l = 0$, $m = -0.2$, and $n = 1$.

there exists a limit cycle around $A(0, 0)$. The simulation is presented in Figure 7.

In system (6), if we take values of the parameters as $k = 0.25$, $l = -0.25$, $m = -0.125$ and $n = 1$, then by Theorem 6 Case 5, there exists a limit cycle around $A(0, 0)$. The simulation is presented in Figure 8.

In system (6), if we take values of parameters as $k = -0.25$, $l = -0.25$, $m = 0.125$, and $n = 1$, then by Theorem 6 Case 6, there exists a limit cycle around $A(0, 0)$. The simulation is presented in Figure 9.

FIGURE 7: Existence of limit cycle of system (6), when $k = -1, l = 0, m = 1$, and $n = 1$.

FIGURE 8: Existence of limit cycle of system (6), when $k = 0.25, l = -0.25, m = -0.125$, and $n = 1$.

FIGURE 9: Existence of limit cycle of system (6), when $k = -0.25, l = -0.25, m = 0.125$, and $n = 1$.

4. Conclusions

In this paper, we studied the existence of the limit cycle and used the formal series method to determine the center-focus point. By the Hopf bifurcation theory, we obtained the sufficient condition for the existence of the limit cycles for

this system, which bifurcate from the equilibrium point. We provided some examples for illustration.

Data Availability

All data required for this paper are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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