1. Introduction

Let $H$ be a real Hilbert space and $S \in H$ be a nonempty closed convex subset. Let $f : H \rightarrow H$ be an operator. In this work, we investigate the following variational inequality problem (VIPs):

\[ \text{find a point } u^* \in S, \quad \text{s.t.} \langle f(u^*), x - u^* \rangle \geq 0, \forall x \in S. \quad (1) \]

Denote by $\text{Sol}(S, f)$ the solution set of $(1)$. The VIPs is an important tool to study various problems in the domain of mechanics, optimization, transportation, fixed point, economics equilibrium, contract problems in elasticity, and other branches of mathematics, see [1–17]. Therefore, VIPs have received much attention by many scholars, see [18–30]. There are a variety of methods to solve the VIPs, such as regularization method and projection method [31–39]. In this work, we focus on projection method.

As we all know that the gradient projection algorithm is the simplest and oldest method ([40, 41]), the method is defined as follows:

\[ u^{k+1} = P_S(u^k - \gamma f(u^k)), \quad (2) \]

where $P_S : H \rightarrow S$ is the metric projection and $\gamma$ is some positive number.

In order to obtain a convergent result, this algorithm requires that the operator $f$ is strongly monotone. In order to avoid the strong monotonicity hypothesis, Korpelevich [42] proposed the extragradient algorithm which is stated as follows:

\[ \begin{align*}
& x^k = P_S(u^k - \gamma f(u^k)), \\
& u^{k+1} = P_S(u^k - \gamma f(x^k)),
\end{align*} \quad (3) \]

where $\gamma \in (0, (1/L))$ and operator $f$ is monotone and Lipschitz continuous in $H$.

Note that the algorithm $(3)$ calculates two projections on $S$ in each iteration. If the set $S$ is more complicated, there will be a lot of calculations. In order to overcome this difficulty, Censor et al. [43] constructed a half space with subdifferentiation and proposed subgradient extragradient method which is defined by
\[
\begin{align*}
\begin{cases}
x^k = P_{\delta}(u^k - \gamma f(u^k)), \\
T^k = \{z \in \mathcal{H} | \langle u^k - y f(u^k) - x^k, z - x^k \rangle \leq 0\}, \\
u^{k+1} = P_T(u^k - \gamma f(x^k)).
\end{cases}
\end{align*}
\]

Recently, Dong et al. [44] improved the algorithm (4) with self-adaptive stepsize which generates a sequence \{u^k\} by the following form:

\[
\begin{align*}
\begin{cases}
x^k = P_{\delta}(u^k - \gamma f(u^k)), \\
y^k f(u^k) - f(x^k) \leq \sigma \|u^k - x^k\|, \quad \forall \sigma \in (0, 1), \\
T^k = \{z \in \mathcal{H} | \langle u^k - y f(u^k) - x^k, z - x^k \rangle \leq 0\}, \\
u^{k+1} = P_T(u^k - \tau \eta f(x^k)),
\end{cases}
\end{align*}
\]

where \( \zeta^k = ((u^k - x^k, \kappa(u^k, x^k))/\|\kappa(u^k, x^k)\|^2) \) and \( \kappa(u^k, x^k) = (u^k - x^k) - \gamma f(u^k - f(x^k)). \)

Weak convergence of Algorithm (5) has been obtained. Motivated and inspired by the above work, in this paper, we continue to investigate iterative algorithms for solving the monotone variational inequality in Hilbert spaces. We construct two modified subgradient extragradient algorithms for finding the solution of the monotone variational inequality. Our algorithms combine self-adaptive technique and inerital method. Under some mild conditions, we prove that the proposed algorithms converge strongly to a solution of the monotone variational inequality.

The organization of this paper is as follows. In Section 2, we present some definitions and preliminary results, which will be used in further analysis of the proposed algorithms. In Section 3, we proposed two modified subgradient extragradient algorithms and prove strong convergence theorems.

2. Preliminaries

Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Use \( \rightharpoonup \) and \( \longrightarrow \) to denote weak and strong convergence, respectively. Let \( \{x^k\} \) be a sequence in \( H \). We use \( \omega_w(x^k) \) to denote the set of all weak cluster points of \( \{x^k\} \), i.e.,

\[
\omega_w(x^k) = \{x^* : \exists \{x^k_i\} \subset \{x^k\} \text{ such that } x^{k_i} \rightharpoonup x^* \text{ as } i \rightarrow \infty\}.
\]

For \( \forall u, v \in H \), and \( \lambda \in \mathbb{R} \), the following results hold

\[
\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v \rangle,
\]

\[
\|\lambda u + (1 - \lambda) v\|^2 = \lambda\|u\|^2 + (1 - \lambda)\|v\|^2 - \lambda(1 - \lambda)\|u - v\|^2.
\]

**Definition 1.** Let \( f : H \longrightarrow H \) be an operator. Recall that the operator \( f \) is said to be

(i) Monotone if

\[
\langle f(u) - f(v), u - v \rangle \geq 0, \quad \forall u, v \in H.
\]

(ii) Strongly monotone if there exists \( \gamma > 0 \) s.t.

\[
\langle f(u) - f(v), u - v \rangle \geq \gamma\|u - v\|^2, \quad \forall u, v \in H.
\]

(iii) L-Lipschitz continuous if there exists \( L > 0 \) s.t.

\[
\|f(u) - f(v)\| \leq L\|u - v\|, \quad \forall u, v \in H.
\]

If \( L < 1 \), \( f \) is said to be \( L \)-contractive.

Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). For any \( x \in H \), there exists a unique point \( P_C(x) \in C \) such that

\[
\|x - P_C(x)\| \leq \|y - x\|, \quad \forall y \in C.
\]

It is well known that \( P_C \) satisfies [45]

\[
\|x - P_C(x), y - P_C(x)\| \leq \|x - y\|^2,
\]

for all \( x \in H \) and \( y \in C \).

**Lemma 1** (see [46]). Let \( \{b^k\} \) be a real number sequence. Suppose that there exists a subsequence \( \{b^{k_m}\} \) of \( \{b^k\} \) such that \( b^{k_m} \rightarrow b^{r+1} \) for all \( m \in \mathbb{N} \). Define the sequence \( \{y(k)\} \) as follows:

\[
y(k) = \max\{n \in \mathbb{N} | \|x_n\| \leq r, b^n \leq b^{r+1}\},
\]

for each \( k \geq r_0 \). Then, the following inequality holds:

\[
0 \leq b^k \leq b^{y(k)+1}, \quad k \geq r_0,
\]

for each \( k \geq r_0 \). Further, for all \( k \geq r_0 \), the sequence \( \{y(k)\} \) is nondecreasing and

\[
\lim_{k \rightarrow \infty} y(k) = +\infty.
\]

**Lemma 2** (see [33]). Suppose that the sequence \( \{\delta^k\} \) of real numbers is nonnegative and there exists \( k_0 \in \mathbb{N} \) such that

\[
\delta^{k+1} \leq (1 - y^k)\delta^k + y^k\delta^k,
\]

for each \( k \geq k_0 \), where the sequences \( \{y^k\} \) and \( \{\delta^k\} \) satisfy the following conditions:

\[
\{y^k\} \subset (0, 1),
\]

\[
\lim_{k \rightarrow \infty} y^k = 0,
\]

\[
\sum_{k=1}^{\infty} y^k = \infty,
\]

\[
\limsup_{k \rightarrow \infty} \delta^k \leq 0.
\]

Then, \( \lim_{k \rightarrow \infty} \delta^k = 0 \).
Lemma 3. If \( x^k = v^k \) or \( \kappa(v^k, x^k) = 0 \) in Algorithm 1, then \( x^k \in \text{Sol}(S, f) \).

Proof. Since \( f \) is \( L \)-Lipschitz continuous, we obtain

\[
\|\kappa(v^k, x^k)\| = \|v^k - x^k - y^k(f(v^k) - f(x^k))\| \\
\geq \|v^k - x^k\| - \|y^k(f(v^k) - f(x^k))\| \\
\geq \|v^k - x^k\| - \|y^k\|\|f(v^k) - f(x^k)\| \\
\geq \|v^k - x^k\| - \|y^k\|L\|v^k - x^k\|. \\
\|\kappa(v^k, x^k)\| = \|v^k - x^k - y^k(f(v^k) - f(x^k))\| \\
\leq \|v^k - x^k\| + \|y^k\|\|f(v^k) - f(x^k)\| \\
\leq \|v^k - x^k\| + \|y^k\|L\|v^k - x^k\| = (1 + y^kL)\|v^k - x^k\|. 
\]

(21)

It follows that

\[
(1 - y^kL)\|v^k - x^k\| \leq \|\kappa(v^k, x^k)\| \leq (1 + y^kL)\|v^k - x^k\|. 
\]

(22)

Consequently, \( v^k = x^k \Leftrightarrow \kappa(v^k, x^k) = 0 \). Furthermore, if \( v^k = x^k \) or \( \kappa(v^k, x^k) = 0 \), we have

\[
x^k = P_S(x^k - y^k f(x^k)). 
\]

(23)

Combining (13) and (23), we get

\[
\langle x^k - y^k f(x^k) - x^k, x^k - z \rangle \geq 0, \quad \forall z \in S, 
\]

(24)

which implies that

\[
\langle f(x^k), z - x^k \rangle \geq 0, \quad \forall z \in S. 
\]

(25)

This completes the proof. \( \square \)

Lemma 4. The sequence \( \{y^k\}^\infty_{k=0} \) generated by Algorithm 1 is monotonically decreasing, and \( y^k \leq \min\{y^0, (\sigma/L)\} \) for each \( k \geq 0 \).

Proof. Obviously, by the definition of \( \{y^{k+1}\} \), we have \( y^k \) is a monotonically decreasing sequence. Then, \( y^k \geq y^0, \forall n > 0 \). Since \( f \) is Lipschitz continuous, we have

\[
\|f(u^k) - f(x^k)\| \leq L\|u^k - x^k\|. 
\]

(26)

In the case of \( f(u^k) \neq f(x^k) \), we have

\[
\sigma\|u^k - x^k\| \geq \sigma. 
\]

(27)

Obviously, the lower bound of \( \{y^k\} \) is \( \min\{y^0, (\sigma/L)\} \). This completes the proof. \( \square \)

Lemma 5. Let \( \{z^k\} \) be the sequence generated by Algorithm 1. Then, we have

\[
z^k \geq \frac{1 - \sigma}{1 + \sigma^2}. 
\]

(28)

Proof. Combining Lemma 4 and Cauchy–Schwarz inequality, we have

\[
\langle v^k - x^k, \kappa(v^k, x^k) \rangle = \langle v^k - x^k, v^k - x^k - y^k(f(v^k) - f(x^k)) \rangle \\
= \|v^k - x^k\|^2 - y^k\langle v^k - x^k, f(v^k) - f(x^k) \rangle \\
\geq \|v^k - x^k\|^2 - y^k\|v^k - x^k\|\|f(v^k) - f(x^k)\| \\
\geq \|v^k - x^k\|^2 - \|v^k - x^k\|^2 \|\|f(v^k) - f(x^k)\| \\
= (1 - y^k L)\|v^k - x^k\|^2 \\
\geq (1 - \sigma)\|v^k - x^k\|^2. 
\]

(29)

Since \( f \) is monotone and Lipschitz continuous, then we obtain

\[
\|\kappa(v^k, x^k)\|^2 = \|v^k - x^k - y^k(f(v^k) - f(x^k))\|^2 \\
= \|v^k - x^k\|^2 + (y^k)^2\|f(v^k) - f(x^k)\|^2 \\
\geq \|v^k - x^k\|^2 + (y^k)^2\|f(v^k) - f(x^k)\|^2 \\
= (1 + (y^k)^2 L)^2\|v^k - x^k\|^2 \\
\leq (1 + \sigma^2)\|v^k - x^k\|^2. 
\]

(30)

From (29) and (30), we have

\[
z^k = \frac{\langle v^k - x^k, \kappa(v^k, x^k) \rangle}{\|\kappa(v^k, x^k)\|^2} \geq \frac{1 - \sigma}{1 + \sigma^2}. 
\]

(31)
This completes the proof.

\begin{lemma}
Let \( u^k \in \text{Sol}(S, f) \). Then,
\[
\left\| t^k - u^k \right\|^2 \leq \left\| v^k - u^k \right\|^2 - \left\| v^k - t^k \right\|^2 - \tau \zeta \kappa(v^k, x^k)^2 + \tau(2 - \tau)(\zeta^2)\left\| \kappa(v^k, x^k) \right\|^2.
\]
\end{lemma}

\begin{proof}
From (13) and Algorithm 1, we have
\[
\left\| t^k - u^k \right\|^2 \leq \left\| P_{x^k}(v^k - \tau \zeta \kappa f(x^k) - P_{u^k}u^k \right\|^2
\]
\[
\leq \left\langle t^k - u^k, v^k - \tau \zeta \kappa f(x^k) - u^k \right\rangle
\]
\[
= \frac{1}{2}\left\| t^k - u^k \right\|^2 + \frac{1}{2}\left\| v^k - \tau \zeta \kappa f(x^k) - u^k \right\|^2
\]
\[
- \frac{1}{2}\left\| v^k - t^k \right\|^2 - \frac{1}{2}\tau \zeta \kappa f(x^k)^2
\]
\[
= \frac{1}{2}\left\| t^k - u^k \right\|^2 + \frac{1}{2}\left\| v^k - u^k \right\|^2
\]
\[
+ \frac{1}{2}\tau \zeta \kappa f(x^k)^2
\]
\[
- \left\langle v^k - u^k, \tau \zeta \kappa f(x^k) \right\rangle - \frac{1}{2}\left\| v^k - v^k \right\|^2
\]
\[
- \frac{1}{2}\tau \zeta \kappa f(x^k)^2
\]
\[
= \frac{1}{2}\left\| t^k - u^k \right\|^2 + \frac{1}{2}\left\| v^k - u^k \right\|^2 - \frac{1}{2}\left\| v^k - v^k \right\|^2
\]
\[
- \left\langle v^k - u^k, \tau \zeta \kappa f(x^k) \right\rangle.
\]
\end{proof}

It follows that
\[
2\left\| t^k - u^k \right\|^2 \leq \left\| t^k - u^k \right\|^2 + \left\| v^k - u^k \right\|^2 - \left\| v^k - v^k \right\|^2 - 2\tau \zeta \kappa \left\langle t^k - u^k, f(x^k) \right\rangle.
\]
\end{equation}

or equivalently
\[
\left\| t^k - u^k \right\|^2 \leq \left\| v^k - u^k \right\|^2 - \left\| v^k - v^k \right\|^2 - 2\tau \zeta \kappa \left\langle t^k - u^k, f(x^k) \right\rangle.
\]
\end{equation}

We deduce from \( x^k \in C \) and \( u^k \in \text{Sol}(S, f) \) that \( \left\langle t^k - u^k \right\|^2 \geq 0 \). It follows from the monotonicity of operator \( f \) that \( \left\langle f(x^k) - f(u^k), x^k - u^k \right\rangle \geq 0 \). Then, \( \left\langle f(x^k), x^k - u^k \right\rangle \geq 0 \). It equates that \( \left\langle f(x^k), t^k - u^k \right\rangle \geq \left\langle f(x^k), t^k - x^k \right\rangle \). Thus,
\begin{equation}
-2\tau \zeta \kappa \left\langle f(x^k), t^k - u^k \right\rangle \leq -2\tau \zeta \kappa \left\langle f(x^k), t^k - x^k \right\rangle.
\end{equation}

On the other hand, combining \( t^k \in T^k \) and Algorithm 1, we obtain
\begin{equation}
\left\langle \kappa(v^k, x^k), t^k - x^k \right\rangle \leq \gamma \left\langle f(x^k), t^k - x^k \right\rangle.
\end{equation}

This implies that
\begin{equation}
-2\tau \zeta \kappa \left\langle f(x^k), t^k - u^k \right\rangle \leq -2\tau \zeta \kappa \left\langle \kappa(v^k, x^k), t^k - x^k \right\rangle.
\end{equation}

Hence, we obtain
\begin{equation}
-2\tau \zeta \kappa \left\langle f(x^k), t^k - u^k \right\rangle \leq -2\tau \zeta \kappa \left\langle \kappa(v^k, x^k), t^k - x^k \right\rangle
\end{equation}
\begin{equation}
\leq -2\tau \zeta \kappa \left\langle \kappa(v^k, x^k), v^k - x^k \right\rangle
\end{equation}
\begin{equation}
+ 2\tau \zeta \kappa \left\langle \kappa(v^k, x^k), v^k - t^k \right\rangle.
\end{equation}

Now, we calculate \(-2\tau \zeta \kappa \left\langle \kappa(v^k, x^k), v^k - x^k \right\rangle \) and \(2\tau \zeta \kappa \left\langle \kappa(v^k, x^k), v^k - t^k \right\rangle \) separately. From the definition of \( \{v^k\} \), we get
\begin{equation}
-2\tau \zeta \kappa \left\langle \kappa(v^k, x^k), v^k - x^k \right\rangle = -2\tau \zeta \kappa \left\langle \kappa(v^k, x^k) \right\rangle^2.
\end{equation}

Meanwhile,
\begin{equation}
2\tau \zeta \kappa \left\langle \kappa(v^k, x^k), v^k - t^k \right\rangle = \left\| v^k - t^k \right\|^2 + \tau \zeta \kappa \left\langle \kappa(v^k, x^k) \right\rangle^2
\end{equation}
\begin{equation}
- \left\| v^k - t^k - \tau \zeta \kappa(v^k, x^k) \right\|^2.
\end{equation}

This implies that
\begin{equation}
-2\tau \zeta \kappa \left\langle f(x^k), t^k - u^k \right\rangle \leq -2\tau \zeta \kappa \left\langle \kappa(v^k, x^k) \right\rangle^2
\end{equation}
\begin{equation}
+ \left\| v^k - t^k \right\|^2 + \tau \zeta \kappa \left\langle \kappa(v^k, x^k) \right\rangle^2
\end{equation}
\begin{equation}
\left\| v^k - t^k - \tau \zeta \kappa(v^k, x^k) \right\|^2 = \left\| v^k - t^k \right\|^2 - \left\| v^k - t^k - \tau \zeta \kappa(v^k, x^k) \right\|^2
\end{equation}
\begin{equation}
- \tau(2 - \tau)(\zeta^2)\left\| \kappa(v^k, x^k) \right\|^2.
\end{equation}

So, we get
\begin{equation}
\left\| t^k - u^k \right\|^2 \leq \left\| v^k - u^k \right\|^2 - \left\| v^k - t^k - \tau \zeta \kappa(v^k, x^k) \right\|^2
\end{equation}
\begin{equation}
- \tau(2 - \tau)(\zeta^2)\left\| \kappa(v^k, x^k) \right\|^2.
\end{equation}

This completes the proof.
\end{proof}

\begin{theorem}
The sequence \( \{u^k\} \) generated by Algorithm 1 converges strongly to \( u^f \in \text{Sol}(S, f) \).
\end{theorem}

\begin{proof}
We divide the proof into four claims.

Claim 1. We prove the boundedness of the sequences \( \{u^k\} \) and \( \{t^k\} \). Indeed, from Algorithm 1 and Lemma 6, we get
\begin{equation}
\left\| u^{k+1} - u^k \right\| \leq \left\| (1 - \theta^k + \lambda^k t^k - u^k \right\|
\end{equation}
\begin{equation}
\leq \left\| (1 - \theta^k - \lambda^k)(v^k - u^k) + \lambda^k \left( t^k - u^k \right) \right\|
\end{equation}
\begin{equation}
\leq \left\| (1 - \theta^k - \lambda^k)(v^k - u^k) \right\| + \lambda^k \left\| t^k - u^k \right\| + \theta^k \left\| u^k \right\|
\end{equation}
\begin{equation}
\leq (1 - \theta^k - \lambda^k)\left\| v^k - u^k \right\| + \lambda^k \left\| t^k - u^k \right\| + \theta^k \left\| u^k \right\|.
\end{equation}
Algorithm 1: Strong convergence algorithm with contractive technique.

\[
\begin{align*}
\|u^{k+1} - u^k\| &\leq (1 - \theta_k)\|\nu^k - u^k\| + \lambda_k\|\nu^k - u^k\| + \theta\|u^k\| \\
&= (1 - \theta_k)\|\nu^k - u^k\| + \theta\|u^k\|. 
\end{align*}
\]

Combining Algorithm 1 and (44), we obtain
\[
\begin{align*}
\|u^{k+1} - u^k\| &= (1 - \theta_k)\|\nu^k - u^k\| + \theta\|u^k\| \\
&= (1 - \theta_k)\|\nu^k - u^k\| + \theta\|u^k\| \\
&\leq (1 - \theta_k)\|\nu^k - u^k\| + \theta\|u^k\| \\
&= (1 - \theta_k)\|\nu^k - u^k\| + \theta\|\|u^k\|\|. 
\end{align*}
\]

where
\[
\|\|u^k\|\| = (1 - \theta_k)\|\nu^k - u^k\|. 
\]

Taking into account \(e^k = o(\theta^k)\) and the definition of \(\rho^k\), we get
\[
\lim_{k \to \infty} \|e^k\| = 0. 
\]

Then, the sequence \(\{e^k\}\) is bounded. Let \(M = \sup_{k \geq 1} (\|e^k\| + \|u^k\|)\). We obtain from (45) that
\[
\begin{align*}
\|u^{k+1} - u^k\| &\leq (1 - \theta_k)\|\nu^k - u^k\| + \theta\|u^k\| \\
&\leq \max\left\{\|u^k - u^k\|, M\right\}. 
\end{align*}
\]

For all \(k \geq k_0\), we have
\[
\|u^{k+1} - u^k\| \leq \max\left\{\|u^k - u^k\|, M\right\}. 
\]

It follows that the sequence \(\{u^k\}\) is bounded. Therefore, the sequence \(\{r^k\}\) is bounded.

Claim 2. We prove that the following holds:
\[
\begin{align*}
\|u^{k+1} - u^k\|^2 &\leq (1 - \theta_k)\|\nu^k - u^k\|^2 + 2\lambda_k^k \theta_k^k \\
&\quad \leq \|u^{k+1} - u^k\|^2 + 2\theta_k^k \langle u^k - u^k, u^{k+1} - u^k\rangle. 
\end{align*}
\]

Set \(z^k = (1 - \lambda_k^k)\nu^k + \lambda_k^k r^k\). Then, \(v^k - z^k = \lambda_k^k (v^k - r^k)\).

Therefore,
\[
\begin{align*}
u^k = (1 - \theta_k^k)\nu^k + \lambda_k^k r^k \\
&= (1 - \theta_k^k)z^k - \theta_k^k (v^k - z^k) \\
&= (1 - \theta_k^k)z^k - \theta_k^k \lambda_k^k (v^k - r^k). 
\end{align*}
\]

From Lemma 6, we have
\[
\|z^k - u^k\|^2 \leq \|v^k - u^k\|^2, 
\]

which implies that
\[
\|z^k - u^k\|^2 \leq (1 - \lambda_k^k)\|v^k - u^k\|^2 + \lambda_k^k\|r^k - u^k\|^2 \leq \|v^k - u^k\|^2. 
\]

By (7), (51), and (53), we get
Claim 3. By (8) and Algorithm 1, we obtain

\[
\|v^k - u^i\|^2 = \|u^k + \rho^k(u^k - u^{k-1}) - u^i\|^2 \\
= (1+\rho^k)(u^k - u^i)^2 - \rho^k(u^{k-1} - u^i)^2 + \rho^k(1+\rho^k)(u^k - u^{k-1})^2 \\
\leq (1+\rho^k)(u^k - u^i)^2 - \rho^k(u^{k-1} - u^i)^2 + 2\rho^k(u^k - u^{k-1})^2 \\
= \|u^k - u^i\|^2 + \rho^k(\|u^k - u^i\|^2 - \|u^{k-1} - u^i\|^2) + 2\rho^k(\|u^k - u^{k-1}\|^2).
\]

(55)

Using Lemma (8) and (52), we get

\[
\|u^{k+1} - u^i\|^2 = (1 - \theta^k)(v^k - u^i)^2 + \theta^k(i^k - u^i) + \theta^k(-u^i)^2 \\
\leq (1 - \theta^k)(v^k - u^i)^2 + \theta^k(i^k - u^i)^2 + \theta^k(u^i)^2 \\
\leq (1 - \theta^k)(v^k - u^i)^2 + \theta^k(1 - \theta^k)(v^k - u^{k-1})^2 \\
+ \theta^k(2-\tau)(\xi^k)^2(\|v^k, x^k\|^2) \\
\leq (1 - \theta^k)(v^k - u^i)^2 + \theta^k(u^i)^2 - \lambda^k(\|v^k - i^k - \tau\xi^k\|)^2 \\
- \lambda^k(2-\tau)(\xi^k)^2(\|v^k, x^k\|^2).
\]

(56)

From (55) and (56), we get

\[
\|u^{k+1} - u^i\|^2 \leq (1 - \theta^k)(v^k - u^{k-1})^2 + \theta^k(1-\theta^k)(u^k - u^{k-1})^2 \\
- \lambda^k(\|v^k - i^k - \tau\xi^k\|)^2 - \lambda^k(2-\tau)(\xi^k)^2(\|v^k, x^k\|^2) \\
\leq (1 - \theta^k)(v^k - u^{k-1})^2 + \theta^k(1-\theta^k)(u^k - u^{k-1})^2 \\
+ 2\rho^k(1-\theta^k)(u^k - u^{k-1})^2 + \theta^k(u^i)^2 \\
- \lambda^k(\|v^k - i^k - \tau\xi^k\|)^2 - \lambda^k(2-\tau)(\xi^k)^2(\|v^k, x^k\|^2) \\
\leq \|v^k - u^{k-1}\|^2 + \rho^k(1-\theta^k)(u^k - u^{k-1})^2 \\
+ 2\rho^k(1-\theta^k)(u^k - u^{k-1})^2 + \theta^k(u^i)^2 \\
- \lambda^k(\|v^k - i^k - \tau\xi^k\|)^2 - \lambda^k(2-\tau)(\xi^k)^2(\|v^k, x^k\|^2).
\]

(57)

Claim 4. Next, we will consider two different cases to prove the strong convergence of the sequence \(\{\|u^k - u^i\|^2\}\).

Case 1. There exists an \(N \in \mathbb{N}\) s.t. \(\|u^{k+1} - u^i\|^2 \leq \|u^k - u^i\|^2\), \(\forall k \geq N\). Obviously, the limit of the sequence \(\{\|u^k - u^i\|^2\}\) exists which implies that
\[
\lim_{k \to \infty} \|u^{k+1} - u^k\| = 0. \quad \text{(57)}
\]
In (57), taking the limit as \(k \to \infty\), we deduce
\[
\lim_{k \to \infty} k(v^k, x^k) = 0,
\]
\[
\lim_{k \to \infty} \left\| v^k - t^k - \tau \zeta^k k(v^k, x^k) + \tau \zeta^k k(v^k, x^k) \right\|^2 = 0.
\]
(59)

On the other hand, we have
\[
\left\| v^k - t^k \right\| \leq \left\| v^k - t^k - \tau \zeta^k k(v^k, x^k) + \tau \zeta^k k(v^k, x^k) \right\|. 
\]
(60)

So, we have \(\lim_{k \to \infty} \|v^k - t^k\| = 0\).

Combining Lemma 3 and (58), we obtain
\[
\lim_{k \to \infty} \|v^k - x^k\| = 0.
\]
(61)

Now, we show that \(u_{n_k}(u^k) \subset \text{Sol}(S, f)\). Choose \(\rho^k \in u_{n_k}(u^k)\). It implies that there exists a subsequence \(\{u^{n_k}\}\) of \(\{u^k\}\) which converges weakly to \(\rho^k\). Therefore, \(v^{n_k} \to \rho^k\). Due to \(\lim_{k \to \infty} \|v^k - x^k\| = 0\), we obtain \(x^{n_k} \to \rho^k\). By Algorithm 1, we have
\[
\left\langle x^k - v^k + y^k f(v^k), u - x^k \right\rangle \geq 0, \quad \forall u \in C.
\]
(62)

Since \(f\) is monotone, we have
\[
0 \leq \left\langle x^k - v^k, u - x^k \right\rangle + y^k \left\langle f(v^k), u - x^k \right\rangle \\
= \left\langle x^k - v^k, u - x^k \right\rangle + y^k \left\langle f(v^k), u - v^k \right\rangle \\
+ y^k \left\langle f(v^k), v^k - x^k \right\rangle \\
\leq \left\langle x^k - v^k, u - x^k \right\rangle + y^k \left\langle f(u), u - v^k \right\rangle \\
+ y^k \left\langle f(v^k), v^k - x^k \right\rangle.
\]
(63)

Taking the limit in (63) as \(k \to \infty\), we get
\[
\left\langle f(u), u - \rho^k \right\rangle \geq 0, \quad \forall u \in C.
\]
(64)

which implies that \(u_{n_k}(u^k) \subset \text{Sol}(S, f)\).

Set \(b^k = \|u^k - u^1\|^2\) for all \(k \geq 0\). By (65) for \(q = u^1\), we obtain
\[
b^{k+1} \leq (1 - \theta^k) \left\| v^k - q \right\|^2 + \theta^k \left[ 2 \lambda^k \left\| v^k - t^k \right\| u^{k+1} - q \right\|
+ 2 \left\langle -q, u^{k+1} - q \right\rangle.
\]
(65)

We deduce from Algorithm 1 that
\[
\left\| v^k - q \right\|^2 \leq \left\langle u^k - q + \rho^k \left\| u^k - u^{k-1} \right\| \right\|^2
= \left\| u^k - q \right\|^2 + 2 \rho^k \| u^k - q \left\|^2 \left\| u^k - u^{k-1} \right\|
+ 2 \rho^k \left\| u^k - q \left\| u^k - u^{k-1} \right\|^2
\leq \left\| u^k - q \right\|^2 + 2 \rho^k \left\| u^k - q \left\| u^k - u^{k-1} \right\|^2
+ 2 \rho^k \left\| u^k - q \left\| u^k - u^{k-1} \right\|^2
\leq b^k + 3 \lambda^k \rho^k \left\| u^k - u^{k-1} \right\|^2.
\]
(66)

where
\[
K = \sup_{k \geq 1} \left\{ \left\| u^k - u^{k-1} \right\|, \left\| u^k - q \right\| \right\}.
\]
(67)

By virtue of (65) and (66), we have
\[
b^{k+1} \leq (1 - \theta^k) b^k + \delta^k,
\]
(68)

where
\[
\delta^k = \theta^k \left[ 3 \lambda^k \left( 1 - \theta^k \right) \rho^k \left\| u^k - u^{k-1} \right\|^2 - 2 \lambda^k \left\| v^k - t^k \right\|
\leq \left\| u^{k+1} - q \right\|^2 + 2 \left\langle -q, u^{k+1} - q \right\rangle.
\]
(69)

So, we get
\[
\limsup_{k \to \infty} \left\langle -q, u^{k+1} - q \right\rangle = \sup_{u \in u_{n_k}} \left\langle u^k \right\rangle \left\langle -q, u^1 - q \right\rangle \leq 0.
\]
(70)

From (70), we deduce that \(q \in P_{\text{Sol}(S, f)}(0)\). Combining the property of projection, \(\lim_{k \to \infty} \|v^k - t^k\|^2 = 0\) and \(\lim_{k \to \infty} (\rho^k / \theta^k) \left\| u^k - u^{k-1} \right\| = 0\), we have \(\limsup_{k \to \infty} \delta^k \leq 0\). By Lemma 2, we obtain \(b^k = \left\| u^k - u^1 \right\|^2 \to 0 \quad (k \to \infty)\). Therefore, the sequence \(\{u^k\}\) converges strongly to \(u^*\).

Case 2. There exists a subsequence \(\{b^{k_i}\} \subset \{b^k\}_{k \geq 1}\) s.t. \(b^{k_i} \leq b^{k+1}\) for \(\forall i \geq 0\). From Lemma 2, we can deduce
\[
b^{k+1} \leq b^{k+1},
\]
(71)

for each \(k \geq k_0\), where \(\gamma(k) = \max \{ n \in \mathbb{N} | k_0 \leq n \leq k, b^{n+1} \leq b^{n+1} \}\). Further, the sequence \(\{\gamma(k)\}_{k \geq k_0}\) is nondecreasing (i.e., \(\lim_{k \to \infty} \gamma(k) = \infty\)). Let \(b^k = \left\| u^k - u^1 \right\|^2\). By (71) and Claim 3 for \(q = u^1\), we obtain
\[
\lambda^{\gamma(k)} \left\langle u^{\gamma(k)} - t^{\gamma(k)} - \gamma^{\gamma(k)} d(u^{\gamma(k)}, x^{\gamma(k)}) \right\rangle
\]
\[
+ \tau (2 - \tau) (\gamma^{\gamma(k)})^2 \left\| d(u^{\gamma(k)}, x^{\gamma(k)}) \right\|^2
\leq \rho^{\gamma(k)} (1 - \theta^{\gamma(k)}) (b^{\gamma(k)} - b^{\gamma(n)-1})
\]
\[
+ 2 \rho^{\gamma(k)} (1 - \theta^{\gamma(k)}) \left\| u^{\gamma(k)} - u^{\gamma(n)-1} \right\|^2 + \theta^{\gamma(k)} \| q \|^2.
\]
(72)

We deduce from the definition of \(b^k\) that
Combining (72) and (73), we have
\[
\lambda^y(k) \left[ \| u^y(k) - t^y(k) - y^z(t^y(k), u^y(k)) \| \right] + \tau (2 - \tau) \left( \lambda^y(k) \right)^2 \| d(u^y(k), u^y) \|^2 \\
\leq \rho^y(k) \left( 1 - \theta^y(k) \right) \left( \| u^y(k) - t^y(k) - y^z(t^y(k), u^y(k)) \| + 2 \lambda^y(k) \left( 1 - \theta^y(k) \right) \| u^y(k) - u^y(k-1) \| \right) + \rho^y(k) \| q \| \| q \| .
\]
(74)

Similarly, we have
\[
\rho^y(k) \left( 1 - \theta^y(k) \right) \| u^y(k) - u^y(k-1) \| \rightarrow 0.
\]
It follows that
\[
\lim_{k \rightarrow \infty} \| u^y(k) - u^y \|^2 = \lim_{k \rightarrow \infty} \| u^y(k) - v^y \|^2 = 0,
\]
and
\[
\| b^{y(k)} \|_1 \leq \left( 1 - \theta^y(k) \right) \rho^{y(k)} + \theta^y(k).
\]
(75)

Since \( b^{y(k)} \leq b^{y(k)+1} \) and \( \theta^y(k) > 0 \), from (76), we have
\[
\| b^{y(k)} \| \leq 3 \left( 1 - \theta^y(k) \right) \rho^{y(k)} \| u^y(k) - u^y(k-1) \| \\
- 2 \lambda^y(k) \| y^z(t^y(k)) \| \| u^y(k) - u^y(k+1) \| - 2 \lambda^y(k) \| t^y(k) - y^z(t^y(k)) \| + 2 \lambda^y(k) \| q, u^y(k+1) - q \|.
\]
(76)

Since \( b^{y(k)} \leq b^{y(k)+1} \) and \( \theta^y(k) > 0 \), from (76), we have
\[
\| b^{y(k)} \| \leq 3 \left( 1 - \theta^y(k) \right) \rho^{y(k)} \| u^y(k) - u^y(k-1) \| \\
- 2 \lambda^y(k) \| y^z(t^y(k)) \| \| u^y(k) - u^y(k+1) \| - 2 \lambda^y(k) \| t^y(k) - y^z(t^y(k)) \| + 2 \lambda^y(k) \| q, u^y(k+1) - q \|.
\]
(77)

Since \( q \in \text{P_{Sol(S,f)}}(0) \) and \( u_{\text{w}}(u^y(k)) \subset \text{Sol}(S, f) \), we have
\[
\limsup_{k \rightarrow \infty} \left( -q, u^y(k-1) - q \right) = \sup_{k \rightarrow \infty} \left( -q, u^y(k) - q \right) \leq 0.
\]
By (75), (77), and \( (\rho^y(k)/\theta^y(k)) \| u^y(k) - u^y(k-1) \| \rightarrow 0 \), we get
\[
\limsup_{k \rightarrow \infty} b^{y(k)} \leq 2 \sup_{q, u_{\text{w}}} \left( -q, u^y(k) - q \right) \leq 0.
\]
(78)

It follows from (76) that
\[
\limsup_{k \rightarrow \infty} b^{y(k)+1} \leq 0
\]
(79)
or \( \lim_{k \rightarrow \infty} b^{y(k)+1} = 0. \)

Hence, \( \lim_{k \rightarrow \infty} b^k = 0 \). Therefore, the sequence \( \{ u^k \} \) converges strongly to \( u^1 \). This completes the proof.

Suppose that \( g : H \rightarrow H \) is a \( \rho \)-contractive operator. Next, we propose an iterative algorithm with viscosity item.

**Theorem 2.** The sequence \( \{ u^k \} \) generated by Algorithm 1 converges strongly to \( u^1 = \text{P_{Sol(S,f)}}(g(u^1)). \)

**Proof.** We divide the proof into 4 claims.

Claim 1. We prove the boundedness of the sequences \( \{ g(v^k) \}, \{ x^k \} \) and \( \{ t^k \} \). From Algorithm 1, we get
\[
\| v^k - u^1 \| = \| u^k - \rho^y(k)(u^k - u^k-1) - u^1 \| \\
\leq \| u^k - u^1 \| + \rho^k \| u^k - u^k-1 \| \\
= \| u^k - u^1 \| + \rho^k \| u^k - u^k-1 \|. \]
(80)

From Algorithm 1, we obtain \( (\rho^y(k)/\theta^y(k))u^k - u^k-1 \| \rightarrow 0, (k \rightarrow \infty) \). Then, \( \exists M_1 > 0 \) s.t.
\[
\frac{\rho^k}{\theta^y(k)} \| u^k - u^k-1 \| \leq M_1, \quad \forall k > 0.
\]
(81)

By Algorithm 1 and (81), we have
Initialization. Choose $u^0, u^1 \in H$ arbitrarily.

Step 1. Choose $\rho^k$ s.t. $0 < \rho^k \leq \overline{\rho}^k$, where $\overline{\rho}^k = \left\{ \begin{array}{ll} \min\{\rho, (\kappa^k - u^k - u^{k-1})\} & \text{if } u^k \neq u^{k-1} \\ \text{otherwise} & \end{array} \right.$

Calculate $v^1 = u^k + \rho^k(u^k - u^{k-1})$.

Step 2. Calculate $x^1 = P^k_{\nu}(v^1 - \tau^k f_k(x^1))$, where $\tau^k = \left\{ \begin{array}{ll} \min\{\nu^k, (\|u^k - x^1\|/\|f^k - f x^1\|)\} & \text{if } u^k \neq f x^1 \\ \text{otherwise} & \end{array} \right.$

Step 3. Construct the half space $T^k$ as follows $T^k = \{ z \in H | (v^1 - \tau^k f_k(x^1)) z \leq 0 \}$.

Calculate $t^k = P_{\tau^k}(v^1 - \tau^k f_k(x^1))$.

Step 4. Compute $u^{k+1} = \theta^k g(v^1) + (1 - \theta^k) t^k$.

If $x^k = v^1$, then stop and $x^k \in \text{Sol}(S, f)$. Otherwise, set $k = k + 1$ and return to step 1.

**Algorithm 2:** Strong convergence algorithm with viscosity term.

\[
\begin{align*}
\|u^{k+1} - u^1\| &\leq \left(1 - (1 - \rho^k)\theta^k\right)\|u^k - u^1\| + (1 - (1 - \rho^k)\theta^k)\rho^k \|u^k - u^{k-1}\| + \theta^k \|g(u^1) - u^1\| \\
&\leq \left(1 - (1 - \rho^k)\theta^k\right)\|u^k - u^1\| + (1 - (1 - \rho^k)\theta^k)\rho^k M_1 + \theta^k \|g(u^1) - u^1\| \\
&\leq \left(1 - (1 - \rho^k)\theta^k\right)\|u^k - u^1\| + (1 - (1 - \rho^k)\theta^k)\rho^k M_1 + \theta^k \|g(u^1) - u^1\| \\
&\leq \max\left\{\|u^k - u^1\|, M_1 + \|g(u^1) - u^1\|\right\} \leq \cdots \leq \max\left\{\|u^k - u^1\|, M_1 + \|g(u^1) - u^1\|\right\}.
\end{align*}
\]

From (81) and (82), we have

Claim 2. From (7) and Algorithm 1, we have

\[
\begin{align*}
\|u^{k+1} - u^1\|^2 &\leq \left(1 - (1 - \rho^k)\theta^k\right)\|u^k - u^1\|^2 + (1 - (1 - \rho^k)\theta^k)\rho^k \|u^k - u^{k-1}\|^2 + \theta^k \|g(u^1) - u^1\|^2 \\
&\leq \left(1 - (1 - \rho^k)\theta^k\right)\|u^k - u^1\|^2 + (1 - (1 - \rho^k)\theta^k)\rho^k M_1 + \theta^k \|g(u^1) - u^1\|^2 \\
&\leq \left(1 - (1 - \rho^k)\theta^k\right)\|u^k - u^1\|^2 + (1 - (1 - \rho^k)\theta^k)\rho^k M_1 + \theta^k \|g(u^1) - u^1\|^2 \\
&\leq \max\left\{\|u^k - u^1\|^2, M_1 + \|g(u^1) - u^1\|^2\right\} \leq \cdots \leq \max\left\{\|u^k - u^1\|^2, M_1 + \|g(u^1) - u^1\|^2\right\}.
\end{align*}
\]

It is obvious that the sequence $\{u^k\}$ is bounded. Furthermore, the sequences $\{g(v^k)\}$, $\{x^k\}$ and $\{t^k\}$ are bounded.
Claim 3. By (8) and (55), we obtain

\[ \|u^{k+1} - u^i\|^2 = \|\theta^k (g(v^k) - u^i) + (1 - \theta^k) (t^k - u^i)\|^2 \]
\[ \leq \theta^k \|g(v^k) - u^i\|^2 + (1 - \theta^k) \|t^k - u^i\|^2 \]
\[ \leq \theta^k \|g(v^k) - u^i\|^2 + (1 - \theta^k) \|t^k - u^i\|^2. \]  

From (85) and (55), we obtain

\[ \|u^{k+1} - u^i\|^2 \leq \theta^k \|g(v^k) - u^i\|^2 + (1 - \theta^k) \|u^k - u^i\|^2 \]
\[ + 2(1 - \theta^k) \rho^k \|u^k - u^{k-1}\|^2 \]
\[ + (1 - \theta^k) \rho^k \|u^k - u^{k-1}\|^2 - \tau \kappa(v^k, x^k) \]
\[ \leq \theta^k \|g(v^k) - u^i\|^2 + \|u^k - u^i\|^2, \]  

This implies that

\[ (1 - \theta^k) \|v^k - t^k - \tau \kappa(v^k, x^k)\|^2 \leq \|u^k - u^i\|^2, \]
\[ (1 - \theta^k) \|v^k - t^k - \tau \kappa(v^k, x^k)\|^2 \leq \|u^k - u^i\|^2. \]

Claim 4. According to Claim 3, we can see that there are two possible cases.

Case 1. There exists an \( N \in \mathbb{N} \), s.t. \( \|u^{k+1} - u^i\|^2 \leq \|u^k - u^i\|^2 \) for \( \forall k > N \). It follows that \( \lim_{k \to \infty} \|u^k - u^i\| = 0 \). From (86) and \( \lim_{k \to \infty} \theta^k = 0 \), we have

\[ \lim_{k \to \infty} \|g(v^k) - u^i\|^2 = 0. \]

Case 2. There exists an \( N \in \mathbb{N} \), s.t. \( \|u^k - u^i\|^2 \leq \|u^{k+1} - u^i\|^2 \) for \( \forall k > N \). It follows that \( \lim_{k \to \infty} \|u^k - u^i\| = 0 \). From (86) and \( \lim_{k \to \infty} \theta^k = 0 \), we have

\[ \lim_{k \to \infty} \|v^k - t^k - \tau \kappa(v^k, x^k)\| = 0. \]
\[ \|v^k - t^k\|^2 \leq \|v^k - t^k - \tau \xi_k (v^k, x^k)\|^2 + \|\tau \xi_k (v^k, x^k)\|^2, \] (89)

So,
\[ \lim_{k \to \infty} \|v^k - t^k\| = 0. \] (90)

Similarly, we can obtain
\[ u^w(u^k) \subset \text{Sol}(S, f). \] (91)

Set \( b^k = \|u^k - q\|^2 \) for all \( k \geq 0 \). By (84) for \( q = u^t \), we get
\[ b^{k+1} \leq (1 - (1 - \rho)\theta^k) b^k + \delta^k, \] (92)
where
\[ \delta^k = 3K(1 - (1 - \rho)\theta^k)\|u^k - u^{k-1}\|^2 + 2\theta^k \langle g(q) - q, u^{k+1} - q\rangle. \] (93)

It follows from (66) and (92) that
\[ b^{k+1} \leq (1 - (1 - \rho)\theta^k)b^k + \delta^k, \] (94)

where
\[ \delta^k = 3K(1 - (1 - \rho)\theta^k)\|u^k - u^{k-1}\|^2 + 2\theta^k \langle g(q) - q, u^{k+1} - q\rangle. \] (95)

Hence, we deduce \( q \in P_{\text{Sol}(S,f)}(0) \). Combining the property of projection, \( \lim_{k \to \infty} \|v^k - t^k\|^2 = 0 \) and \( \lim_{k \to \infty} (\rho^k \theta^k) \|u^k - u^{k-1}\|^2 = 0 \), we have \( \limsup_{k \to \infty} \delta^k = 0 \). By Lemma 2, we obtain \( b^k = \|u^k - u^t\|^2 \to 0 \) \((k \to \infty)\). Therefore, the sequence \( \{u^k\} \) converges strongly to \( u^t \).

Case 2. There exists a subsequence \( \{b^{k_j}\} \subset \{b^k\}_{k \geq \bar{k}_0} \), s.t. \( b^{k_j} \leq b^{k_j+1} \) for each \( j \geq 0 \). From Lemma 1, we deduce that
\[ b^{y(k)} \leq b^{y(k)+1}, \] (96)
for all \( k \geq \bar{k}_0 \), where \( y(k) = \max\{n \in \mathbb{N} | \bar{k}_0 \leq n \leq k, b^n \leq b^{n+1}\} \). Therefore, the sequence \( \{y(k)\}_{k \geq \bar{k}_0} \) is nondecreasing (i.e., \( \lim_{k \to \infty} y(k) = \infty \)). By (96) and Claim 3 for \( q = u^t \), we obtain
\[ (1 - \theta^{y(k)}) \|\langle v^{y(k)} - t^{y(k)}\rangle - \tau \xi^{y(k)} (v^{y(k)}, x^{y(k)})\|^2 + (1 - \theta^{y(k)}) \tau (2 - \tau)(\xi^{y(k)})^2 \|\langle v^{y(k)}, x^{y(k)}\rangle\|^2 \leq \|u^{y(k)} - q\|^2 - \|u^{y(k)+1} - u^t\|^2 + \theta^{y(k)} \|\langle v^{y(k)} - q\rangle\|^2 + (1 - \theta^{y(k)}) \rho^{y(k)} \|\langle b^{y(k)} - b^{y(k)-1}\rangle + 2(1 - \theta^{y(k)}) \rho^{y(k)} \|\langle u^{y(k)} - u^{y(k)-1}\rangle\|^2. \] (97)

From (73) and (97), we get
\[ (1 - \theta^{y(k)}) \|\langle v^{y(k)} - t^{y(k)}\rangle - \tau \xi^{y(k)} (v^{y(k)}, x^{y(k)})\|^2 + (1 - \theta^{y(k)}) \tau (2 - \tau)(\xi^{y(k)})^2 \|\langle v^{y(k)}, x^{y(k)}\rangle\|^2 \leq \|u^{y(k)} - u^t\|^2 - \|u^{y(k)+1} - u^t\|^2 + \theta^{y(k)} \|\langle v^{y(k)} - u^{y(k)-1}\rangle\|^2 + 2(1 - \theta^{y(k)}) \rho^{y(k)} \|\langle u^{y(k)} - u^{y(k)-1}\rangle\|^2 + (1 - \theta^{y(k)}) \rho^{y(k)} \|\langle u^{y(k)} - u^{y(k)-1}\rangle\|^2. \] (98)

Using Claim 1 and (98), we have \( \lim_{k \to \infty} (1 - \theta^{y(k)}) \rho^{y(k)} \|\langle v^{y(k)} - u^{y(k)-1}\rangle\| = 0 \). Therefore, \( u^w(u^{y(k)}) \subset \text{Sol}(S, f) \),
\[ \lim_{k \to \infty} \|t^{y(k)} - u^{y(k)}\|^2 = \lim_{k \to \infty} \|t^{y(k)} - u^{y(k)-1}\|^2 = 0, \] \[ b^{y(k)+1} \leq (1 - (1 - \rho)\theta^{y(k)})b^{y(k)} + 3K \] (99)
\[ + 2\theta^{y(k)} \langle g(q) - q, u^{y(k)+1} - q\rangle. \]

Since \( b^{y(k)} \leq b^{y(k)+1} \) and \( b^{y(k)} \geq 0 \), we receive
\[ b^{y(k)} \leq 3K(1 - (1 - \rho)\theta^{y(k)})b^{y(k)} + 3K \] (99)
\[ + 2\theta^{y(k)} \langle g(q) - q, u^{y(k)+1} - q\rangle. \]

Note that \( q \in P_{\text{Sol}(S,f)}(0) \) and \( w_u(u^{y(k)}) \subset \text{Sol}(S, f) \). By the property of projection, we have
\[ \limsup_{k \to \infty} \langle g(q) - q, u^{y(k)+1} - q\rangle = \sup_{u \in w_u(u^{y(k)})} \langle u^{y(k)}\rangle \] (99)
\[ \langle g(q) - q, u^t - q\rangle \leq 0. \] (99)

Since \( (1 - \theta^{y(k)}) \rho^{y(k)} \|\langle v^{y(k)} - u^{y(k)-1}\rangle\| \to 0 \), we deduce
\[ \limsup_{k \to \infty} a_{y(k)} \leq 2 \sup_{u \in w_u(u^{y(k)})} \langle x_{y(k)}\rangle \langle g(q) - q, u^t - q\rangle \leq 0. \] (99)

Since \( y(k) \leq \bar{k}_0 \leq b^{y(k)+1} \), we have \( b^{y(k)} \leq b^{y(k)+1} \) for all \( k \geq \bar{k}_0 \).

For all \( k \geq \bar{k}_0 \), \( b^{y(k)} \leq b^{y(k)+1} \) and \( b^{y(k)} \geq 0 \), we receive
\[ b^{y(k)} \leq 3K(1 - (1 - \rho)\theta^{y(k)})b^{y(k)} + 3K \] (99)
\[ + 2\theta^{y(k)} \langle g(q) - q, u^{y(k)+1} - q\rangle. \]

So, \( \limsup_{k \to \infty} b^{y(k)+1} \leq 0 \) or \( \lim_{k \to \infty} b^{y(k)+1} = 0 \). Hence, \( \lim_{k \to \infty} b^k = 0 \) which implies that the sequence \( \{u^k\} \) converges strongly to \( u^t \). This completes the proof. \( \square \)
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors’ Contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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