Research Article

Weaving Frames in Hilbert C^* -Modules

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In this paper, we investigate weaving frames in Hilbert C^* -modules. We show that the equivalence of woven and weakly woven frames is still true for modular frames under certain conditions. By using the analysis operators of frames and frame operators of canonical duals, we obtain several perturbation results for given weaving frames and different weaving frame pairs. When the C^* -algebra is nonunital, we derive a correspondence of adjointable operators which is bounded below woven families. Finally, we discuss the redundancy of weaving frames in Hilbert C^* -modules.

1. Introduction

Due to the useful applications in the characterization of function spaces, signal processing, and many other fields of applications, the theory of frames has developed rather rapidly in recent years. Various generalizations of frames have been developed. For example, g -frames [1], fusion frames [2], pseudoframes [3], and operator-valued frames [4]. Recently, Bemrose et al. [5] introduced a new concept of “weaving frames” in separable Hilbert spaces. This notion has potential applications in distributed signal processing and wireless sensor networks. Two frames $\{x_j\}_{j \in \mathbb{J}}$ and $\{y_j\}_{j \in \mathbb{J}}$ are said to be *woven* if there exist constants $0 < C \leq D$ such that, for every subset $\sigma \subset \mathbb{J}$ ($\mathbb{J}$ is a finite or countable index set), the family $\{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in \sigma^c}$ is a frame with frame bounds (C, D) . Over the years, various extensions of weaving frames have been investigated, see [6–10].

By allowing the inner product to take values in a C^* -algebra, Hilbert C^* -modules are natural generalizations of Hilbert spaces. Note that the theory of Hilbert C^* -modules is quite different from that of Hilbert spaces. For example, not all bounded linear operators on Hilbert C^* -modules are adjointable. Thus, there are many essential differences between Hilbert space frames and modular frames. The problems about modular frames are more complicated than those in Hilbert spaces. In this paper, we

investigate the weaving properties of modular frames. We refer to [11–13] for more information on frames in Hilbert C^* -modules.

The paper is organized as follows. Section 2 contains the definitions and some basic results about frames in Hilbert C^* -modules. In Section 3, we introduce the weaving frames in Hilbert C^* -modules and show that the equivalence of woven and weakly woven frames is still effective for Hilbert C^* -modules under certain conditions. Sufficient conditions for perturbations of weaving frames are given in Section 4. In Section 5, we give some weaving results for the nonunital case, and we investigate the redundancy property of weaving frames in Section 6.

2. Preliminaries

We first recall some basics about Hilbert C^* -modules.

Definition 1. Let $\mathcal{A}$ be a C^* -algebra. A pre-Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is a linear space and an algebraic (left) $\mathcal{A}$ -module with compatible scalar multiplication: $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$, and $x \in \mathcal{H}$, together with a map $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$ that possesses the following properties:

- (i) $\langle x, x \rangle \geq 0$, for every $x \in \mathcal{H}$

- (ii) $\langle x, x \rangle = 0$ if and only if $x = 0$
- (iii) $\langle x, y \rangle = \langle y, x \rangle^*$, for every $x, y \in \mathcal{H}$
- (iv) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$, for every $a, b \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$

The map $\|\cdot\|: x \mapsto \|\langle x, x \rangle\|^{1/2}$ defines a norm on $\mathcal{H}$, and $\mathcal{H}$ is called a Hilbert $\mathcal{A}$ -module if it is complete with respect to this norm. Denote by $\langle \mathcal{H}, \mathcal{H} \rangle \subset \mathcal{A}$ the closure of the linear span of all $\langle x, y \rangle$, $x, y \in \mathcal{H}$. A Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is called full if $\langle \mathcal{H}, \mathcal{H} \rangle = \mathcal{A}$.

Let $\mathcal{H}$ and $\mathcal{M}$ be Hilbert $\mathcal{A}$ -modules. A map $F: \mathcal{H} \rightarrow \mathcal{M}$ is called adjointable if there exists a map $F^*: \mathcal{M} \rightarrow \mathcal{H}$ such that

$$\langle x, F^*y \rangle = \langle Fx, y \rangle, \quad x \in \mathcal{H}, y \in \mathcal{M}. \quad (1)$$

Denote the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{M}$ by $\mathcal{L}(\mathcal{H}, \mathcal{M})$. The adjointable operator is automatically linear and bounded. In the case $\mathcal{H} = \mathcal{H}$, we write $\mathcal{L}(\mathcal{H})$ for simplicity. For more information on Hilbert C^* -modules, see [14–18].

Now, we review the definition of modular frames:

Definition 2. Let $\mathcal{A}$ be a unital C^* -algebra and $\mathbb{J}$ be a finite or countable index set. A sequence $\{x_j\}_{j \in \mathbb{J}}$ in a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is called a (standard) frame if there exist constants $C, D > 0$ such that, for every $x \in \mathcal{H}$,

$$C\langle x, x \rangle \leq \sum_{j \in \mathbb{J}} \langle x, x_j \rangle \langle x_j, x \rangle \leq D\langle x, x \rangle, \quad (2)$$

where the sum in the middle of the inequality converges in norm. The constants C and D in (2) are called the lower and upper frame bounds, respectively. In this case, we say $\{x_j\}_{j \in \mathbb{J}}$ is a (C, D) -frame for convenience. Moreover, we call $\{x_j\}_{j \in \mathbb{J}}$ a Bessel sequence if only the right inequality in (2) holds; a C -tight frame if $C = D$ and a Parseval frame if $C = D = 1$.

Recall that a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is finitely generated if there exists a finite set $\{x_1, \dots, x_n\} \subset \mathcal{H}$ such that every element $x \in \mathcal{H}$ can be expressed as an $\mathcal{A}$ -linear combination $x = \sum_{i=1}^n a_i x_i$, $a_i \in \mathcal{A}$. A Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is countably generated if there exists a countable set of generators. It follows from Kasparov's stabilization theorem (Theorem 2 in [14]) that every finitely or countably generated Hilbert C^* -module has a frame. Meanwhile, frames need not exist in general Hilbert C^* -modules (cf. [19]), so it is naturally restricting our consideration to finitely or countably generated Hilbert C^* -modules.

Next, we give an equivalent definition of modular frames which is easy to be applied.

Proposition 1 (see [13]). *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}} \subseteq \mathcal{H}$ a sequence. Then, $X = \{x_j\}_{j \in \mathbb{J}}$ is a frame of $\mathcal{H}$ with bounds (C, D) if and only if*

$$C\|x\|^2 \leq \left\| \sum_{j \in \mathbb{J}} \langle x, x_j \rangle \langle x_j, x \rangle \right\| \leq D\|x\|^2, \quad (3)$$

for all $x \in \mathcal{H}$.

For a unital C^* -algebra $\mathcal{A}$, let $\ell^2(\mathcal{A})$ be a Hilbert $\mathcal{A}$ -module defined by

$$\ell^2(\mathcal{A}) = \left\{ \{a_j\}_{j \in \mathbb{J}} \in \mathcal{A}: \sum_{j \in \mathbb{J}} a_j a_j^* \text{ converges in norm} \right\}, \quad (4)$$

with the inner product $\langle \{a_j\}_{j \in \mathbb{J}}, \{b_j\}_{j \in \mathbb{J}} \rangle = \sum_{j \in \mathbb{J}} a_j b_j^*$. Note that $\ell^2(\mathcal{A})$ possesses a canonical basis $\{e_j\}_{j \in \mathbb{J}}$, where e_j takes value $1_{\mathcal{A}}$ ($1_{\mathcal{A}}$ denotes the unit element of $\mathcal{A}$) at j and $0_{\mathcal{A}}$ everywhere else. For a given Bessel sequence $X = \{x_j\}_{j \in \mathbb{J}}$ in $\mathcal{H}$, its analysis operator,

$$\begin{aligned} U_X: \mathcal{H} &\rightarrow \ell^2(\mathcal{A}), \\ U_X x &= \sum_{j \in \mathbb{J}} \langle x, x_j \rangle e_j, \end{aligned} \quad (5)$$

is well defined and adjointable. The adjoint operator T_X of U_X fulfills $T_X e_j = x_j$ for every j and is called the synthesis operator. By composing U_X and T_X , we obtain the frame operator S_X :

$$\begin{aligned} S_X: \mathcal{H} &\rightarrow \mathcal{H}, \\ S_X x &= T_X U_X x = \sum_{j \in \mathbb{J}} \langle x, x_j \rangle x_j, \end{aligned} \quad (6)$$

where S_X is invertible and each $x \in \mathcal{H}$ can be represented as

$$x = \sum_{j \in \mathbb{J}} \langle x, S_X^{-1} x_j \rangle x_j, \quad (7)$$

where $\{S_X^{-1} x_j\}_{j \in \mathbb{J}}$ is called the canonical dual frame of $\{x_j\}_{j \in \mathbb{J}}$.

At the end of this section, we present some fundamental results which will be used in the following sections.

Proposition 2 (see [17]). *For an operator $F: \mathcal{H} \rightarrow \mathcal{H}$, the following conditions are equivalent:*

- (i) F is a positive element of $\mathcal{L}(\mathcal{H})$
- (ii) For every $x \in \mathcal{H}$, $\langle Fx, x \rangle$ is positive in $\mathcal{A}$

Proposition 3 (see [20]). *Let $F \in \mathcal{L}(\mathcal{H})$. Then, for any $x \in \mathcal{H}$, we have $\langle Fx, Fx \rangle \leq \|F\|^2 \langle x, x \rangle$.*

3. Weaving Frames in Hilbert C^* -Modules

In this section, we investigate the weaving frames in Hilbert C^* -modules. It can be observed from (7) that each frame for a Hilbert C^* -module $\mathcal{H}$ generates $\mathcal{H}$. This means that a Hilbert C^* -module that admits frames is necessarily countably generated.

Throughout the rest of the paper, for ease of notation, let

$$[m] = \{1, \dots, m\}, \quad (8)$$

for a given natural number m . First, we give the definition of weaving frames in Hilbert C^* -modules.

Definition 3. A finite family of frames $\left\{\{x_{ij}\}_{j \in \mathbb{J}}, i \in [m]\right\}$ in a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is said to be woven if there are universal constants C and D such that, for every partition $\{\sigma_1, \dots, \sigma_m\}$ of $\mathbb{J}$, the family $\{x_{ij}\}_{i=1, j \in \sigma_i}^m (= \cup_{i \in [m]} \{x_{ij}\}_{j \in \sigma_i})$ is a frame for $\mathcal{H}$ with lower and upper frame bounds C and D , respectively. In this case, we usually call $\left\{\{x_{ij}\}_{j \in \mathbb{J}}, i \in [m]\right\}$ woven with universal bounds (C, D) . Each family $\{x_{ij}\}_{i=1, j \in \sigma_i}^m$ is called a weaving.

It is known from Proposition 3.1 in [5] that every weaving for a woven family automatically has a universal upper frame bound. By the positivity of the summand in (2), we can also derive a universal Bessel bound for every weaving in a Hilbert C^* -module.

Proposition 4. Suppose that $\left\{\{x_{ij}\}_{j \in \mathbb{J}}, i \in [m]\right\}$ are Bessel sequences for a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ with Bessel bounds D_i , respectively. Then, every weaving is a Bessel sequence with a Bessel bound $\sum_{i=1}^m D_i$.

Proof. For every partition $\{\sigma_1, \dots, \sigma_m\}$ of $\mathbb{J}$ and any $x \in \mathcal{H}$, we have

$$\sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle \langle x_{ij}, x \rangle \leq \sum_{i=1}^m \sum_{j \in \mathbb{J}} \langle x, x_{ij} \rangle \langle x_{ij}, x \rangle \leq \sum_{i=1}^m D_i \langle x, x \rangle, \quad (9)$$

as required.

Consequently, we only need to check the lower frame bound when studying the property of woven frames. The following result implies that the woven property is preserved under some adjointable operators. $\square$

Proposition 5. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $\left\{\{x_{ij}\}_{j \in \mathbb{J}}, i \in [m]\right\}$ be a woven family of frames for $\mathcal{H}$ with universal bounds (C, D) .

- (1) If $F \in \mathcal{L}(\mathcal{H})$ is invertible, then the family $\{Fx_{ij}\}_{i=1, j \in \sigma_i}^m$ is woven with universal bounds $(C\|F^{-1}\|^{-2}, D\|F\|^2)$
- (2) If $G \in \mathcal{L}(\mathcal{H})$ is a co-isometry, then $\{Gx_{ij}\}_{i=1, j \in \sigma_i}^m$ is woven with universal bounds (C, D)

Proof.

- (1) For every partition $\{\sigma_1, \dots, \sigma_m\}$ of $\mathbb{J}$ and any $x \in \mathcal{H}$,

$$C \langle F^* x, F^* x \rangle \leq \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, Fx_{ij} \rangle \langle Fx_{ij}, x \rangle \leq D \langle F^* x, F^* x \rangle. \quad (10)$$

It follows from Proposition 3 that

$$\|F^{-1}\|^{-2} \langle x, x \rangle \leq \|F^* x, F^* x\| \leq \|F\|^2 \langle x, x \rangle. \quad (11)$$

Hence, $\{Fx_{ij}\}_{i=1, j \in \sigma_i}^m$ is a frame with bounds $(C\|F^{-1}\|^{-2}, D\|F\|^2)$.

(2) Since G is a co-isometry, we know $\|G^* x\| = \|x\|$. Then, by E. C. Lance's theorem [16], $\langle G^* x, G^* x \rangle = \langle x, x \rangle$. Now, the conclusion follows from a similar discussion of (1).

We show that multiplying the frames in the woven family by individual elements still preserves the woven property. $\square$

Proposition 6. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $\left\{\{x_{ij}\}_{j \in \mathbb{J}}, i \in [m]\right\}$ be a woven family of frames for $\mathcal{H}$ with universal frame bounds (C, D) . If there exist $a_{ij} \in \mathcal{A}$ and $0 < \alpha \leq \beta < \infty$ which satisfies $\alpha \cdot 1_{\mathcal{A}} \leq a_{ij}^* a_{ij} \leq \beta \cdot 1_{\mathcal{A}}$, then $\{a_{ij}x_{ij}\}_{i=1, j \in \sigma_i}^m$ is also woven with universal bounds $(\alpha C, \beta D)$.

Proof. Recall that $a \leq b$ implies $c^* a c \leq c^* b c$, for any $a, b \in \mathcal{A}_{sa}$ ($\mathcal{A}_{sa}$ denotes the set of all self-adjoint elements in $\mathcal{A}$) and $c \in \mathcal{A}$. On the one hand, we have for any partition $\{\sigma_1, \dots, \sigma_m\}$ of $\mathbb{J}$ and any $x \in \mathcal{H}$,

$$\begin{aligned} \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, a_{ij}x_{ij} \rangle \langle a_{ij}x_{ij}, x \rangle &= \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle a_{ij}^* a_{ij} \langle x_{ij}, x \rangle \\ &\leq \beta \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle \langle x_{ij}, x \rangle \leq \beta D \langle x, x \rangle. \end{aligned} \quad (12)$$

On the other hand, we also have

$$\begin{aligned} \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, a_{ij}x_{ij} \rangle \langle a_{ij}x_{ij}, x \rangle &= \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle a_{ij}^* a_{ij} \langle x_{ij}, x \rangle \\ &\geq \alpha \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle \langle x_{ij}, x \rangle \geq \alpha C \langle x, x \rangle. \end{aligned} \quad (13)$$

This gives the desired result.

Now, we give a definition of weakly woven frames. $\square$

Definition 4. A finite family of frames $\left\{\{x_{ij}\}_{j \in \mathbb{J}}, i \in [m]\right\}$ in a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is called weakly woven if, for every partition $\{\sigma_1, \dots, \sigma_m\}$ of $\mathbb{J}$, the family $\{x_{ij}\}_{i=1, j \in \sigma_i}^m$ is a frame for $\mathcal{H}$.

Bemrose et al. presented a characterization of the equivalence of woven and weakly woven frames in [5]. At the end of this section, we consider whether this equivalence is still effective for Hilbert C^* -modules. As a preparation, we need the following proposition.

Proposition 7. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $\left\{\{x_{ij}\}_{j \in \mathbb{J}}, i \in [m]\right\}$ be a family of frames for $\mathcal{H}$. Assume, for every partition $\{\Lambda_i\}_{i=1}^m$ of a finite subset of $\mathbb{J}$ and every $\varepsilon > 0$, there exists a $\Lambda_i \subset \Gamma_i$ ($i \in [m]$) partition $\{\Gamma_i\}_{i=1}^m$ of $\mathbb{J}$ such that and the lower frame bound of

$\{x_{ij}\}_{i=1,j \in \Gamma_i}^m$ is less than ε . Then, there exists a partition $\{\sigma_i\}_{i=1}^m$ of $\mathbb{J}$ such that $\{x_{ij}\}_{i=1,j \in \sigma_i}^m$ is not a frame.

Proof. Assume without loss of generality that $\mathbb{J} = \mathbb{N}$. Choose $\varepsilon = 1/2$ and $\Lambda_i^1 = \emptyset$ for $i \in [m]$. Then, by assumption, there exists a partition $\{\Gamma_i^1\}_{i=1}^m$ of $\mathbb{N}$ such that $\Lambda_i^1 \subset \Gamma_i^1$ for all $i \in [m]$ and the lower frame bound of $\{x_{ij}\}_{i=1,j \in \Gamma_i^1}^m$ is less than $1/2$. So, there exists a element y_1 such that

$$\sum_{i=1}^m \sum_{j \in \Gamma_i^1} \langle y_1, x_{ij} \rangle \langle x_{ij}, y_1 \rangle < \frac{1}{2} \langle y_1, y_1 \rangle. \quad (14)$$

By the convergence of the series $\sum_{i=1}^m \sum_{j \in \mathbb{N}} \langle y_1, x_{ij} \rangle \langle x_{ij}, y_1 \rangle$ for each $x \in \mathcal{H}$, we can find a $k_1 \in \mathbb{N}$ such that $\sum_{i=1}^m \sum_{j \geq k_1} \langle y_1, x_{ij} \rangle \langle x_{ij}, y_1 \rangle < 1/2 \langle y_1, y_1 \rangle$.

Let $\Lambda_i^2 = \Gamma_i^1 \cap [k_1]$ and $\varepsilon = 1/2^2$. Then, there exists a partition $\{\Gamma_i^2\}_{i=1}^m$ of $\mathbb{N}$ and an element $y_2 \in \mathcal{H}$ such that $\Lambda_i^2 \subset \Gamma_i^2$ for any $i \in [m]$ and

$$\sum_{i=1}^m \sum_{j \in \Gamma_i^2} \langle y_2, x_{ij} \rangle \langle x_{ij}, y_2 \rangle < \frac{1}{2^2} \langle y_2, y_2 \rangle. \quad (15)$$

Since $\sum_{i=1}^m \sum_{j \in \mathbb{N}} \langle y_2, x_{ij} \rangle \langle x_{ij}, y_2 \rangle$ converges, there exists a $k_2 > k_1$ such that

$$\sum_{i=1}^m \sum_{j \geq k_2} \langle y_2, x_{ij} \rangle \langle x_{ij}, y_2 \rangle < \frac{1}{2^2} \langle y_2, y_2 \rangle. \quad (16)$$

Continuing in this way, for a partition $\{\Lambda_i^{n+1} = \Gamma_i^n \cap [k_n], i \in [m]\}$ of $[k_n]$, we can find a partition $\{\Gamma_i^{n+1}\}_{i=1}^m$ of $\mathbb{N}$ and $y_{n+1} \in \mathcal{H}$ such that $\Lambda_i^{n+1} \subset \Gamma_i^{n+1}$ for every $i \in [m]$ and

$$\sum_{i=1}^m \sum_{j \in \Gamma_i^{n+1}} \langle y_{n+1}, x_{ij} \rangle \langle x_{ij}, y_{n+1} \rangle < \frac{1}{2^{n+1}} \langle y_{n+1}, y_{n+1} \rangle. \quad (17)$$

Similar to above, there exists a $k_{n+1} > k_n$ such that

$$\sum_{i=1}^m \sum_{j \geq k_{n+1}} \langle y_{n+1}, x_{ij} \rangle \langle x_{ij}, y_{n+1} \rangle < \frac{1}{2^{n+1}} \langle y_{n+1}, y_{n+1} \rangle. \quad (18)$$

Now, set $\sigma_i = \cup_{n=1}^{\infty} \Lambda_i^{n+1}$ for $i \in [m]$ and observe that $\{\sigma_i\}_{i=1}^m$ is a partition of $\mathbb{N}$. It follows from construction along with (17) and (18) that

$$\begin{aligned} & \sum_{i=1}^m \sum_{j \in \sigma_i} \langle y_{n+1}, x_{ij} \rangle \langle x_{ij}, y_{n+1} \rangle \\ &= \sum_{i=1}^m \sum_{j \in \Lambda_i^{n+1}} \langle y_{n+1}, x_{ij} \rangle \langle x_{ij}, y_{n+1} \rangle + \sum_{i=1}^m \sum_{j \in \sigma_i \cap [k_n]^c} \langle y_{n+1}, x_{ij} \rangle \langle x_{ij}, y_{n+1} \rangle \\ &\leq \sum_{i=1}^m \sum_{j \in \Gamma_i^{n+1}} \langle y_{n+1}, x_{ij} \rangle \langle x_{ij}, y_{n+1} \rangle + \sum_{i=1}^m \sum_{j \geq k_{n+1}} \langle y_{n+1}, x_{ij} \rangle \langle x_{ij}, y_{n+1} \rangle \\ &< \frac{1}{2^n} \langle y_{n+1}, y_{n+1} \rangle, \end{aligned} \quad (19)$$

so that a lower frame bound of $\{x_{ij}\}_{i=1,j \in \sigma_i}^m$ is zero. Hence, it is not a frame and we have the result. $\square$

Corollary 1. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $\{\{x_{ij}\}_{j \in \mathbb{J}}, i \in [m]\}$ be a weakly woven family of frames for $\mathcal{H}$. Then, there exist disjoint finite sets $\Lambda_1, \dots, \Lambda_m \subset \mathbb{J}$ and a constant $C > 0$ such that, for any partition $\{\sigma_i\}_{i=1}^m$ of $\mathbb{J}$ with $\Lambda_i \subset \sigma_i$ for $i \in [m]$, the family $\{x_{ij}\}_{i=1,j \in \sigma_i}^m$ has a lower frame bound C .

Every bounded sequence in a Hilbert space has a weakly convergent subsequence. However, this is usually not true for bounded sequences in general Hilbert C^* -modules. Thus, the computations will be quite different than given in Proposition 4.5 in [5] and more conditions will be needed when investigating the analogous result.

Theorem 1. Let $\mathcal{H}$ be a full Hilbert $\mathcal{A}$ -module over a finite-dimensional C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames for $\mathcal{H}$. Then, the following are equivalent:

- (i) X and Y are woven
- (ii) X and Y are weakly woven

Proof. (i) $\Rightarrow$ (ii) is clear. For the proof of (ii) $\Rightarrow$ (i), we see from Proposition 4 that only a universal lower bound for X and Y needs to be shown. By Corollary 1, there exist disjoint finite sets $\Lambda_0, \Gamma_0 \subset \mathbb{J}$ and a constant $C > 0$ such that, for any subset σ_0 of $\mathbb{N}$ with $\Lambda_0 \subset \sigma_0$ and $\Gamma_0 \subset \sigma_0^c$ and every $x \in \mathcal{H}$,

$$\left\| \sum_{j \in \sigma_0} \langle x, x_j \rangle \langle x_j, x \rangle + \sum_{j \in \sigma_0^c} \langle x, y_j \rangle \langle y_j, x \rangle \right\| \geq C \|x\|^2. \quad (20)$$

Since permuting both frames simultaneously does not affect weaving, we permute X and Y such that $\Lambda_0 \cup \Gamma_0 = [m]$.

If we can prove that, for any partition $\{\Lambda_\alpha, \Gamma_\alpha\}$ of $[m]$, there exist $C_\alpha > 0$ such that, for every subset σ_α of $\mathbb{N}$ with $\Lambda_\alpha \subset \sigma_\alpha$ and $\Gamma_\alpha \subset \sigma_\alpha^c$, the family $\{x_j\}_{j \in \sigma_\alpha} \cup \{y_j\}_{j \in \sigma_\alpha^c}$ has a lower frame bound C_α ; then, the frames X and Y are woven with a universal lower bound $C_0 = \min\{C_\alpha : \alpha\}$, and we will obtain the desired result. Now, assume the above result is not true, and we will show that this yields a contradiction.

First, by hypothesis, we can find a partition $\{\Lambda_1, \Gamma_1\}$ of $[m]$ such that, for any $\varepsilon > 0$, there exists a subset σ_1 of $\mathbb{N}$ with $\Lambda_1 \subset \sigma_1$ and $\Gamma_1 \subset \sigma_1^c$ such that the family $\{x_j\}_{j \in \sigma_1} \cup \{y_j\}_{j \in \sigma_1^c}$ has a lower frame bound less than ε . Then, for all $n \in \mathbb{N}$, there exist subsets $\sigma_n \subset \mathbb{N}$ with $\Lambda_1 \subset \sigma_n$ and $\Gamma_1 \subset \sigma_n^c$ and $h_n \in \mathcal{H}$ with $\|h_n\| = 1$ such that

$$\left\| \sum_{j \in \sigma_n} \langle h_n, x_j \rangle \langle x_j, h_n \rangle + \sum_{j \in \sigma_n^c} \langle h_n, y_j \rangle \langle y_j, h_n \rangle \right\| < \frac{1}{n}, \quad (21)$$

and the sets σ_n satisfy the following properties:

- (1) For every $k = 1, 2, \dots$, either $m + k \in \sigma_n$, for all $n \geq k$, or $m + k \in \sigma_n^c$, for all $n \geq k$
- (2) There exist a subset σ of $\mathbb{N}$ with $\Lambda_1 \subset \sigma$ and $\Gamma_1 \subset \sigma^c$ such that $m + k \in \sigma$ implies that $m + k \in \sigma_n$, for all $n \geq k$, or if $m + k \in \sigma^c$, then $m + k \in \sigma_n^c$, for all $n \geq k$

Since $\mathcal{H}$ is a Hilbert C^* -module over a finite-dimensional C^* -algebra $\mathcal{A}$, we see from Proposition 2.1 in [21] that, for the norm-bounded sequence $\{h_n\}_{n=1}^\infty$, there exists a subsequence $\{h_{n_i}\}_{i=1}^\infty$ of $\{h_n\}_{n=1}^\infty$ and $h \in \mathcal{H}$ such that

$$\|\langle y, h_{n_i} \rangle - \langle y, h \rangle\| \longrightarrow 0 \text{ (as } n \longrightarrow \infty) \text{ for any } y \in \mathcal{H}. \quad (22)$$

We reindex $h_{n_i} \longrightarrow h_i$ and $\sigma_{n_i} \longrightarrow \sigma_i$ and notice that (21), (1), and (2) are still satisfied in this construction. Now, fix $k \in \mathbb{N}$ so that $k > 2/C$, where C is the constant in (20). The fact that $\mathcal{H}$ is full (Theorem 2.5 in [22]) and that finite-dimensional C^* -algebras are unital imply that $\{h_n\}_{n=1}^\infty$ converges in norm to h on finite-dimensional subspaces of $\mathcal{H}$. Hence, we can find an $N_k \in \mathbb{N}$ such that, for all $n \geq N_k > k$,

$$\left\| \sum_{j \in [m+k]} \langle h - h_n, x_j \rangle \langle x_j, h - h_n \rangle + \sum_{j \in [m+k]^c} \langle h - h_n, y_j \rangle \langle y_j, h - h_n \rangle \right\| < \frac{1}{2k}. \quad (23)$$

Denote $\delta = \Lambda_0 \cup (\sigma \cap [m]^c)$ and $\delta_n = \Lambda_0 \cup (\sigma_n \cap [m]^c)$. It follows from (20) that

$$\left\| \sum_{j \in \delta_n} \langle h_n, x_j \rangle \langle x_j, h_n \rangle + \sum_{j \in \delta_n^c} \langle h_n, y_j \rangle \langle y_j, h_n \rangle \right\| \geq C. \quad (24)$$

By definition of σ and δ , for $n \geq N_k > k$, $\sigma \cap [m, k] = \sigma_n \cap [m, k]$, where $[m, k] = \{m+1, m+2, \dots, m+k\}$. Then, $\delta \cap [m+k] = \Lambda_0 \cup (\sigma \cap [m, k]) = \Lambda_0 \cup (\sigma_n \cap [m, k]) = \delta_n \cap [m+k]$, and thus, inequalities (21), (23), and (24) imply that

$$\begin{aligned} & \left\| \sum_{j \in \delta} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \delta^c} \langle h, y_j \rangle \langle y_j, h \rangle \right\| \\ & \geq \left\| \sum_{j \in \delta \cap [m+k]} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \delta^c \cap [m+k]} \langle h, y_j \rangle \langle y_j, h \rangle \right\| \\ & = \left\| \sum_{j \in \delta_n \cap [m+k]} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \delta_n^c \cap [m+k]} \langle h, y_j \rangle \langle y_j, h \rangle \right\| \\ & \geq \frac{1}{2} \left\| \sum_{j \in \delta_n \cap [m+k]} \langle h_n, x_j \rangle \langle x_j, h_n \rangle + \sum_{j \in \delta_n^c \cap [m+k]} \langle h_n, y_j \rangle \langle y_j, h_n \rangle \right\| \\ & \quad - \left\| \sum_{j \in \delta_n \cap [m+k]} \langle h - h_n, x_j \rangle \langle x_j, h - h_n \rangle + \sum_{j \in \delta_n^c \cap [m+k]} \langle h - h_n, y_j \rangle \langle y_j, h - h_n \rangle \right\| \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2} \left\| \sum_{j \in \delta_n} \langle h_n, x_j \rangle \langle x_j, h_n \rangle + \sum_{j \in \delta_n^c} \langle h_n, y_j \rangle \langle y_j, h_n \rangle - \sum_{j \in \delta_n \cap [m+k]^c} \langle h_n, x_j \rangle \langle x_j, h_n \rangle \right. \\
&\quad \left. + \sum_{j \in \delta_n^c \cap [m+k]^c} \langle h_n, y_j \rangle \langle y_j, h_n \rangle \right\| \\
&\quad - \left\| \sum_{j \in [m+k]} \langle h - h_n, x_j \rangle \langle x_j, h - h_n \rangle + \sum_{j \in [m+k]} \langle h - h_n, y_j \rangle \langle y_j, h - h_n \rangle \right\| \\
&\geq \frac{1}{2} \left\| \sum_{j \in \delta_n} \langle h_n, x_j \rangle \langle x_j, h_n \rangle + \sum_{j \in \delta_n^c} \langle h_n, y_j \rangle \langle y_j, h_n \rangle \right\| \\
&\quad - \frac{1}{2} \left\| \sum_{j \in \delta_n \cap [m+k]^c} \langle h_n, x_j \rangle \langle x_j, h_n \rangle + \sum_{j \in \delta_n^c \cap [m+k]^c} \langle h_n, y_j \rangle \langle y_j, h_n \rangle \right\| \\
&\quad - \frac{1}{2k} \geq \frac{1}{2} \cdot C - \frac{1}{2} \cdot \frac{1}{n} - \frac{1}{2k} \\
&\geq \frac{C}{2} - \frac{1}{k} > 0.
\end{aligned} \tag{25}$$

This gives $h \neq 0$.

Finally, we will show that $\{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in \sigma^c}$ is not a frame for $\mathcal{H}$. Similar to the above, for the set σ given in (2), $\sigma \cap [m+k] = \sigma_{N_k} \cap [m+k]$. Thus, using (21) and (23),

$$\begin{aligned}
&\left\| \sum_{j \in \sigma \cap [m+k]} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \sigma^c \cap [m+k]} \langle h, y_j \rangle \langle y_j, h \rangle \right\| \\
&= \left\| \sum_{j \in \sigma_{N_k} \cap [m+k]} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \sigma_{N_k}^c \cap [m+k]} \langle h, y_j \rangle \langle y_j, h \rangle \right\| \\
&\leq 2 \left\| \sum_{j \in \sigma_{N_k} \cap [m+k]} \langle h_{N_k}, x_j \rangle \langle x_j, h_{N_k} \rangle + \sum_{j \in \sigma_{N_k}^c \cap [m+k]} \langle h_{N_k}, y_j \rangle \langle y_j, h_{N_k} \rangle \right\| \\
&\quad + 2 \left\| \sum_{j \in \sigma_{N_k} \cap [m+k]} \langle h - h_{N_k}, x_j \rangle \langle x_j, h - h_{N_k} \rangle + \sum_{j \in \sigma_{N_k}^c \cap [m+k]} \langle h - h_{N_k}, y_j \rangle \langle y_j, h - h_{N_k} \rangle \right\| \\
&\leq 2 \left\| \sum_{j \in \sigma_{N_k}} \langle h_{N_k}, x_j \rangle \langle x_j, h_{N_k} \rangle + \sum_{j \in \sigma_{N_k}^c} \langle h_{N_k}, y_j \rangle \langle y_j, h_{N_k} \rangle \right\| \\
&\quad + 2 \left\| \sum_{j \in [m+k]} \langle h - h_{N_k}, x_j \rangle \langle x_j, h - h_{N_k} \rangle + \sum_{j \in [m+k]} \langle h - h_{N_k}, y_j \rangle \langle y_j, h - h_{N_k} \rangle \right\| \\
&\leq 2 \cdot \frac{1}{N_k} + 2 \cdot \frac{1}{2k}.
\end{aligned} \tag{26}$$

Letting $k \rightarrow \infty$ in the above formula,

$$\left\| \sum_{j \in \sigma} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \sigma^c} \langle h, y_j \rangle \langle y_j, h \rangle \right\| = 0, \quad (27)$$

implying that $\{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in \sigma^c}$ is not a frame. Thus, a contradiction is met, concluding the proof. $\square$

4. Perturbations of Weaving Frames

Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ be a Bessel sequence for $\mathcal{H}$ with the analysis operator U_X . To characterize weaving frames, we need to define the analysis operator relevant to the partitions. For any given subset σ of $\mathbb{J}$, denoted by P_σ , the orthogonal projection is onto $\overline{\text{span}}\{e_j\}_{j \in \sigma}$, where $\{e_j\}_{j \in \mathbb{J}}$ is the standard orthonormal basis of $\ell^2(\mathcal{A})$. Then, the analysis operator relevant to σ is defined by

$$\begin{aligned} U_X^\sigma: \mathcal{H} &\longrightarrow \ell^2(\mathcal{A}), \\ U_X^\sigma x &= P_\sigma U_X x = \sum_{j \in \sigma} \langle x, x_j \rangle e_j. \end{aligned} \quad (28)$$

Similarly, the synthesis operator T_X^σ relevant to σ is given by

$$\begin{aligned} T_X^\sigma: \ell^2(\mathcal{A}) &\longrightarrow \mathcal{H}, \\ T_X^\sigma \{c_j\}_{j \in \mathbb{J}} &= T_X P_\sigma \{c_j\}_{j \in \mathbb{J}} = \sum_{j \in \sigma} c_j x_j. \end{aligned} \quad (29)$$

By composing U_X^σ and T_X^σ , we obtain the frame operator S_X^σ relevant to σ :

$$\begin{aligned} S_X^\sigma: \mathcal{H} &\longrightarrow \mathcal{H}, \\ S_X^\sigma x &= T_X^\sigma U_X^\sigma x = \sum_{j \in \sigma} \langle x, x_j \rangle x_j. \end{aligned} \quad (30)$$

Let $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames for $\mathcal{H}$. We compute, for any $\sigma \subset \mathbb{J}$ and any $x \in \mathcal{H}$,

$$\|U_X^\sigma x + U_Y^{\sigma^c} x\|^2 = \left\| \sum_{j \in \sigma} \langle x, x_j \rangle \langle x_j, x \rangle + \sum_{j \in \sigma^c} \langle x, y_j \rangle \langle y_j, x \rangle \right\|. \quad (31)$$

Then, by Proposition 1, we know X and Y are woven with universal bounds (C, D) if and only if, for any $\sigma \subset \mathbb{J}$,

$$\sqrt{C}\|x\| \leq \|U_X^\sigma x + U_Y^{\sigma^c} x\| \leq \sqrt{D}\|x\|. \quad (32)$$

We begin investigating the perturbations of weaving frames.

Theorem 2. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames for $\mathcal{H}$ with frame bounds (C_1, D_1) and (C_2, D_2) , respectively. Assume that there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ such that

$$\lambda_1 \sqrt{C_1} + \lambda_2 \sqrt{D_2} + \mu < \sqrt{C_1}, \quad (33)$$

and for any $x \in \mathcal{H}$,

$$\|U_X x - U_Y x\| \leq \lambda_1 \|U_X x\| + \lambda_2 \|U_Y x\| + \mu \|x\|, \quad (34)$$

where U_X and U_Y denote the analysis operators for X and Y , respectively. Then, X and Y are woven frames with universal bounds $(\sqrt{C_1} - \lambda_1 \sqrt{C_1} - \lambda_2 \sqrt{D_2} - \mu)^2$ and $\min\{(\sqrt{D_1} + \lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2} + \mu)^2, D_1 + D_2\}$.

Proof. For any given $\sigma \subset \mathbb{J}$, we denote $\{\eta_j\}_{j \in \mathbb{J}} = \{x_j\}_{j \in \sigma^c} \cup \{y_j\}_{j \in \sigma}$ and define an operator $U_\eta: \mathcal{H} \rightarrow \ell^2(\mathcal{A})$ by

$$U_\eta x = U_X^{\sigma^c} x + U_Y^\sigma x = \sum_{j \in \sigma^c} \langle x, x_j \rangle e_j + \sum_{j \in \sigma} \langle x, y_j \rangle e_j. \quad (35)$$

It is clear that U_η is the analysis operator for $\{\eta_j\}_{j \in \mathbb{J}}$. Hence, for any $x \in \mathcal{H}$,

$$\begin{aligned} \|U_X x - U_Y x\| &= \|U_X x - U_X^{\sigma^c} x - U_Y^\sigma x\| \\ &= \left\| \sum_{j \in \mathbb{J}} \langle x, x_j \rangle e_j - \sum_{j \in \sigma^c} \langle x, x_j \rangle e_j - \sum_{j \in \sigma} \langle x, y_j \rangle e_j \right\| \\ &= \left\| \sum_{j \in \sigma} \langle x, x_j - y_j \rangle e_j \right\| \\ &= \left\| \sum_{j \in \sigma} \langle x, x_j - y_j \rangle \langle x_j - y_j, x \rangle \right\|^{1/2} \\ &\leq \left\| \sum_{j \in \mathbb{J}} \langle x, x_j - y_j \rangle \langle x_j - y_j, x \rangle \right\|^{1/2} \\ &= \|U_X x - U_Y x\|. \end{aligned} \quad (36)$$

On the one hand, we see from (34) that

$$\|U_X x\| - \|U_Y x\| \leq \|U_X x - U_Y x\| \leq \lambda_1 \|U_X x\| + \lambda_2 \|U_Y x\| + \mu \|x\|. \quad (37)$$

Hence,

$$\begin{aligned} \|U_Y x\| &\geq (1 - \lambda_1) \|U_X x\| - \lambda_2 \|U_Y x\| - \mu \|x\| \\ &\geq (1 - \lambda_1) \sqrt{C_1} \|x\| - \lambda_2 \sqrt{D_2} \|x\| - \mu \|x\| \\ &= (\sqrt{C_1} - \lambda_1 \sqrt{C_1} - \lambda_2 \sqrt{D_2} - \mu) \|x\|. \end{aligned} \quad (38)$$

On the other hand, we also have

$$\|U_\eta x\| - \|U_X x\| \leq \|U_X x - U_Y x\| \leq \lambda_1 \|U_X x\| + \lambda_2 \|U_Y x\| + \mu \|x\|. \quad (39)$$

Therefore,

$$\begin{aligned} \|U_\eta x\| &\leq (\lambda_1 + 1) \|U_X x\| + \lambda_2 \|U_Y x\| + \mu \|x\| \\ &\leq (\lambda_1 + 1) \sqrt{D_1} \|x\| + \lambda_2 \sqrt{D_2} \|x\| + \mu \|x\| \\ &= (\sqrt{D_1} + \lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2} + \mu) \|x\|, \end{aligned} \quad (40)$$

as required. $\square$

By choosing $\lambda_1 = \lambda_2 = 0$ in Theorem 2, we see that X and Y are woven with universal bounds $(\sqrt{C_1} - \mu)^2$ and $\min\{(\sqrt{D_1} + \mu)^2, D_1 + D_2\}$, when $\mu < \sqrt{C_1}$ and $\|U_X x - U_Y x\| \leq \mu \|x\|$. If the value for the universal bounds is irrelevant, we can get the following result.

Corollary 2. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames for $\mathcal{H}$ with frame bounds (C_1, D_1) and (C_2, D_2) , respectively. If, for any $x \in H$,

$$\|U_X x - U_Y x\| < \sqrt{C_1} \|x\|, \text{ or } \|U_X x - U_Y x\| < \sqrt{C_2} \|x\|, \quad (41)$$

then X and Y are woven.

The property of woven frames is preserved under an adjointable invertible operator. However, applying two different operators to woven frames does not always preserve the weaving property. Using Corollary 2, we consider under the given frames are woven under different operators.

Corollary 3. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames for $\mathcal{H}$ with frame bounds (C_1, D_1) and (C_2, D_2) , respectively. Assume $F_1, F_2 \in \mathcal{L}(\mathcal{H})$ are invertible operators on $\mathcal{H}$. Then, $\{F_1 x_j\}$ and $\{F_2 y_j\}$ are woven frames when $(\sqrt{D_1} \|F_1\| + \sqrt{D_2} \|F_2\|) \|F_1^{-1}\| < \sqrt{C_1}$ or $(\sqrt{D_1} \|F_1\| + \sqrt{D_2} \|F_2\|) \|F_1^{-1}\| < \sqrt{C_2}$.

Proof. Note that $\{F_1 x_j\}_{j \in \mathbb{J}}$ (resp. $\{F_2 y_j\}_{j \in \mathbb{J}}$) is a frame with lower frame bound $C_1 \|F_1^{-1}\|^{-2}$ (resp. $C_2 \|F_2^{-1}\|^{-2}$) and analysis operator $U_X F_1^*$ (resp. $U_Y F_2^*$). Thus, by assumption,

$$\begin{aligned} \|U_X F_1^* x - U_Y F_2^* x\| &\leq \|U_X F_1^* x\| + \|U_Y F_2^* x\| \\ &\leq \sqrt{D_1} \|F_1\| \|x\| + \sqrt{D_2} \|F_2\| \|x\| \\ &< \sqrt{C_1} \|F_1^{-1}\|^{-2} \|x\|. \end{aligned} \quad (42)$$

Using Corollary 2, we obtain the desired result. $\square$

Canonical dual frames play a fundamental role in the study of woven frames. Using Corollary 3, we now derive a perturbation result for canonical dual frames.

Theorem 3. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames for $\mathcal{H}$ with frame bounds (C_1, D_1) and (C_2, D_2) , respectively. If, for any $x \in H$,

$$\|U_X x - U_Y x\| < \frac{C_1 C_2}{C_2 \sqrt{D_1} + D_1 \sqrt{D_2} + D_2 \sqrt{D_1}} \|x\|, \quad (43)$$

or

$$\|U_X x - U_Y x\| < \frac{C_1 C_2}{C_1 \sqrt{D_2} + D_1 \sqrt{D_2} + D_2 \sqrt{D_1}} \|x\|, \quad (44)$$

then $\{S_X^{-1} x_j\}_{j \in \mathbb{J}}$ and $\{S_Y^{-1} y_j\}_{j \in \mathbb{J}}$ are woven.

Proof. Let μ denote $C_1 C_2 / C_2 \sqrt{D_1} + D_1 \sqrt{D_2} + D_2 \sqrt{D_1}$, and we have

$$\|U_X x - U_Y x\| < \mu \|x\|, \quad (45)$$

$$\frac{\mu}{C_1} + \frac{\mu}{C_1 C_2} (\sqrt{D_1 D_2} + D_2) = \sqrt{\frac{1}{D_1}}. \quad (46)$$

Now, using (45), we get, for any $x \in \mathcal{H}$,

$$\begin{aligned} \|S_X x - S_Y x\| &= \|T_X U_X x - T_Y U_Y x\| \\ &\leq \|T_X U_X x - T_X U_Y x\| + \|T_X U_Y x - T_Y U_Y x\| \\ &< \sqrt{D_1} \|U_X x - U_Y x\| + \mu \|U_Y x\| \\ &< \mu (\sqrt{D_1} + \sqrt{D_2}) \|x\|. \end{aligned} \quad (47)$$

Note that $\{S_X^{-1} x_j\}_{j \in \mathbb{J}}$ and $\{S_Y^{-1} y_j\}_{j \in \mathbb{J}}$ are frames with frame bounds $(1/D_1, 1/C_1)$ and $(1/D_2, 1/C_2)$, respectively. Hence, for any $x \in \mathcal{H}$,

$$\|S_X^{-1} x - S_Y^{-1} x\| = \|S_Y^{-1} (S_Y - S_X) S_X^{-1} x\| < \frac{\mu}{C_1 C_2} (\sqrt{D_1} + \sqrt{D_2}). \quad (48)$$

Now, denote $\{S_X^{-1} x_j\}_{j \in \mathbb{J}}$ and $\{S_Y^{-1} y_j\}_{j \in \mathbb{J}}$ by $S_X^{-1} X$ and $S_Y^{-1} Y$, respectively. Putting (45), (46), and (48) together, we get, for any $x \in H$,

$$\begin{aligned} \|U_{S_X^{-1} X} x - U_{S_Y^{-1} Y} x\| &= \left\| \sum_{j \in \mathbb{J}} \langle x, S_X^{-1} x_j \rangle e_j - \sum_{j \in \mathbb{J}} \langle x, S_Y^{-1} y_j \rangle e_j \right\| \\ &\leq \left\| \sum_{j \in \mathbb{J}} \langle x, S_X^{-1} x_j \rangle e_j - \sum_{j \in \mathbb{J}} \langle x, S_X^{-1} y_j \rangle e_j \right\| \\ &\quad + \left\| \sum_{j \in \mathbb{J}} \langle x, S_X^{-1} y_j - S_Y^{-1} y_j \rangle e_j \right\| \\ &= \|U_X S_X^{-1} x - U_Y S_X^{-1} x\| + \|U_Y (S_X^{-1} x - S_Y^{-1} x)\| \\ &< \mu \|S_X^{-1} x\| + \sqrt{D_2} \|S_X^{-1} x - S_Y^{-1} x\| \\ &< \frac{\mu}{C_1} \|x\| + \frac{\mu}{C_1 C_2} (\sqrt{D_1 D_2} + D_2) \|x\| \\ &= \sqrt{\frac{1}{D_1}} \|x\|. \end{aligned} \quad (49)$$

Now, the conclusion follows from Corollary 3.

Canonical dual frames can be used in investigating the perturbation results of weaving frames. $\square$

Proposition 8. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames for

$\mathcal{H}$ with frame bounds (C_1, D_1) and (C_2, D_2) , respectively. If, for any $x \in H$,

$$\sum_{j \in \mathbb{J}} \|x_j - y_j\| \cdot \|S_X^{-1} x_j\| = \mu < 1, \quad (50)$$

then $\{x_j\}_{j \in \mathbb{J}}$ and $\{y_j\}_{j \in \mathbb{J}}$ are woven frames with universal bounds $(C_1(1 - \mu)^2, D_1 + D_2)$.

Proof. For any $\sigma \subset \mathbb{J}$, we denote $\{\eta_j\}_{j \in \mathbb{J}} = \{x_j\}_{j \in \sigma^c} \cup \{y_j\}_{j \in \sigma}$ and define an operator $L: \mathcal{H} \rightarrow \mathcal{H}$ by

$$Lx = \sum_{j \in \mathbb{J}} \langle x, S_X^{-1} x_j \rangle \eta_j. \quad (51)$$

Note that $\{S_X^{-1} x_j\}$ is a $(1/D, 1/C)$ -frame. For any $x \in \mathcal{H}$, we compute

$$\begin{aligned} \|x - Lx\| &= \left\| \sum_{j \in \mathbb{J}} \langle x, S_X^{-1} x_j \rangle (x_j - \eta_j) \right\| \\ &= \left\| \sum_{j \in \sigma} \langle x, S_X^{-1} x_j \rangle (x_j - y_j) \right\| \\ &\leq \sum_{j \in \sigma} \|x_j - y_j\| \cdot \|S_X^{-1} x_j\| \cdot \|x\| \\ &= \mu \|x\|. \end{aligned} \quad (52)$$

Hence, (50) implies that L is invertible and

$$\|L\| \leq 1 + \mu, \quad \|L^{-1}\| \leq \frac{1}{1 - \mu}. \quad (53)$$

Every $x \in \mathcal{H}$ can be written as

$$x = LL^{-1}x = \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_X^{-1} x_j \rangle \eta_j, \quad (54)$$

implying that

$$\begin{aligned} \|x\|^4 &= \|\langle x, x \rangle\|^2 \\ &= \left\| \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_X^{-1} x_j \rangle \langle \eta_j, x \rangle \right\|^2 \\ &\leq \left\| \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_X^{-1} x_j \rangle \langle S_X^{-1} x_j, L^{-1}x \rangle \right\| \cdot \left\| \sum_{j \in \mathbb{J}} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \right\| \\ &\leq \frac{1}{C_1} \|L^{-1}x\|^2 \cdot \left\| \sum_{j \in \mathbb{J}} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \right\| \\ &\leq \frac{1}{C_1(1 - \mu)^2} \|x\|^2 \cdot \left\| \sum_{j \in \mathbb{J}} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \right\|. \end{aligned} \quad (55)$$

Hence, we have obtained the claimed lower frame bound:

$$\left\| \sum_{j \in \mathbb{J}} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \right\| \geq C_1(1 - \mu)^2 \|x\|^2, \quad (56)$$

as required. $\square$

We have known frames that small perturbations of each other are woven. In this section, we consider the sufficient conditions under which every perturbation of a given woven frame pair $(\{x_j\}_{j \in \mathbb{J}}, \{y_j\}_{j \in \mathbb{J}})$ is still woven.

Theorem 4. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$. Suppose $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ are woven frames with universal bounds (M, N) and $W = \{w_j\}_{j \in \mathbb{J}}$ and $Z = \{z_j\}_{j \in \mathbb{J}}$ are frames for $\mathcal{H}$. Assume that there exist $\mu, \mu' \geq 0$ such that

$$\lambda_1 \sqrt{M} + \lambda'_1 \sqrt{M} + \mu + \mu' < \sqrt{M}, \quad (57)$$

and for any $x \in \mathcal{H}$, we have

$$\|U_X x - U_W x\| \leq \lambda_1 \|U_X x\| + \lambda_2 \|U_W x\| + \mu \|x\|, \quad (58)$$

$$\|U_Y x - U_Z x\| \leq \lambda'_1 \|U_Y x\| + \lambda'_2 \|U_Z x\| + \mu' \|x\|. \quad (59)$$

Then, W and Z are woven with lower universal bound $(\sqrt{M} - \lambda_1 \sqrt{M} - \lambda'_1 \sqrt{M} - \mu - \mu'/1 + \lambda_2 + \lambda'_2)^2$.

Proof. For any given $\sigma \subset \mathbb{J}$, we denote $\{\eta_j\}_{j \in \mathbb{J}} = \{x_j\}_{j \in \sigma^c} \cup \{y_j\}_{j \in \sigma}$ and $\{\xi_j\}_{j \in \mathbb{J}} = \{w_j\}_{j \in \sigma^c} \cup \{z_j\}_{j \in \sigma}$. Then, by assumption, we know $\{\eta_j\}$ is a (M, N) -frame.

It follows from $U_X^\sigma = P_\sigma U_\eta$ that $\|U_X^\sigma x\| \leq \|U_\eta x\|$. Hence, using (58) and (59),

$$\begin{aligned} \|U_\xi x\| - \|U_\eta x\| &\leq \|U_\xi x - U_\eta x\| \\ &\leq \|U_Z^\sigma x + U_W^\sigma x - U_Y^\sigma x - U_X^\sigma x\| \\ &\leq \|U_Z^\sigma x - U_Y^\sigma x\| + \|U_W^\sigma x - U_X^\sigma x\| \\ &\leq \lambda_1 \|U_Z^\sigma x\| + \lambda_2 \|U_Y^\sigma x\| + \mu \|x\| \\ &\quad + \lambda'_1 \|U_W^\sigma x\| + \lambda'_2 \|U_X^\sigma x\| + \mu' \|x\| \\ &\leq \lambda_1 \|U_\xi x\| + \lambda_2 \|U_\eta x\| + \mu \|x\| \\ &\quad + \lambda'_1 \|U_\xi x\| + \lambda'_2 \|U_\eta x\| + \mu' \|x\| \\ &= (\lambda_1 + \lambda'_1) \|U_\xi x\| + (\lambda_2 + \lambda'_2) \|U_\eta x\| + (\mu + \mu') \|x\|. \end{aligned} \quad (60)$$

Therefore,

$$\|U_\eta x\| \geq \frac{\sqrt{M} - \lambda_1 \sqrt{M} - \lambda'_1 \sqrt{M} - \mu - \mu'}{1 + \lambda_2 + \lambda'_2} \|x\|, \quad (61)$$

and we have the result. $\square$

Theorem 5. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$. Suppose $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ are

woven frames with universal bounds (M, N) and $W = \{w_j\}_{j \in \mathbb{J}}$ and $Z = \{z_j\}_{j \in \mathbb{J}}$ are frames for $\mathcal{H}$. Assume that there exist $\mu, \mu' \geq 0$ such that $\mu + \mu' < 1$, and for any $x \in \mathcal{H}$,

$$\begin{aligned} \sum_{j \in \mathbb{J}} \|x_j - w_j\| \cdot \|S_X^{-1} x_j\| &= \mu, \\ \sum_{j \in \mathbb{J}} \|y_j - z_j\| \cdot \|S_Y^{-1} y_j\| &= \mu'. \end{aligned} \quad (62)$$

Then, W and Z are woven with lower universal bounds $M(1 - \mu - \mu')^2$.

Proof. Since X and Y are woven with universal bounds (M, N) , we obtain that $\{\eta_j\}_{j \in \mathbb{J}} = \{x_j\}_{j \in \sigma^c} \cup \{y_j\}_{j \in \sigma}$ is a (M, N) -frame for any $\sigma \subset \mathbb{J}$. Now, we denote $\{\xi_j\}_{j \in \mathbb{J}} = \{w_j\}_{j \in \sigma^c} \cup \{z_j\}_{j \in \sigma}$ and define an operator $L: \mathcal{H} \rightarrow \mathcal{H}$ by

$$Lx = \sum_{j \in \mathbb{J}} \langle x, S_\eta^{-1} \eta_j \rangle \xi_j. \quad (63)$$

Then, for assumption, we have, for any $x \in \mathcal{H}$,

$$\begin{aligned} \|x - Lx\| &= \left\| \sum_{j \in \mathbb{J}} \langle x, S_\eta^{-1} \eta_j \rangle (\eta_j - \xi_j) \right\| \\ &\leq \left\| \sum_{j \in \sigma} \langle x, S_Y^{-1} y_j \rangle (y_j - z_j) \right\| + \left\| \sum_{j \in \sigma^c} \langle x, S_X^{-1} x_j \rangle (x_j - w_j) \right\| \\ &\leq \sum_{j \in \sigma} \|y_j - z_j\| \cdot \|S_Y^{-1} y_j\| \cdot \|x\| + \sum_{j \in \sigma^c} \|x_j - w_j\| \cdot \|S_X^{-1} x_j\| \cdot \|x\| \\ &\leq (\mu + \mu') \|x\|. \end{aligned} \quad (64)$$

It follows that L is invertible and

$$\|L^{-1}\| \leq \frac{1}{1 - \mu - \mu'}. \quad (65)$$

Now, for any $x \in \mathcal{H}$,

$$x = LL^{-1}x = \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_\eta^{-1} \eta_j \rangle \xi_j. \quad (66)$$

Therefore,

$$\begin{aligned} \|x\|^4 &= \|\langle x, x \rangle\|^2 \\ &= \left\| \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_\eta^{-1} \eta_j \rangle \langle \xi_j, x \rangle \right\|^2 \\ &\leq \left\| \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_\eta^{-1} \eta_j \rangle \langle S_\eta^{-1} \eta_j, L^{-1}x \rangle \right\| \cdot \left\| \sum_{j \in \mathbb{J}} \langle x, \xi_j \rangle \langle \xi_j, x \rangle \right\| \\ &\leq \frac{1}{M} \|L^{-1}x\|^2 \cdot \left\| \sum_{j \in \mathbb{J}} \langle x, \xi_j \rangle \langle \xi_j, x \rangle \right\|. \end{aligned} \quad (67)$$

Putting (65) and (66) together, we have

$$\left\| \sum_{j \in \mathbb{J}} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \right\| \geq M(1 - \mu - \mu')^2 \|x\|^2, \quad (68)$$

as required. $\square$

5. Weaving Frames in Hilbert C^* -Modules over Nonunital C^* -Algebras

The frames for a Hilbert space $\mathcal{H}$ are precisely the families $\{Te_j\}_{j \in \mathbb{J}}$, where $T: \ell^2(\mathbb{J}) \rightarrow \mathcal{H}$ is a bounded surjection and $\{e_j\}_{j \in \mathbb{J}}$ is the canonical basis of $\ell^2(\mathbb{J})$. This correspondence is still true for modular frames when the C^* -algebra $\mathcal{A}$ is unital. Assume $\{x_j\}_{j \in \mathbb{J}}$ is a frame for a Hilbert C^* -module over a unital C^* -algebras with analysis operator U . We can derive from Corollary 2 that any adjointable operators belonging to a small neighborhood of U determine a frame which is woven with $\{x_j\}_{j \in \mathbb{J}}$. However, for the nonunital case, the situation is more complicated. Let $\mathcal{A}$ be an arbitrary C^* -algebra and $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module. An adjointable surjection from $\ell^2(\mathcal{A})$ to $\mathcal{H}$ may not be the analysis operator for any frame. Furthermore, there exist operator whose any neighborhood can contain an operator that is not the analysis operator for any frame. Thus, we need a new notation to obtain analogues of the unital case.

Arambasic et al. introduced the concept of the outer frame in [23] to investigate the bijective correspondence of all surjections from $\ell^2(\mathcal{A})$ to $\mathcal{H}$ and all both frames or outer frames for $\mathcal{H}$. Let $M(\mathcal{H})$ be the multiplier module of $\mathcal{H}$. Note that $M(\mathcal{H})$ is a Hilbert $M(\mathcal{A})$ -module which contains $\mathcal{H}$ as an ideal submodule. Recall that a sequence $\{x_j\}_{j \in \mathbb{J}}$ in $M(\mathcal{H})$ is called an *outer frame* for $\mathcal{H}$ if $x_i \in M(\mathcal{H}) \setminus \mathcal{H}$ for at least one $i \in \mathbb{J}$, and (2) holds for every $x \in \mathcal{H}$ and suitable positive constants (C, D) . Let $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames or outer frames for $\mathcal{H}$, and we say X and Y are *woven* if there are universal constants C and D such that, for any subset σ of $\mathbb{J}$, $\{x_j\}_{j \in \sigma^c} \cup \{y_j\}_{j \in \sigma}$ is a frame or an outer frame for $\mathcal{H}$ with bounds (C, D) (Theorem 3.19 in [23]) shows that, for any surjection $T \in \mathcal{L}(\ell^2(\mathcal{A}), \mathcal{H})$, there exists a unique frame or an outer frame $\{x_j\}_{j \in \mathbb{J}}$ whose synthesis operator coincides with T . Now, we give the key result of this section.

Theorem 6. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a C^* -algebra $\mathcal{A}$ and $U, V \in \mathcal{L}(\ell^2(\mathcal{A}), \mathcal{H})$. Suppose U is bounded below and

$$\|U - V\| < \max\left\{\|(U^*U)^{-(1/2)}\|^{-1}, \|(V^*V)^{-(1/2)}\|^{-1}\right\}. \quad (69)$$

Then, U and V are analysis operators for a woven family of frames or outer frames.

Proof. Without loss of generality, we assume $\|U - V\| < \|(U^*U)^{-(1/2)}\|^{-1}$. Since U is bounded below,

$$\begin{aligned}
\|(U^*U)^{-(1/2)}\|^{-1} &= \left(\min \left\{ m \geq 0 : \|(U^*U)^{-(1/2)}x\| \leq m\|x\|, x \in \mathcal{H} \right\} \right)^{-1} \\
&= \left(\min \left\{ m \geq 0 : \|x\| \leq m\|(U^*U)^{1/2}x\|, x \in \mathcal{H} \right\} \right)^{-1} \\
&= \left(\min \left\{ m \geq 0 : \frac{1}{m}\|x\| \leq \|Ux\|, x \in \mathcal{H} \right\} \right)^{-1} \\
&= \max \{ m \geq 0 : m\|x\| \leq \|Ux\|, x \in \mathcal{H} \}.
\end{aligned} \tag{70}$$

Write $\mu := \|U - V\|$. Then, by assumption,

$$\|Vx\| \geq \|Ux\| - \|Ux - Vx\| \geq \left(\|(U^*U)^{-(1/2)}\|^{-1} - \mu \right) \|x\|, \quad \forall x \in \mathcal{H}, \tag{71}$$

implying that V is bounded below. Hence, U and V are analysis operators of frames or outer frames denoted by $\{x_j\}_{j \in \mathbb{J}}$ and $\{y_j\}_{j \in \mathbb{J}}$, respectively.

For any $\sigma \subset \mathbb{J}$, we denote $\{\eta_j\}_{j \in \mathbb{J}} = \{x_j\}_{j \in \sigma^c} \cup \{y_j\}_{j \in \sigma}$ and define an operator $U_\eta: \mathcal{H} \rightarrow \ell^2(\mathcal{A})$ by

$$U_\eta x = \sum_{j \in \sigma^c} \langle x, x_j \rangle e_j + \sum_{j \in \sigma} \langle x, y_j \rangle e_j. \tag{72}$$

Then, for any $x \in \mathcal{H}$,

$$\|U_\eta x - Ux\| = \left\| \sum_{j \in \sigma} \langle x, y_j - x_j \rangle e_j \right\| \leq \|Ux - Vx\|. \tag{73}$$

It follows that

$$\|U_\eta x\| \geq \|Ux\| - \|U_\eta x - Ux\| \geq \left(\|(U^*U)^{-(1/2)}\|^{-1} - m \right) \|x\|. \tag{74}$$

Thus, $\{\eta_j\}_{j \in \mathbb{J}}$ is a frame or an outer frame for $\mathcal{H}$. We can prove the result similarly using $\|U - V\| < \|(V^*V)^{-(1/2)}\|^{-1}$, and we have the result. $\square$

When $\mathcal{A}$ is unital, $M(\mathcal{H}) = \mathcal{H}$. There are no outer frames for $\mathcal{H}$, and clearly, Theorem 6 holds for ordinary frames in the Hilbert $\mathcal{A}$ -module.

6. Redundancy of Weaving Frames

In this section, we study the redundancy of frames in Hilbert C^* -modules. Recall that a frame $\{x_j\}_{j \in \mathbb{J}}$ for a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is said to be a (standard) Riesz basis for $\mathcal{H}$ if it satisfies the following:

- (i) $x_j \neq 0$, for all j
- (ii) If an $\mathcal{A}$ -linear combination $\sum_{j \in S} c_j x_j$ with coefficients $\{c_j : j \in S\} \subset \mathcal{A}$ and $S \subset \mathbb{J}$ is equal to zero, then every summand $c_j x_j$ is zero

It is shown in [5] that a frame which is woven with a Riesz basis in a Hilbert space must be a Riesz basis and [24] shows that woven frames have the same excess. However, the redundancy property of modular frames is quite different

from that of frames in Hilbert spaces. As was pointed out in Section 2, every countably generated Hilbert C^* -module admits a (standard) frame. However, there exist countably generated Hilbert C^* -modules that contain no orthonormal basis, even a Riesz basis, cf. [25].

We first state the following characterization for Riesz bases in Hilbert $\mathcal{A}$ -modules.

Proposition 9 (see [12]). *Suppose that $\{x_j\}_{j \in \mathbb{J}}$ is a frame of $\mathcal{H}$; then, $\{x_j\}_{j \in \mathbb{J}}$ is a Riesz basis if and only if*

- (i) $x_j \neq 0$ for all $j \in \mathbb{J}$
- (ii) $\sum_{j \in \mathbb{J}} c_j x_j = 0$ for some sequence $\{c_j\}_{j \in \mathbb{J}} \in \ell^2(\mathcal{A})$, then $c_j x_j = 0$, for all j

Now, we derive the following result.

Theorem 7. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be woven frames for $\mathcal{H}$. If X is a Riesz basis with $\text{ran } \theta_X = \ell^2(\mathcal{A})$, then Y is a Riesz basis with $\text{ran } \theta_Y = \ell^2(\mathcal{A})$.*

Proof. Let $\{e_j\}_{j \in \mathbb{J}}$ be the canonical basis of $\ell^2(\mathcal{A})$. Note that every modular frame can be written as $\{Te_j\}_{j \in \mathbb{J}}$ for a surjective adjointable operator T . Since $\text{ran } \theta_X = \ell^2(\mathcal{A})$, the synthesis operator T_X is invertible. Then, by Proposition 5, we can assume that $X = \{e_j\}_{j \in \mathbb{J}}$ and $Y = \{Te_j\}_{j \in \mathbb{J}}$, where T is a surjective operator in $\mathcal{L}(\ell^2(\mathcal{A}))$.

By way of contradiction, assume that $\text{ran } \theta_Y \neq \ell^2(\mathcal{A})$. This means that $\ker T_Y \neq \{0\}$. Then, there exists $\{c_j\}_{j \in \mathbb{J}}$ with $c_i \neq 0$ for some i , such that $\sum_{j \in \mathbb{J}} c_j Te_j = 0$. This implies that $Te_i = \sum_{j \neq i} (-c_j/c_i) Te_j$, and thus, $Te_i \in \overline{\text{span}}\{Te_j\}_{j \neq i}$. Without loss of generality it may be assumed that $Te_1 \in \overline{\text{span}}\{Te_j\}_{j=2}^\infty$. For given any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(Te_1, \text{span}\{Te_j\}_{j=2}^n) < \varepsilon$. Now, choose $x \in \text{span}\{e_j\}_{j=1}^n \cap (\text{span}\{Te_j\}_{j=2}^n)^\perp$ with $\|x\| = 1$. Then, for any $y \in \text{span}\{Te_j\}_{j=2}^n$, we compute

$$\begin{aligned}
\sum_{j=1}^n |\langle x, e_j \rangle|^2 + \sum_{j=n+1}^\infty |\langle x, Te_j \rangle|^2 &= |\langle x, Te_1 \rangle|^2 = |\langle x, Te_1 - y \rangle|^2 \\
&\leq \|Te_1 - y\|^2 < \varepsilon.
\end{aligned} \tag{75}$$

So, $\{e_j\}_{j=1}^n \cup \{Te_j\}_{j=n+1}^\infty$ is not a frame. Then, we have $\text{ran } \theta_Y = \ell^2(\mathcal{A})$. Hence, $T_Y\{c_j\}_{j \in \mathbb{J}} = 0$ can imply $c_j = 0$ for all j . Thus, $c_j x_j = 0$ for all j , and it follows from Proposition 9 that Y is a Riesz basis.

Theorem 4.15 in [23] proved that a frame or an outer frame for a full Hilbert $\mathcal{A}$ -module has a unique dual if and only if the analysis operator is surjective. Combined with Theorem 7, we have the following. $\square$

Corollary 4. *Let $\mathcal{H}$ be a full Hilbert $\mathcal{A}$ -module and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be a woven family of frames or outer frames. If X has a unique dual, then Y has a unique dual.*

Now, we derive a sufficient condition for the situation that a frame which is woven to a Riesz basis is a Riesz basis itself.

Proposition 10. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -module over a unital C^* -algebra $\mathcal{A}$ and $X = \{x_j\}_{j \in \mathbb{J}}$ and $Y = \{y_j\}_{j \in \mathbb{J}}$ be frames for $\mathcal{H}$ with frame bounds (C_1, D_1) and (C_2, D_2) , respectively. Assume that X is a Riesz basis, and there exist constants $\lambda_1, \lambda_2 \geq 0$ with*

$$\lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2} < \min\{\sqrt{C_1}, \sqrt{C_2}\}, \quad (76)$$

such that, for any $\{c_j\}_{j \in \mathbb{J}} \in \ell^2(\mathcal{A})$,

$$\|T_X\{c_j\}_{j \in \mathbb{J}} - T_Y\{c_j\}_{j \in \mathbb{J}}\| \leq \lambda_1 \|T_X\{c_j\}_{j \in \mathbb{J}}\| + \lambda_2 \|T_Y\{c_j\}_{j \in \mathbb{J}}\|, \quad (77)$$

where T_X and T_Y are the synthesis operators for X and Y , respectively. Then, X and Y are woven frames and Y is a Riesz basis.

Proof. Define an operator L and a sequence $\{\eta_j\}_{j \in \mathbb{J}}$ in the same way as the proof of Theorem 9. Using (77), on the sequence $\{c_j\}_{j \in \mathbb{J}} := \{\langle x, S_X^{-1} x_j \rangle\}_{j \in \mathbb{J}}$, we obtain that

$$\begin{aligned} \|x - Lx\| &= \left\| \sum_{j \in \mathbb{J}} \langle x, S_X^{-1} x_j \rangle (x_j - \eta_j) \right\| \\ &= \left\| \sum_{j \in \mathbb{J}} c_j (x_j - y_j) \right\| \\ &\leq \lambda_1 \|T_X^\sigma\{c_j\}_{j \in \mathbb{J}}\| + \lambda_2 \|T_Y^\sigma\{c_j\}_{j \in \mathbb{J}}\| \\ &\leq \lambda_1 \|T_X\| \cdot \|\{c_j\}_{j \in \mathbb{J}}\| + \lambda_2 \|T_Y\| \cdot \|\{c_j\}_{j \in \mathbb{J}}\| \\ &\leq (\lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2}) \|\{c_j\}_{j \in \mathbb{J}}\| \\ &\leq \frac{\lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2}}{\sqrt{C_1}} \|x\|. \end{aligned} \quad (78)$$

Hence, (76) implies that L is invertible and $\|L^{-1}\| \leq \sqrt{C_1}/\sqrt{C_1} - \lambda_1 \sqrt{D_1} - \lambda_2 \sqrt{D_2}$. Every $x \in \mathcal{H}$ can be written as

$$x = LL^{-1}x = \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_X^{-1}x_j \rangle \eta_j, \quad (79)$$

implying that

$$\begin{aligned} \|x\|^4 &= \|\langle x, x \rangle\|^2 \\ &= \left\| \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_X^{-1}x_j \rangle \langle \eta_j, x \rangle \right\|^2 \\ &\leq \left\| \sum_{j \in \mathbb{J}} \langle L^{-1}x, S_X^{-1}x_j \rangle \langle S_X^{-1}x_j, L^{-1}x \rangle \right\| \cdot \left\| \sum_{j \in \mathbb{J}} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \right\| \\ &\leq \frac{1}{C_1} \|L^{-1}x\|^2 \cdot \left\| \sum_{j \in \mathbb{J}} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \right\|. \end{aligned} \quad (80)$$

Therefore,

$$\left\| \sum_{j \in \mathbb{J}} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \right\| \geq (\sqrt{C_1} - \lambda_1 \sqrt{D_1} - \lambda_2 \sqrt{D_2})^2 \|x\|^2, \quad (81)$$

and thus, we get X and Y are woven.

Now, let us prove Y is a Riesz basis. Assume $y_i = 0$ for some i . It follows from (76) that $\lambda_1, \lambda_2 < 1$ and $\|x_i\| \leq \lambda_1 \|x_i\|$. So, $x_i = 0$, and it is a contradiction. Hence, $y_j \neq 0$, for each $j \in \mathbb{J}$. Now, let $\{c_j\}_{j \in \mathbb{J}} \in \ker T_X$. Using (77), we obtain that $\|T_Y\{c_j\}_{j \in \mathbb{J}}\| \leq \lambda_2 \|T_Y\{c_j\}_{j \in \mathbb{J}}\|$, which implies that $\{c_j\}_{j \in \mathbb{J}} \in \ker T_Y$. So, $\text{ran } \theta_X = \text{ran } \theta_Y$, and it follows from Theorem 3.1 in [12] that Y is a Riesz basis. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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