Research Article

The Normalized Laplacians, Degree-Kirchhoff Index, and the Complexity of Möbius Graph of Linear Octagonal-Quadrilateral Networks

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The normalized Laplacian plays an indispensable role in exploring the structural properties of irregular graphs. Let $L_n^{8,4}$ represent a linear octagonal-quadrilateral network. Then, by identifying the opposite lateral edges of $L_n^{8,4}$, we get the corresponding Möbius graph $MQ_n(8, 4)$. In this paper, starting from the decomposition theorem of polynomials, we infer that the normalized Laplacian spectrum of $MQ_n(8, 4)$ can be determined by the eigenvalues of two symmetric quasi-triangular matrices $\mathcal{L}_A$ and $\mathcal{L}_S$ of order $4n$. Next, owing to the relationship between the two matrix roots and the coefficients mentioned above, we derive the explicit expressions of the degree-Kirchhoff indices and the complexity of $MQ_n(8, 4)$.

1. Introduction

It is well established that networks can be represented by graphs. The graphs we consider in this paper are simple, undirected, and connected. Let us first recall some definitions commonly used in graph theory. Suppose G represents a simple undirected graph with $|V_G| = n$ and $|E_G| = m$. For more notations, the readers are referred to [1].

Note that $D(G) = \text{diag}\{d_1, d_2, \dots, d_n\}$ represents a degree matrix, where d_p is the degree of v_p . $A(G)$ is the adjacency matrix of G . The Laplacian matrix of G is $L(G) = D(G) - A(G)$. The (p, q) -entry of the normalized Laplacian matrix is given by

$$(\mathcal{L}(G))_{pq} = \begin{cases} 1, & p = q, \\ -\frac{1}{\sqrt{d_p d_q}}, & p \neq q \text{ and } v_p \sim v_q, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

As a matter of fact, there are many parameters that can be used to describe the structure and properties of molecular graphs in graph networks. One of the parameters based on resistance distance is defined as the Wiener index [2, 3], which is

$$W(G) = \sum_{i < j} d_{ij}, \quad (2)$$

where $d_{ij} = d_G(v_i, v_j)$ represents the length of the shortest path between two vertices v_i and v_j in G . The Wiener index is widely used in chemical and mathematical research. For details, see [4–7].

The parameter of resistance distance was first proposed by Klein and Randić [8] in 1993. It means that if every edge of a graph G is regarded as a unit resistance, then the distance between any two vertices i and j in G is called resistance distance, which is denoted as r_{ij} . Similar to the Wiener index, we give the expression of the Kirchhoff index [9, 10] according to the resistance distance, namely,

$$Kf(G) = \sum_{i < j} r_{ij} = n \sum_{i=2}^n \frac{1}{\mu_i}. \quad (3)$$

In 2007, Chen and Zhang [11] proposed that the eigenvalues and eigenvectors of normalized Laplacian spectrum can be used to describe the resistance distance, and the concept of Kirchhoff index is put forward. However, it is very difficult to calculate the degree-Kirchhoff index from the complexity division of graphs, so it is important to find the explicit expression of degree-Kirchhoff index. In recent years, many scholars have devoted themselves to the study of Kirchhoff index of various graphs. Huang et al. [12, 13] proved the Kirchhoff index of linear hexagonal chains and linear polyomino chains successively. Ma and Bian [14] determined the normalized Laplacians and degree-Kirchhoff index of cylinder phenylene chain. Liu et al. [15] described the normalized Laplacian and degree-Kirchhoff index of linear octagonal-quadrilateral networks. For more excellent results, refer to [16–21]. After learning the excellent works of scholars, in this paper, we use the correlation properties of Laplace matrix to calculate the degree-Kirchhoff index and the complexity of Möbius graph of linear octagonal-quadrilateral networks. The investigation of complex graph and network has gone through a spectacular development in the past decades. Especially in organic chemistry, more and more attention has been paid to the application of polyomino in polycyclic aromatic compounds. Many scholars are interested in the study of linear octagonal networks and related molecular graphs. We all know that linear octagonal network is an octagonal system without branch compression. It is constructed by regularly inserting some new points on the straight line of the linear polyomino network. The research on the structure and properties of this kind of natural graph network lays a solid foundation for the advancement of theoretical chemistry, as well as for the development of applied mathematics.

Let $L_n^{8,4}$ be the linear octagonal-quadrilateral networks, and octagons and quadrilaterals are connected by a common edge, which are depicted in Figure 1. Then, the corresponding Möbius graph $MQ_3(8, 4)$ of octagonal-quadrilateral networks is obtained by the reverse identification of the opposite edge by $L_n^{8,4}$ (see Figure 2). Obviously, we can obtain that $|V_{MQ_n}(8, 4)| = 8n$, $|E_{MQ_n}(8, 4)| = 10n$.

The rest of the paper will be divided into the following sections. In Section 2, we put forward some basic notations and related lemmas. In Section 3, we determine the normalized Laplacian spectrum of $MQ_n(8, 4)$. In Section 4, we present Kemeny's constant, the degree-Kirchhoff index, and the complexity of $MQ_n(8, 4)$.

2. Preliminary

In this section, we introduce some common symbols and related calculation methods [1], which are applied to the rest of the article.

The characteristic polynomial of matrix R of order n is defined as $P_R(x) = \det(xI - R)$. It is not difficult to find that

π is an automorphism of G , and we can write the product of disjoint 1-cycles and transposition, namely,

$$\pi = (\bar{1})(\bar{2}), \dots, (\bar{m})(1, 1')(2, 2'), \dots, (k, k'). \quad (4)$$

Then, one has $|V(G)| = m + 2k$, and let $v_0 = \{\bar{1}, \bar{2}, \dots, \bar{m}\}$, $v_1 = \{1, 2, \dots, k\}$, $v_2 = \{1', 2', \dots, k'\}$. Thus, the Laplacian matrix can be expressed in the form of block matrix, that is,

$$\mathcal{L}(G) = \begin{pmatrix} \mathcal{L}_{V_0V_0} & \mathcal{L}_{V_0V_1} & \mathcal{L}_{V_0V_2} \\ \mathcal{L}_{V_1V_0} & \mathcal{L}_{V_1V_1} & \mathcal{L}_{V_1V_2} \\ \mathcal{L}_{V_2V_0} & \mathcal{L}_{V_2V_1} & \mathcal{L}_{V_2V_2} \end{pmatrix}, \quad (5)$$

where

$$\begin{aligned} \mathcal{L}_{V_0V_1} &= \mathcal{L}_{V_0V_2}, \\ \mathcal{L}_{V_1V_2} &= \mathcal{L}_{V_2V_1}, \\ \mathcal{L}_{V_1V_1} &= \mathcal{L}_{V_2V_2}. \end{aligned} \quad (6)$$

Let

$$P = \begin{pmatrix} I_m & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}I_k & \frac{1}{\sqrt{2}}I_k \\ 0 & \frac{1}{\sqrt{2}}I_k & -\frac{1}{\sqrt{2}}I_k \end{pmatrix}, \quad (7)$$

and then

$$P' \mathcal{L}(G) P = \begin{pmatrix} \mathcal{L}_A(G) & 0 \\ 0 & \mathcal{L}_S(G) \end{pmatrix}. \quad (8)$$

Note that P' is the transposition of P , where

$$\begin{aligned} \mathcal{L}_A &= \begin{pmatrix} \mathcal{L}_{V_0V_0} & \sqrt{2} \mathcal{L}_{V_0V_1} \\ \sqrt{2} \mathcal{L}_{V_1V_0} & \mathcal{L}_{V_1V_1} + \mathcal{L}_{V_1V_2} \end{pmatrix}, \\ \mathcal{L}_S &= \mathcal{L}_{V_1V_1} - \mathcal{L}_{V_1V_2}. \end{aligned} \quad (9)$$

Lemma 1 (see [12]). Let $\mathcal{L}(L_n)(G)$, $\mathcal{L}_A(G)$, $\mathcal{L}_S(G)$ be determined as above; then,

$$P_{\mathcal{L}(L_n)}(G) = P_{\mathcal{L}_A}(G) P_{\mathcal{L}_S}(G). \quad (10)$$

Lemma 2. Let G be a graph with $|V_G| = n$ and $|E_G| = m$, and $0 = \mu_1 < \mu_2 \leq \dots \leq \mu_n (n \geq 2)$ are the eigenvalues of $\mathcal{L}(G)$. Then, we can quickly confirm that the following formulas hold.

FIGURE 1: Linear octagonal-quadrilateral networks.

FIGURE 2: Graph $MQ_3(8,4)$.

(a) (see [22]). Kemeny's constant of G can be denoted as

$$K_c(G) = \sum_{i=2}^n \frac{1}{\mu_i}. \quad (11)$$

(b) (see [11]). The degree-Kirchhoff index of G is defined as

$$Kf^*(G) = 2m \sum_{k=2}^n \frac{1}{\mu_k}. \quad (12)$$

(c) (see [1]). The number of spanning trees of G can also be called the complexity of G . Then, the complexity of G is

$$\prod_{i=1}^n d_i \sum_{k=2}^n \lambda_k = 2m\tau(G). \quad (13)$$

3. The Normalized Laplacian Spectrum of $MQ_n(8,4)$

In this section, we focus on obtaining the normalized Laplacian spectrum of $MQ_n(8,4)$ by Lemma 1.

Given a square matrix T of order n . We will use $T[\{p_1, p_2, \dots, p_k\}]$ to denote the submatrix obtained by deleting the $p_1^{\text{th}}, p_2^{\text{th}}, \dots, p_k^{\text{th}}$ rows and corresponding columns of T . With a suitable labeling, the vertices of $MQ_n(8,4)$ are shown in Figure 2. Apparently, $\pi = (1, 1')(2, 2'), \dots, (4n, (4n)')$ is an automorphism of $MQ_n(8,4)$. Then, $v_0 = \emptyset, v_1 = \{1, 2, 3, \dots, 4n\}$ and $v_2 = \{1', 2', 3', \dots, (4n)'\}$. Besides, we express $\mathcal{L}_A(MQ_n(8,4))$ and $\mathcal{L}_S(MQ_n(8,4))$ as $\mathcal{L}_A$ and $\mathcal{L}_S$. Then, one can get

$$\begin{aligned} \mathcal{L}_A &= \mathcal{L}_{V_1 V_1} + \mathcal{L}_{V_1 V_2}, \\ \mathcal{L}_S &= \mathcal{L}_{V_1 V_1} - \mathcal{L}_{V_1 V_2}. \end{aligned} \quad (14)$$

In view of equation (1), we have

$$\begin{aligned}
\mathcal{L}_{V_1 V_1} &= \begin{pmatrix} 1 & \frac{-1}{\sqrt{6}} & & & & & & & \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & \\ & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & \\ & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{3} & & & & \\ & & & \frac{-1}{3} & 1 & \frac{-1}{\sqrt{6}} & & & \\ & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & \\ & & & & & \ddots & & & \\ & & & & & \frac{-1}{3} & 1 & \frac{-1}{\sqrt{6}} & \\ & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \\ & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & \\ & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \\ & & & & & & & & & \end{pmatrix}_{(4n) \times (4n)}, \\
\mathcal{L}_{V_1 V_2} &= \begin{pmatrix} \frac{-1}{3} & & & & & & & & \frac{-1}{3} \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & \frac{-1}{3} & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \frac{-1}{3} & \\ & & & & & & & & 0 \\ & & & & & & & & & 0 \\ \frac{-1}{3} & & & & & & & & & \frac{-1}{3} \end{pmatrix}_{(4n) \times (4n)}.
\end{aligned} \tag{15}$$

Hence,

$$\mathcal{L}_A = \begin{pmatrix} \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & & \frac{-1}{3} \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & \\ & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & \\ & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & \\ & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & \\ & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \\ & & & & & \ddots & & \\ & & & & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} \\ & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} \\ & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\ & & & & & & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} \end{pmatrix}_{(4n) \times (4n)}, \quad (16)$$

$$\mathcal{L}_S = \begin{pmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & \frac{-1}{3} \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & \\ & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & \\ & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & \\ & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & \\ & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \\ & & & & & \ddots & & \\ & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} \\ & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} \\ & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\ & & & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} \end{pmatrix}_{(4n) \times (4n)}$$

Assume that $0 = \eta_1 < \eta_2 \leq \eta_3 \leq \dots \leq \eta_{4n}$ are the roots of $P_{\mathcal{L}_A}(x) = 0$ and $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \dots \leq \varphi_{4n}$ are the roots of $P_{\mathcal{L}_S}(x) = 0$, respectively. Then, according to Lemma 1, we can get that the spectrum of MQ_n is just $\eta_1, \eta_2, \dots, \eta_{4n}, \varphi_1, \varphi_2, \dots, \varphi_{4n}$, and it is easy to check that $\eta_1 = 0, \eta_p > 0 (p = 2, 3, \dots, 4n)$, and $\varphi_q > 0, (q = 1, 2, \dots, 4n)$.

Next, we calculate some main results of MQ_n related to the normalized Laplacian spectrum.

4. The Degree-Kirchhoff Index and the Complexity of $MQ_n(8, 4)$

In this section, we first introduce some theorems, which are obtained by describing the eigenvalues and eigenvectors of normalized Laplacian matrix. Then, we obtain Kemeny's constant, the degree-Kirchhoff index, and the complexity of $MQ_n(8, 4)$ based on these theorems, where $\varrho(n) = (41n\sqrt{14}/28)[((15 + 4\sqrt{14})^n - (15 - 4\sqrt{14})^n) / ((15 + 4\sqrt{14})^n + (15 - 4\sqrt{14})^n) + 2]$.

Theorem 1

$$\sum_{p=2}^{4n} \frac{1}{\eta_p} = \frac{200n^2 - 11}{60}. \quad (17)$$

Proof. Let

$$\begin{aligned} P_{\mathcal{L}_S}(x) &= \det(xI - \mathcal{L}_A) = x^{4n} + a_1x^{4n-1} + \dots + a_{4n-1}x + a_{4n} \\ &= x(x^{4n-1} + a_1x^{4n-2} + \dots + a_{4n-2}x + a_{4n-1}), \quad a_{4n-1} \neq 0. \end{aligned} \quad (18)$$

Then, we can exactly get that $\eta_1, \eta_2, \dots, \eta_{4n}$ are the roots of the following equation:

$$x^{4n-1} + a_1x^{4n-2} + \dots + a_{4n-2}x + a_{4n-1} = 0. \quad (19)$$

Based on Vieta's theorem of $P_{\mathcal{L}_A}(x)$, it is easy to get

$$\sum_{p=2}^{4n} \frac{1}{\eta_p} = \frac{(-1)^{4n-2}a_{4n-2}}{(-1)^{4n-1}a_{4n-1}}. \quad (20)$$

Before calculating $(-1)^{4n-2}a_{4n-2}$ and $(-1)^{4n-1}a_{4n-1}$, we must determine p th order principal submatrices, R_p^0, R_p^1, R_p^2 , and R_p^3 , which consist of the first p rows and columns of the following matrices $\mathcal{L}_A^0, \mathcal{L}_A^1, \mathcal{L}_A^2$, and $\mathcal{L}_A^3$, respectively, $p = 1, 2, \dots, 4n$. Let

$$\mathcal{L}_A^0 = \begin{pmatrix} \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & & & \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & \\ & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & \\ & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & & \\ & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & \\ & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & \\ & & & & \ddots & & & & \\ & & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & \\ & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & \\ & & & & & \ddots & & & \\ & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & \\ & & & & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} \end{pmatrix}, \quad (4n) \times (4n)$$

$$\begin{aligned} \mathcal{L}_A^1 &= \left(\begin{array}{ccccccc} 1 & -\frac{1}{2} & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & & & & \\ & -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{\sqrt{6}} & & \\ & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} \\ & & & & & \ddots & \\ & & & & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\ & & & & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} \\ & & & & & & & -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{3} \\ & & & & & & & & -\frac{1}{3} & \frac{2}{3} \end{array} \right)_{(4n) \times (4n)}, \\ \mathcal{L}_A^2 &= \left(\begin{array}{ccccccc} 1 & -\frac{1}{\sqrt{6}} & & & & & \\ -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{3} & & & & \\ & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{\sqrt{6}} & & & \\ & & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & & \\ & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & \\ & & & & -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{3} \\ & & & & & \ddots & \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} \\ & & & & & & -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{3} \\ & & & & & & & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{\sqrt{6}} \\ & & & & & & & & -\frac{1}{\sqrt{6}} & 1 \end{array} \right)_{(4n) \times (4n)}. \end{aligned}$$

$$r_p^3 = \frac{2}{3}r_{p-1}^0 - \frac{1}{9}r_{p-2}^1. \quad (25)$$

Proof. of Fact 1. Take $r_p^0 = \det R_p^0$, $r_p^1 = \det R_p^1$, $r_p^2 = \det R_p^2$, and $r_p^3 = \det R_p^3$. By a straightforward calculation, one can get the following values (see Table 1).

For $4 \leq p \leq 4n-1$, we can get expansion formula of $\det R_p^0$ with respect to its last row:

$$r_p^0 = \begin{cases} \frac{2}{3}r_{p-1}^0 - \frac{1}{6}r_{p-2}^0, & \text{if } p \equiv 0 \pmod{4}, \\ \frac{2}{3}r_{p-1}^0 - \frac{1}{9}r_{p-2}^0, & \text{if } p \equiv 1 \pmod{4}, \\ r_{p-1}^0 - \frac{1}{6}r_{p-2}^0, & \text{if } p \equiv 2 \pmod{4}, \\ r_{p-1}^0 - \frac{1}{4}r_{p-2}^0, & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (26)$$

For $1 \leq p \leq n-1$, let $a_p = r_{4p}^0$; for $0 \leq p \leq n-1$, let $b_p = r_{4p+1}^0$, $c_p = r_{4p+2}^0$, $d_p = r_{4p+3}^0$. Then, we can get $a_1 = (5/36)$, $b_0 = (2/3)$, $c_0 = (1/2)$, $d_0 = (5/36)$, $b_1 = (3/54)$, $c_1 = (7/216)$, $d_1 = (1/54)$, and for $p \geq 2$, we have

$$\begin{cases} a_p = \frac{2}{3}d_{p-1} - \frac{1}{6}c_{p-1}, \\ b_p = \frac{2}{3}a_p - \frac{1}{9}d_{p-1}, \\ c_p = b_p - \frac{1}{6}a_p, \\ d_p = c_p - \frac{1}{4}b_p. \end{cases} \quad (27)$$

Then, it is not difficult to obtain that

$$\begin{cases} a_p = 18c_p - 24d_p, \\ b_p = 4c_p - 4d_p, \\ c_p = \frac{1}{18}c_{p-1} - \frac{1}{1296}c_{p-2}, \\ d_p = \frac{1}{18}d_{p-1} - \frac{1}{1296}d_{p-2}. \end{cases} \quad (28)$$

According to the equation of d_p in (28), it is evident to see that $x^2 - (1/18)x + (1/1296) = 0$, and its two roots are $(1/36)$ and $(1/36)$. Therefore, $d_p = (x_p + y)(1/36)^p$ is the general solution. Then, we can get

$$\begin{cases} y = \frac{1}{3}, \\ \frac{1}{36}(x + y) = \frac{1}{54}, \\ x = \frac{1}{3}, \\ y = \frac{1}{3}. \end{cases} \quad (29)$$

Thus, we can obtain $d_p = (1/3)(p+1)(1/36)^p$ ($p \geq 1$). Similarly, we have $c_p = ((2p/3) + (1/2))(1/36)^p$ ($p \geq 1$); $a_p = (4p+1)(1/36)^p$ ($p \geq 1$); and $b_p = (2/3)(2p+1)(1/36)^p$ ($p \geq 1$).

The result is obtained as desired. $\square$

By similar consideration, Fact 2 is available. Then, based on the conclusion of Facts 1 and 2, we quickly get Facts 3 and 4.

Now, we will further calculate $(-1)^{4n-1}a_{4n-1}$ and $(-1)^{4n-2}a_{4n-2}$ in equation (20). For the sake of discussion, it is assumed that $r_0 = 1$.

Claim 1. $(-1)^{4n-1}a_{4n-1} = 40n^2(1/36)^n$.

Proof. of Claim 1. Since $(-1)^{4n-1}a_{4n-1}$ is the total of all the principal minors of order $4n-1$ of $\mathcal{L}_A$, we have

$$\begin{aligned} (-1)^{4n-1}a_{4n-1} &= \sum_{p=1}^{4n} \det \mathcal{L}_A[p] = \sum_{p=4, p \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p] + \sum_{p=1, p \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p] \\ &\quad + \sum_{p=2, p \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p] + \sum_{p=3, p \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p], \end{aligned} \quad (30)$$

where

TABLE 1: Initial value.

r_p^0	Value	r_p^0	Value	r_p^0	Value	r_p^0	Value
r_1^0	(2/3)	r_2^0	(1/2)	r_3^0	(1/3)	r_4^0	(5/36)
r_5^0	(3/54)	r_6^0	(7/216)	r_7^0	(1/54)	r_8^0	(1/144)

$$\begin{aligned}
\sum_{p=4, p \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p] &= \sum_{p=4, p \equiv 0 \pmod{4}}^{4n} \left(r_{p-1}^0 r_{4n-p}^0 - \frac{1}{9} r_{p-2}^1 r_{4n-p-1}^0 \right), \\
\sum_{p=1, p \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p] &= \sum_{p=1, p \equiv 1 \pmod{4}}^{4n-3} \left(r_{p-1}^0 r_{4n-p}^1 - \frac{1}{9} r_{p-2}^1 r_{4n-p-1}^1 \right), \\
\sum_{p=2, p \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p] &= \sum_{p=2, p \equiv 2 \pmod{4}}^{4n-2} \left(r_{p-1}^0 r_{4n-p}^2 - \frac{1}{9} r_{p-2}^1 r_{4n-p-1}^2 \right), \\
\sum_{p=3, p \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p] &= \sum_{p=3, p \equiv 3 \pmod{4}}^{4n-1} \left(r_{p-1}^0 r_{4n-p}^3 - \frac{1}{9} r_{p-2}^1 r_{4n-p-1}^3 \right).
\end{aligned} \tag{31}$$

By Facts 1 and 2, we have

$$\begin{aligned}
\sum_{p=4, p \equiv 0 \pmod{4}}^{4n} \left(r_{p-1}^0 r_{4n-p}^0 - \frac{1}{9} r_{p-2}^1 r_{4n-p-1}^0 \right) &= \sum_{p=4, p \equiv 0 \pmod{4}}^{4n} \left[\frac{p}{12} \left(\frac{1}{36} \right)^{((p-4)/4)} (4n-p+1) \frac{1}{12} \left(\frac{1}{36} \right)^{((4n-p)/4)} \right. \\
&\quad \left. - \frac{1}{9} (p-1) \frac{1}{4} \left(\frac{1}{36} \right)^{((p-4)/4)} \frac{1}{12} (4n-p) \left(\frac{1}{36} \right)^{((4n-p-4)/4)} \right] \\
&= \sum_{p=4, p \equiv 0 \pmod{4}}^{4n} 12n \left(\frac{1}{36} \right)^n \\
&= 12n^2 \left(\frac{1}{36} \right)^n.
\end{aligned} \tag{32}$$

Similarly, by Facts 1–4, we can get

$$\begin{aligned}
\sum_{p=1, p \equiv 1 \pmod{4}}^{4n-3} \left(r_{p-1}^0 r_{4n-p}^1 - \frac{1}{9} r_{p-2}^1 r_{4n-p-1}^1 \right) &= 12n^2 \left(\frac{1}{36} \right)^n, \\
\sum_{p=2, p \equiv 2 \pmod{4}}^{4n-2} \left(r_{p-1}^0 r_{4n-p}^2 - \frac{1}{9} r_{p-2}^1 r_{4n-p-1}^2 \right) &= 8n^2 \left(\frac{1}{36} \right)^n, \\
\sum_{p=3, p \equiv 3 \pmod{4}}^{4n-1} \left(r_{p-1}^0 r_{4n-p}^3 - \frac{1}{9} r_{p-2}^1 r_{4n-p-1}^3 \right) &= 8n^2 \left(\frac{1}{36} \right)^n.
\end{aligned} \tag{33}$$

Hence, according to the above results, we have

$$(-1)^{4n-1}a_{4n-1} = \sum_{p=1}^{4n} \det \mathcal{L}_A[p] = 40n^2 \left(\frac{1}{36}\right)^n. \quad (34)$$

The proof of Claim 1 is completed. $\square$

Claim 2. $(-1)^{4n-2}a_{4n-2} = (2/3)(200n^4 - 11n^2)(1/36)^n$.

Proof. of Claim 2. It is not hard to see that $(-1)^{4n-2}a_{4n-2}$ is the total of those principal minors $\mathcal{L}_A$, which have $(4n-2)$ rows and columns. Thus, we have

By equation (35), it can be seen that the change of i and j values will lead to different $\det \mathcal{L}_A[p, q]$ results. Therefore, we will choose different p and q to list the following equations:

$$\begin{aligned} \sum_{1 \leq p < q \leq 4n} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] \\ &+ \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] \\ &+ \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] \\ &+ \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] \\ &+ \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] \\ &+ \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] \\ &+ \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 3 \pmod{4}}^{4n-2} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] \\ &+ \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q]. \end{aligned} \quad (36)$$

By Facts 1–4, we can compute the following results.

Case 1.

$$\begin{aligned} \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 0 \pmod{4}}^{4n} \left(r_{p-1}^0 r_{q-p-1}^0 r_{4n-q}^0 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^0 r_{4n-q-1}^0 \right) \\ &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 0 \pmod{4}}^{4n} 9(q-p)(4n-q+p) \left(\frac{1}{36}\right)^n \\ &= 12(n^4 - n^2) \left(\frac{1}{36}\right)^n. \end{aligned} \quad (37)$$

Case 2.

$$\begin{aligned}
 \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \left(r_{p-1}^0 r_{q-p-1}^0 r_{4n-q}^1 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^0 r_{4n-q-1}^1 \right) \\
 &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 1 \pmod{4}}^{4n-3} 9(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= \frac{3}{2} (8n^4 - 12n^3 + n^2 + 3n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{38}$$

Case 3.

$$\begin{aligned}
 \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \left(r_{p-1}^0 r_{q-p-1}^0 r_{4n-q}^2 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^0 r_{4n-q-1}^2 \right) \\
 &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 2 \pmod{4}}^{4n-2} 6(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= (8n^4 + 8n^3 + 4n^2 + 4n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{39}$$

Case 4.

$$\begin{aligned}
 \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \left(r_{p-1}^0 r_{q-p-1}^0 r_{4n-q}^3 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^0 r_{4n-q-1}^3 \right) \\
 &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 3 \pmod{4}}^{4n-1} 6(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= (8n^4 - 4n^3 + n^2 - 5n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{40}$$

Case 5.

$$\begin{aligned}
 \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 0 \pmod{4}}^{4n} \left(r_{p-1}^0 r_{q-p-1}^1 r_{4n-q}^0 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^1 r_{4n-q-1}^0 \right) \\
 &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 0 \pmod{4}}^{4n} 9(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= \frac{3}{2} (8n^4 + 12n^3 + n^2 - 3n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{41}$$

Case 6.

$$\begin{aligned}
 \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \left(r_{p-1}^0 r_{q-p-1}^1 r_{4n-q}^1 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^1 r_{4n-q-1}^1 \right) \\
 &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 1 \pmod{4}}^{4n-3} 9(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= 12(n^4 - n^2) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{42}$$

Case 7.

$$\begin{aligned}
 \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \left(r_{p-1}^0 r_{q-p-1}^1 r_{4n-q}^2 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^1 r_{4n-q-1}^2 \right) \\
 &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 2 \pmod{4}}^{4n-2} 9(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= (8n^4 + 4n^3 + n^2 + 5n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{43}$$

Case 8.

$$\begin{aligned}
 \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \left(r_{p-1}^0 r_{q-p-1}^1 r_{4n-q}^3 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^1 r_{4n-q-1}^3 \right) \\
 &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 3 \pmod{4}}^{4n-1} 6(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= (8n^4 - 8n^3 + 4n^2 - 4n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{44}$$

Case 9.

$$\begin{aligned}
 \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 0 \pmod{4}}^{4n} \left(r_{p-1}^0 r_{q-p-1}^2 r_{4n-q}^0 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^2 r_{4n-q-1}^0 \right) \\
 &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 0 \pmod{4}}^{4n} 6(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= (8n^4 + 8n^3 + 4n^2 + 4n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{45}$$

Case 10.

$$\begin{aligned}
 \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \left(r_{p-1}^0 r_{q-p-1}^2 r_{4n-q}^1 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^2 r_{4n-q-1}^1 \right) \\
 &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 1 \pmod{4}}^{4n-3} 6(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= (8n^4 - 4n^3 + n^2 - 5n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{46}$$

Case 11.

$$\begin{aligned}
 \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \left(r_{p-1}^0 r_{q-p-1}^2 r_{4n-q}^2 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^2 r_{4n-q-1}^2 \right) \\
 &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 2 \pmod{4}}^{4n-2} 4(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= \frac{16}{3} (n^4 - n^2) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{47}$$

Case 12.

$$\begin{aligned}
 \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \left(r_{p-1}^0 r_{q-p-1}^2 r_{4n-q}^3 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^2 r_{4n-q-1}^3 \right) \\
 &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 3 \pmod{4}}^{4n-1} 4(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= \frac{2}{3} (8n^4 + 4n^3 + n^2 + 5n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{48}$$

Case 13.

$$\begin{aligned}
 \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 0 \pmod{4}}^{4n} \left(r_{p-1}^0 r_{q-p-1}^3 r_{4n-q}^0 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^3 r_{4n-q-1}^0 \right) \\
 &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 0 \pmod{4}}^{4n} 6(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= (8n^4 + 4n^3 + n^2 + 5n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{49}$$

Case 14.

$$\begin{aligned}
 \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \left(r_{p-1}^0 r_{q-p-1}^3 r_{4n-q}^1 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^3 r_{4n-q-1}^1 \right) \\
 &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 1 \pmod{4}}^{4n-3} 6(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= (8n^4 - 8n^3 + 4n^2 - 4n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{50}$$

Case 15.

$$\begin{aligned}
 \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \left(r_{p-1}^0 r_{q-p-1}^3 r_{4n-q}^2 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^3 r_{4n-q-1}^2 \right) \\
 &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 2 \pmod{4}}^{4n-2} 4(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= \frac{2}{3} (8n^4 - 4n^3 + n^2 - 5n) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{51}$$

Case 16.

$$\begin{aligned}
 \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \left(r_{p-1}^0 r_{q-p-1}^3 r_{4n-q}^3 - \frac{1}{9} r_{p-2}^1 r_{q-p-1}^3 r_{4n-q-1}^3 \right) \\
 &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 3 \pmod{4}}^{4n-1} 4(q-p)(4n-q+p) \left(\frac{1}{36} \right)^n \\
 &= \frac{16}{3} (n^4 - n^2) \left(\frac{1}{36} \right)^n.
 \end{aligned} \tag{52}$$

Then, according to the value of p , the above sixteen cases can be divided into the following four categories:

$$\begin{aligned}
 F_0 &= \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] \\
 &\quad + \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 0 \pmod{4}}^{4n-4} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] \\
 &= \frac{1}{2} (80n^4 - 28n^3 - 11n^2 + 7n) \left(\frac{1}{36} \right)^n,
 \end{aligned}$$

$$\begin{aligned}
F_1 &= \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] \\
&+ \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 1 \pmod{4}}^{4n-3} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] \\
&= \frac{1}{2} (80n^4 + 28n^3 - 11n^2 - 7n) \left(\frac{1}{36} \right)^n. \\
F_2 &= \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] \\
&+ \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 2 \pmod{4}}^{4n-2} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] \\
&= \frac{1}{3} (80n^4 + 20n^3 + n^2 + 7n) \left(\frac{1}{36} \right)^n, \\
F_3 &= \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 3 \pmod{4}}^{4n-2} \sum_{q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_A[p, q] \\
&+ \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_A[p, q] + \sum_{p \equiv 3 \pmod{4}}^{4n-1} \sum_{q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_A[p, q] \\
&= \frac{1}{3} (80n^4 - 20n^3 + 10n^2 - 7n) \left(\frac{1}{36} \right)^n.
\end{aligned} \tag{53}$$

Substituting F_0 , F_1 , F_2 , and F_3 into equation (35), one has

$$(-1)^{4n-2} a_{4n-2} = F_0 + F_1 + F_2 + F_3 = \frac{2}{3} (200n^4 - 11n^2) \left(\frac{1}{36} \right)^n. \tag{54}$$

This completes the proof. $\square$

Proof. Let

So, substituting the results of Claims 1 and 2 into equation (20) yields

$$\begin{aligned}
P_{L_S}(x) &= \det(xI - \mathcal{L}_S) = x^{4n} + b_1 x^{4n-1} + \cdots + b_{4n-1} x + b_{4n} \\
&= x(x^{4n-1} + b_1 x^{4n-2} + \cdots + b_{4n-2} x + b_{4n-1}), \quad b_{4n-1} \neq 0.
\end{aligned} \tag{57}$$

Then, we can exactly get that $\varphi_1, \varphi_2, \dots, \varphi_{4n}$ are the roots of the following equation:

$$x^{4n-1} + b_1 x^{4n-2} + \cdots + b_{4n-2} x + b_{4n-1} = 0. \tag{58}$$

Based on Vieta's theorem of $P_{\mathcal{L}_S}(x)$, one has

$$\sum_{q=1}^{4n} \frac{1}{\varphi_q} = \frac{(-1)^{4n-1} b_{4n-1}}{(-1)^{4n} b_{4n}} = \frac{(-1)^{4n-1} b_{4n-1}}{\det \mathcal{L}_S}. \tag{59}$$

Theorem 2.

$$\sum_{q=1}^{4n} \frac{1}{\varphi_q} = \frac{41n\sqrt{14}}{28} \left[\frac{((15 + 4\sqrt{14})^n - (15 - 4\sqrt{14})^n)}{((15 + 4\sqrt{14})^n + (15 - 4\sqrt{14})^n) + 2} \right]. \tag{56}$$

Before calculating $(-1)^{4n-1} b_{4n-1}$ and $\det \mathcal{L}_S$, we must determine i th order principal submatrices S_q^0, S_q^1, S_q^2 , and S_q^3 , which consist of the first q rows and columns of the matrices $\mathcal{L}_S^0, \mathcal{L}_S^1, \mathcal{L}_S^2$, and $\mathcal{L}_S^3$, respectively, $q = 1, 2, \dots, 4n$. Let

$$\mathcal{L}_S^0 = \begin{pmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & & \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & \\ & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & \\ & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & \\ & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & \\ & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & \\ & & & & & \ddots & & & \\ & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} \\ & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} \\ & & & & & & & & \ddots & \\ & & & & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} \\ & & & & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} \\ & & & & & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\ & & & & & & & & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} \end{pmatrix}_{(4n) \times (4n)},$$

$$\mathcal{L}_S^1 = \begin{pmatrix} 1 & \frac{-1}{2} & & & & & & & \\ \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & \\ & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & & \\ & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & \\ & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & \\ & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & \\ & & & & & \ddots & & & \\ & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} \\ & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\ & & & & & & & & \ddots & \\ & & & & & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} \\ & & & & & & & & & & \frac{-1}{3} & \frac{4}{3} \end{pmatrix}_{(4n) \times (4n)},$$

$$\begin{aligned}
\mathcal{L}_S^2 = & \begin{pmatrix} 1 & \frac{-1}{\sqrt{6}} & & & & & & & \\ \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & & & \\ & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & \\ & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & \\ & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & \\ & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & \\ & & & & & \ddots & & & \\ & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & \\ & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & \\ & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} \\ & & & & & & & \frac{-1}{\sqrt{6}} & 1 \end{pmatrix}_{(4n) \times (4n)}, \\
\mathcal{L}_S^3 = & \begin{pmatrix} \frac{4}{3} & \frac{-1}{3} & & & & & & & \\ \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & \\ & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & \\ & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & \\ & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & \\ & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & \\ & & & & & \ddots & & & \\ & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & \\ & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & \\ & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} \\ & & & & & & & \frac{-1}{2} & 1 \end{pmatrix}_{(4n) \times (4n)}.
\end{aligned} \tag{60}$$

n this way, let us start with the following facts.

Fact 5. For $1 \leq q \leq 4n$,

$$s_q^0 = \begin{cases} \left(\frac{4}{8} + \frac{9\sqrt{14}}{56} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(q/4)} + \left(\frac{4}{8} - \frac{9\sqrt{14}}{56} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(q/4)}, & \text{if } q \equiv 0 \pmod{4}, \\ \left(\frac{2}{3} + \frac{31\sqrt{14}}{168} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(q-1/4)} + \left(\frac{2}{3} - \frac{31\sqrt{14}}{168} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(q-1/4)}, & \text{if } q \equiv 1 \pmod{4}, \\ \left(\frac{7}{12} + \frac{53\sqrt{14}}{336} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(q-2/4)} + \left(\frac{7}{12} - \frac{53\sqrt{14}}{336} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(q-2/4)}, & \text{if } q \equiv 2 \pmod{4}, \\ \left(\frac{5}{12} + \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(q-3/4)} + \left(\frac{5}{12} - \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(q-3/4)}, & \text{if } q \equiv 3 \pmod{4}. \end{cases} \quad (61)$$

Fact 6. For $1 \leq q \leq 4n$,

$$s_q^1 = \begin{cases} \left(\frac{1}{2} + \frac{11\sqrt{14}}{56} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(q/4)} + \left(\frac{1}{2} - \frac{11\sqrt{14}}{56} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(q/4)}, & \text{if } q \equiv 0 \pmod{4}, \\ \left(\frac{1}{2} + \frac{17\sqrt{14}}{112} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(q-1/4)} + \left(\frac{1}{2} - \frac{17\sqrt{14}}{112} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(q-1/4)}, & \text{if } q \equiv 1 \pmod{4}, \\ \left(\frac{3}{8} + \frac{23\sqrt{14}}{224} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(q-2/4)} + \left(\frac{3}{8} - \frac{23\sqrt{14}}{224} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(q-2/4)}, & \text{if } q \equiv 2 \pmod{4}, \\ \left(\frac{5}{12} + \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(q-3/4)} + \left(\frac{5}{12} - \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(q-3/4)}, & \text{if } q \equiv 3 \pmod{4}. \end{cases} \quad (62)$$

Fact 7. For $1 \leq q \leq 4n$,

$$s_q^2 = \frac{7}{6}s_{q-2}^0 - \frac{1}{9}s_{q-3}^1. \quad (63)$$

Fact 8. For $1 \leq q \leq 4n$,

$$s_q^3 = \frac{4}{3}s_{q-1}^0 - \frac{1}{9}s_{q-2}^1. \quad (64)$$

Proof. of Fact 5. Take $s_q^0 = \det S_q^0$, $s_q^1 = \det S_q^1$, $s_q^2 = \det S_q^2$, and $s_q^3 = \det S_q^3$. By direct calculation, it is not difficult to get the following values (see Table 2).

For $4 \leq q \leq 4n$, we have $\det S_q^0$

$$s_q^0 = \begin{cases} \frac{4}{3}s_{q-1}^0 - \frac{1}{6}s_{q-2}^0, & \text{if } q \equiv 0 \pmod{4}, \\ \frac{4}{3}s_{q-1}^0 - \frac{1}{9}s_{q-2}^0, & \text{if } q \equiv 1 \pmod{4}, \\ s_{q-1}^0 - \frac{1}{6}s_{q-2}^0, & \text{if } q \equiv 2 \pmod{4}, \\ s_{q-1}^0 - \frac{1}{4}s_{q-2}^0, & \text{if } q \equiv 3 \pmod{4}. \end{cases} \quad (65)$$

For $1 \leq q \leq n$, let $A_q = s_{4q}$; for $0 \leq q \leq n-1$, let $B_q = s_{4q+1}$, $C_q = s_{4q+2}$, $D_q = s_{4q+3}$. Then, we may obtain that

TABLE 2: Initial value.

s_q^0	Value	s_q^0	Value	s_q^0	Value	s_q^0	Value
s_1^0	(4/3)	s_2^0	(7/6)	s_3^0	(5/6)	s_4^0	(33/36)
s_5^0	(61/54)	s_6^0	(211/216)	s_7^0	(25/36)	s_8^0	(989/1296)

$$\begin{cases} A_q = \frac{4}{3}D_{q-1} - \frac{1}{6}C_{q-1}, \\ B_q = \frac{4}{3}A_q - \frac{1}{9}D_{q-1}, \\ C_q = B_q - \frac{1}{6}A_q, \\ D_q = C_q - \frac{1}{4}B_q. \end{cases} \quad (66)$$

From the first three equations in (66), one can get $A_q = (12/13)C_q + (1/78)C_{q-1}$. Next, substituting A_q into the third equation, one has $B_q = (15/13)C_q + (1/468)C_{q-1}$. Then, substituting B_q into the fourth equation, we have $D_q = (37/52)C_q - (1/1872)C_{q-1}$. Finally, substituting A_q and d_q into the first equation, one has $c_q - 30c_{q-1} + c_{q-2} = 0$. Thus,

$$C_q = k_1 \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^q + k_2 \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^q. \quad (67)$$

In view of $C_0 = (7/6), C_1 = (211/216)$, we have

$$\begin{cases} k_1 + k_2 = \frac{7}{6}, \\ k_1 \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right) + k_2 \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right) = \frac{211}{216}, \\ k_1 = \left(\frac{7}{12} + \frac{53\sqrt{14}}{336} \right), \\ k_2 = \left(\frac{7}{12} - \frac{53\sqrt{14}}{336} \right). \end{cases} \quad (68)$$

Thus, it is routine to deduce that

$$\begin{cases} A_q = \left(\frac{4}{8} + \frac{9\sqrt{14}}{56} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^q + \left(\frac{4}{8} - \frac{9\sqrt{14}}{56} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^q, & \text{if } q \equiv 0 \pmod{4}, \\ B_q = \left(\frac{2}{3} + \frac{31\sqrt{14}}{168} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^q + \left(\frac{2}{3} - \frac{31\sqrt{14}}{168} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^q, & \text{if } q \equiv 1 \pmod{4}, \\ C_q = \left(\frac{7}{12} + \frac{53\sqrt{14}}{336} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^q + \left(\frac{7}{12} - \frac{53\sqrt{14}}{336} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^q, & \text{if } q \equiv 1 \pmod{4}, \\ D_q = \left(\frac{5}{12} + \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^q + \left(\frac{5}{12} - \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^q, & \text{if } q \equiv 3 \pmod{4}. \end{cases} \quad (69)$$

In the same way, we can quickly prove the result of Fact 6.

Then, we expand dets_q^2 and dets_q^3 according to the properties of determinant, and we can get Facts 7 and 8.

Now by exploiting the property of determinant, we can get

$$\begin{aligned}
\det \mathcal{L}_S &= \begin{vmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & \frac{1}{3} \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{-1}{\sqrt{6}} \\ \frac{1}{3} & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{6}} & \frac{4}{3} \end{vmatrix}_{4n} = \begin{vmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{-1}{\sqrt{6}} \\ \frac{1}{3} & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{6}} & \frac{4}{3} \end{vmatrix}_{4n} \\
&+ \begin{vmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & \frac{1}{3} \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{6}} & 0 \end{vmatrix}_{4n} \\
&= s_{4n}^0 - \frac{1}{9} s_{4n-2}^1 + 2 \left(\frac{1}{36} \right)^n.
\end{aligned} \tag{70}$$

With Facts 1 and 2, we can obtain one interesting claim.

Claim 3. $\det \mathcal{L}_S = ((5/12) + (\sqrt{14}/9))^n + ((5/12) - (\sqrt{14}/9))^n + 2(1/36)^n$.

Then, we are going to concentrate on calculating $(-1)^{4n-1} b_{4n-1}$.

Claim 4. $(-1)^{4n} - 1 b_{4n-1} = (41n\sqrt{14}/28) [((15 + 4\sqrt{14})^n - (15 - 4\sqrt{14})^n) / ((15 + 4\sqrt{14})^n + (15 - 4\sqrt{14})^n + 2)]$.

Proof. Since $(-1)^{4n-1} b_{4n-1}$ is the total of all the principal minors of order $4n-1$ of $\mathcal{L}_S$, we have

$$\begin{aligned}
(-1)^{4n-1} b_{4n-1} &= \sum_{q=1}^{4n} \det \mathcal{L}_S[q] = \sum_{q=4, q \equiv 0 \pmod{4}}^{4n} \det \mathcal{L}_S[q] + \sum_{q=1, q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_S[q] \\
&+ \sum_{q=2, q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_S[q] + \sum_{q=3, q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_S[q],
\end{aligned} \tag{71}$$

where

$$\det \mathcal{L}_S[q] = \begin{cases} s_{q-1}^0 s_{4n-q}^0 - \frac{1}{9} s_{q-2}^1 s_{4n-q-1}^0, & \text{if } q \equiv 0 \pmod{4}, \\ s_{q-1}^0 s_{4n-q}^1 - \frac{1}{9} s_{q-2}^1 s_{4n-q-1}^1, & \text{if } q \equiv 1 \pmod{4}, \\ s_{q-1}^0 s_{4n-q}^2 - \frac{1}{9} s_{q-2}^1 s_{4n-q-1}^2, & \text{if } q \equiv 2 \pmod{4}, \\ s_{q-1}^0 s_{4n-q}^3 - \frac{1}{9} s_{q-2}^1 s_{4n-q-1}^3, & \text{if } q \equiv 3 \pmod{4}. \end{cases} \quad (72)$$

For $q \equiv 0 \pmod{4}$ and $4 \leq q \leq 4n-4$, in view of (72) and Facts 5–8, one gets

$$\begin{aligned} \det \mathcal{L}_S[q] &= s_{q-1}^0 s_{4n-q}^0 - \frac{1}{9} s_{q-2}^1 s_{4n-q-1}^0 \\ &= \left\{ \left(\frac{5}{12} + \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{(j-4/4)} \right. \\ &\quad \left. + \left(\frac{5}{12} - \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{(j-4/4)} \right. \\ &\quad \times \left[\left(\frac{4}{8} + \frac{25\sqrt{14}}{56} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{((4n-j)/4)} + \left(\frac{4}{8} - \frac{25\sqrt{14}}{56} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{((4n-j)/4)} \right] \\ &\quad - \frac{1}{9} \left[\left(\frac{3}{8} + \frac{23\sqrt{14}}{224} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{((j-4)/4)} + \left(\frac{3}{8} - \frac{23\sqrt{14}}{224} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{((j-4)/4)} \right] \\ &\quad \times \left[\left(\frac{5}{12} + \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^{((4n-j-4)/4)} + \left(\frac{5}{12} - \frac{25\sqrt{14}}{224} \right) \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^{((4n-j-4)/4)} \right] \\ &= \frac{15n\sqrt{14}}{56} \left[\left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^n - \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^n \right]. \end{aligned} \quad (73)$$

Similarly, for $q \equiv 1 \pmod{4}$ and $1 \leq q \leq 4n-3$, we have

$$\begin{aligned} \sum_{q=1, q \equiv 1 \pmod{4}}^{4n-3} \det \mathcal{L}_S[q] &= s_{q-1}^0 s_{4n-q}^1 - \frac{1}{9} s_{q-2}^1 s_{4n-q-1}^1 \\ &= \frac{15n\sqrt{14}}{56} \left[\left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^n - \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^n \right]. \end{aligned} \quad (74)$$

For $q \equiv 2 \pmod{4}$ and $2 \leq q \leq 4n-2$, we have

$$\begin{aligned} \sum_{q=2, q \equiv 2 \pmod{4}}^{4n-2} \det \mathcal{L}_S[q] &= s_{q-1}^0 s_{4n-q}^2 - \frac{1}{9} s_{q-2}^1 s_{4n-q-1}^2 \\ &= \frac{13n\sqrt{14}}{28} \left[\left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^n - \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^n \right]. \end{aligned} \quad (75)$$

For $q \equiv 3 \pmod{4}$ and $3 \leq q \leq 4n-1$, we have

$$\begin{aligned} \sum_{q=3, q \equiv 3 \pmod{4}}^{4n-1} \det \mathcal{L}_S[q] &= s_{q-1}^0 s_{4n-q}^3 - \frac{1}{9} s_{q-2}^1 s_{4n-q-1}^3 \\ &= \frac{13n\sqrt{14}}{28} \left[\left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^n - \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^n \right]. \end{aligned} \quad (76)$$

Thus, one has the following equation:

$$(-1)^{4n-1} b_{4n-1} = \frac{41n\sqrt{14}}{28} \left[\left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^n - \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^n \right]. \quad (77)$$

Therefore, substituting the results of Claims 3 and 4 into (59) yields

$$\sum_{q=1}^{4n} \frac{1}{\varphi_q} = \frac{41n\sqrt{14}}{28} \left[\frac{((15+4\sqrt{14})^n - (15-4\sqrt{14})^n)}{((15+4\sqrt{14})^n + (15-4\sqrt{14})^n) + 2} \right], \quad (78)$$

as desired. $\square$

Note that $|E_{MQ_n}(8,4)| = 10n$. Taking the results of Theorems 1 and 2 to (a) and (b) of Lemma 2, we can immediately get the following two theorems.

Theorem 3. Let $MQ_n(8,4)$ be a Möbius graph with n octagons and n quadrilaterals. Then,

$$\begin{aligned} Kc(MQ_n(8,4)) &= \sum_{p=2}^{4n} \frac{1}{\eta_p} + \sum_{q=1}^{4n} \frac{1}{\varphi_q} \\ &= \frac{200n^2 - 11}{60} + \frac{41n\sqrt{14}}{28} \left[\frac{((15+4\sqrt{14})^n - (15-4\sqrt{14})^n)}{((15+4\sqrt{14})^n + (15-4\sqrt{14})^n) + 2} \right]. \end{aligned} \quad (79)$$

Theorem 4. Let $MQ_n(8,4)$ be a Möbius graph with n octagons and n quadrilaterals. Then,

$$\begin{aligned} Kf^*(MQ_n(8,4)) &= 20n \left(\sum_{p=2}^{4n} \frac{1}{\eta_p} + \sum_{q=1}^{4n} \frac{1}{\varphi_q} \right) \\ &= 20n \left(\frac{200n^2 - 11}{60} + \frac{41n\sqrt{14}}{28} \left[\frac{((15+4\sqrt{14})^n - (15-4\sqrt{14})^n)}{((15+4\sqrt{14})^n + (15-4\sqrt{14})^n) + 2} \right] \right) \\ &= \frac{200n^3 - 11n}{3} + 20nq(n), \end{aligned} \quad (80)$$

Table 3 shows the degree-Kirchhoff indices of Möbius graph of linear octagonal-quadrilateral networks.

Finally, we will concentrate on calculating the complexity of $MQ_n(8,4)$.

Theorem 5. Let $MQ_n(8,4)$ denote a Möbius graph of linear octagonal-quadrilateral networks of length $n \geq 2$. Then,

$$\tau(MQ_n(8,4)) = 4n((15+4\sqrt{14})^n + (15-\sqrt{14})^n + 2). \quad (81)$$

Proof. By Claim 1, one can get

$$\prod_{p=2}^{4n} \eta_p = (-1)^{4n-1} a_{4n-1} = 40n^2 \left(\frac{1}{36} \right)^n. \quad (82)$$

Similarly, according to Claim 3, we have

$$\prod_{q=1}^{4n} \varphi_q = \det \mathcal{L}_S = \left(\frac{5}{12} + \frac{\sqrt{14}}{9} \right)^n + \left(\frac{5}{12} - \frac{\sqrt{14}}{9} \right)^n + 2 \left(\frac{1}{36} \right)^n. \quad (83)$$

Note that $\prod_{i=1}^{8n} d_i(MQ_n) = 2^{4n} 3^{4n}$ and $|E_{MQ_n}(8,4)| = 10n$. By Lemma 2(c), one gets

$$\tau(Q_n(8,4)) = \frac{1}{10n} \prod_{p=2}^{4n} \eta_p \cdot \prod_{q=1}^{4n} \varphi_q = 4n((15+4\sqrt{14})^n + (15-\sqrt{14})^n + 2). \quad (84)$$

TABLE 3: The degree-Kirchhoff indices of $MQ_1(8, 4), MQ_2(8, 4), \dots, MQ_{16}(8, 4)$.

G	$Kf^*(G)$	G	$Kf^*(G)$	G	$Kf^*(G)$	G	$Kf^*(G)$
$MQ_1(8, 4)$	165.50	$MQ_5(8, 4)$	11054.43	$MQ_9(8, 4)$	57442.75	$MQ_{13}(8, 4)$	164937.53
$MQ_2(8, 4)$	963.33	$MQ_6(8, 4)$	18322.78	$MQ_{10}(8, 4)$	77587.71	$MQ_{14}(8, 4)$	204359.11
$MQ_3(8, 4)$	2775.12	$MQ_7(8, 4)$	28210.28	$MQ_{11}(8, 4)$	101951.83	$MQ_{15}(8, 4)$	249599.85
$MQ_4(8, 4)$	6005.23	$MQ_8(8, 4)$	41116.93	$MQ_{12}(8, 4)$	130935.10	$MQ_{16}(8, 4)$	301059.74

TABLE 4: The complexity of $Q_1, Q_2, \dots, Q_{10}$.

G	$\tau(G)$	G	$\tau(G)$
Q_1	128	Q_5	17379554400
Q_2	7200	Q_7	607607778176
Q_3	322944	Q_8	20809093939328
Q_4	12902464	Q_9	701525710449792
Q_5	483303040	Q_{10}	23358178980900000

This completes the proof. $\square$

Thus, we can get the complexity of $MQ_n(8, 4)$, which is listed in Table 4.

Data Availability

The figures, tables, and other data used to support this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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