

## Research Article

# A Strong Convergence to a Common Fixed Point of a Subfamily of a Nonexpansive Evolution Family of Bounded Linear Operators on a Hilbert Space

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In this article, we establish some results for convergence in a strong sense to a common fixed point of a subfamily of a nonexpansive evolution family of bounded linear operators on a Hilbert space. The obtained results generalize some existing ones in the literature for semigroups of operators. An example and an open problem are also given at the end.

## 1. Introduction

In the 19th century, the fixed point theory was started by Poincaré [1]. In the 20th century, many mathematicians, such as Brouwer [2], Schauder [3], Tarski [4], and others, developed the field. The fixed point theory has a wide range of applications. It is one of the most important tools of modern mathematical analysis and is useful in various fields such as mathematics, engineering, physics, economics, and many more. Fixed point theory can be used as a tool to discuss the uniqueness and existence of solutions of many problems such as integral equations [5], differential equations [6, 7], and numerical equations and algebraic systems [8–11]. We refer to [12–19] for a more detailed study on fixed point theory and its applications in metric spaces.

Let  $\mathcal{X} \neq \emptyset$ , and  $\tau: \mathcal{X} \rightarrow \mathcal{X}$  be a self-mapping. The element  $\mathbf{a} \in \mathcal{X}$  is called a fixed point of  $\tau$  if  $\tau(\mathbf{a}) = \mathbf{a}$ . Consider the autonomous system

$$\begin{cases} \dot{\mathbf{h}}(r) = A\mathbf{h}(r), & r \geq 0, \\ \mathbf{h}(0) = \Omega, \end{cases} \quad (1)$$

where  $A$  is a linear operator on a Hilbert space  $\mathcal{H}$ . The solutions of such a system lead us to a class of linear and bounded mappings, called a semigroup. A family  $\mathfrak{S} = \{\mathfrak{S}(\mathfrak{s}): \mathfrak{s} \geq 0\}$  of linear and bounded operators is called a semigroup if it satisfies the following two conditions:

- (1)  $G(0) = I$ , the identity map
- (2)  $G(s+t) = G(s)G(t)$  for all  $s, t \geq 0$

The system becomes more difficult if the operator  $A$  depends on time, i.e., when  $A$  is replaced by  $A(t)$  in the above system. Such a system is called nonautonomous, and its solution leads to the concept of an evolution family. Again, a family  $\mathbf{L} = \{L(\mathfrak{s}, \mathbf{r})\}$  of linear and bounded operators is said to be an evolution family, if the following hold:

- (1)  $L(\mathfrak{s}, \mathfrak{s}) = I$ , for all  $\mathfrak{s} \geq 0$
- (2)  $L(\mathfrak{s}, \mathfrak{t})L(\mathfrak{t}, \mathfrak{r}) = L(\mathfrak{s}, \mathfrak{r})$ , for all  $\mathfrak{s} \geq \mathfrak{t} \geq \mathfrak{r} \geq 0$

*Remark 1* (see [20]). Every semigroup is an evolution family, but the converse is not true in general. In fact, if an evolution family is periodic at every period, then it becomes a semigroup.

The study of fixed points for semigroups is studied by many mathematicians, such as Suzuki [21, 22] and Buthinah et al. [23]. They proved different results concerning a strong convergence to a fixed point of a semigroup and the representation of the set of all common fixed points of the semigroups in a form of intersection of the sets of all common fixed points of only two operators from the family. Such results are of too importance in the field. Recently, such results were generalized to a subfamily of an evolution family acting on different spaces, see [20, 24].

In this paper, we will present some new results for fixed points of an evolution family of operators. We also generalize some other results from semigroups [21], to a subfamily of an evolution family.

## 2. Preliminaries

In this article, we will frequently use the following notations:

- (1) By  $\mathbb{R}$ ,  $\mathbb{R}_+$ ,  $\mathbb{N}$ , and  $\mathbb{Z}_+$ , we will denote the set of all reals, nonnegative reals, natural numbers, and nonnegative integers, respectively.
- (2) The semigroup, evolution family, and its subfamily will be denoted by  $\mathfrak{S}$ ,  $\mathbf{L}$ , and  $\mathbf{L}_s$ , respectively.
- (3) The set of all common fixed points of the semigroup, evolution family, and its subfamily will be denoted and defined as  $\mathcal{F}\mathbf{ix}(G) = \bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s))$ ,  $\mathcal{F}\mathbf{ix}(\mathbf{L}) = \bigcap_{\mathfrak{s} \geq \mathfrak{r} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, \mathfrak{r}))$ , and  $\mathcal{F}\mathbf{ix}(\mathbf{L}_s) = \bigcap_{\mathfrak{s} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0))$ , respectively.
- (4) By  $\mathcal{C}$ , we will denote a closed and convex subset of the Hilbert space  $\mathcal{H}$ .

A map  $\tau: \mathcal{C} \rightarrow \mathcal{C}$  is nonexpansive if  $\|\tau_x - \tau_y\| \leq \|x - y\|$  for all  $x, y \in \mathcal{C}$ .

We denote by  $\mathcal{F}\mathbf{ix}(\tau)$  the set of all fixed points of  $\tau$ . If  $\tau$  is a nonexpansive self-map on  $\mathcal{C}$ , then  $\mathcal{F}\mathbf{ix}(\tau)$  is nonempty, see [25].

For a fixed  $\zeta$  in  $\mathcal{C}$  and  $\varepsilon \in (0, 1)$ , there is a unique point  $x_\varepsilon \in \mathcal{C}$  such that  $x_\varepsilon = (1 - \varepsilon)\mathcal{T}_{x_\varepsilon} + \varepsilon\zeta$ . We see that the map  $x \rightarrow (1 - \varepsilon)\mathcal{T}_x + \varepsilon\zeta$  is a contraction. Indeed,

$$\left\| ((1 - \varepsilon)\mathcal{T}_x + \varepsilon\zeta) - ((1 - \varepsilon)\mathcal{T}_y + \varepsilon\zeta) \right\| = (1 - \varepsilon) \left\| \mathcal{T}_x - \mathcal{T}_y \right\|. \quad (2)$$

The nonexpansiveness of  $\mathcal{T}$  ensures that the map is a contraction.

In 1967, Browder [26] provided the following result for self-mappings.

**Theorem 1.** *Let  $\tau$  be a self-mapping on  $\mathcal{C}$  and  $\{\gamma_n\} \in (0, 1)$  be a sequence such that  $\lim_{n \rightarrow \infty} \gamma_n = 0$ . Then, for a fixed  $\zeta$  in  $\mathcal{C}$ , the sequence*

$$\zeta_n = (1 - \gamma_n)\tau_{\zeta_n} + \gamma_n\zeta. \quad (3)$$

*converges to a fixed point of  $\tau$  nearest to  $\zeta$  in a strong sense.*

In this paper, we will prove a theorem of Suzuki [21] for a subfamily  $\mathbf{L}_s$  of nonexpansive evolution operators on Hilbert spaces. Such a family needs not be a semigroup. The following example will illustrate this fact.

*Example 1.* The family defined by  $\mathbf{L} = \{L(\mathfrak{s}, \mathfrak{r}) = (\mathfrak{r} + 1/\mathfrak{s} + 1) : \mathfrak{s} \geq \mathfrak{r} \geq 0\}$  is clearly an evolution family acting on  $\mathbb{R}_+$ . Since  $L(\mathfrak{s}, \mathfrak{s}) = 1$  (the identity on  $\mathbb{R}_+$ ), and

$$L(\mathfrak{s}, \mathfrak{t})L(\mathfrak{t}, \mathfrak{r}) = \left(\frac{\mathfrak{t} + 1}{\mathfrak{s} + 1}\right)\left(\frac{\mathfrak{r} + 1}{\mathfrak{t} + 1}\right) = \frac{\mathfrak{r} + 1}{\mathfrak{s} + 1} = L(\mathfrak{s}, \mathfrak{r}), \quad (4)$$

by setting  $\mathfrak{r} = 0$ , we have  $\mathbf{L}_s = \{L(\mathfrak{s}, 0) = \mathfrak{s} + 1\}$  which is a subfamily of  $\mathbf{L}$ , but not a semigroup.

However, if we put a condition as given in Remark 1, then such a family becomes a semigroup.

## 3. Main Results

In this section, we will present our main results. The following lemma states that the set of all common fixed points of a semigroup can be represented on the closed unit interval in place of  $\mathbb{R}_+$ .

**Lemma 1.** *Let  $G = \{G(s) : s \geq 0\}$  be a semigroup on a Hilbert space  $\mathcal{H}$ , then*

$$\mathcal{F}\mathbf{ix}(G) = \bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) = \bigcap_{0 \leq s \leq 1} \mathcal{F}\mathbf{ix}(G(s)). \quad (5)$$

*Proof.* The following inclusion

$$\bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) \subseteq \bigcap_{0 \leq s \leq 1} \mathcal{F}\mathbf{ix}(G(s)), \quad (6)$$

is obvious, and we will show the reverse inclusion.

Let  $\mathbf{u} \in \bigcap_{0 \leq s \leq 1} \mathcal{F}\mathbf{ix}(G(s))$ , then  $G(s)\mathbf{u} = \mathbf{u}$  for all  $0 \leq s \leq 1$ . Let  $s \geq 0$ , then it can be written as  $s = \mathbf{n} + \varrho$ , for some  $0 \leq \varrho \leq 1$  and some  $\mathbf{n} \in \mathbb{Z}_+$ .

Now, consider

$$\begin{aligned} G(s)\mathbf{u} &= G(\mathbf{n} + \varrho)\mathbf{u} \\ &= G(\mathbf{n})G(\varrho)\mathbf{u} \\ &= G^{\mathbf{n}}(1)G(\varrho)\mathbf{u} = G^{\mathbf{n}}(1)\mathbf{u} = \mathbf{u}. \end{aligned} \quad (7)$$

That is,  $\mathbf{u} \in \bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s))$ . Thus, we conclude that

$$\bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) = \bigcap_{0 \leq s \leq 1} \mathcal{F}\mathbf{ix}(G(s)). \quad (8)$$

□

In [22], it is proved that the set of all common fixed points of a semigroup can be represented by the intersection of only two operators from the family.

**Theorem 2.** Let  $G = \{G(s) : s \geq 0\}$  be a semigroup on a Hilbert space  $\mathcal{H}$ , then

$$\mathcal{F}\mathbf{ix}(G) = \bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) = \mathcal{F}\mathbf{ix}(G(\alpha)) \cap \mathcal{F}\mathbf{ix}(G(\beta)),$$

where  $\alpha, \beta$  are positive such that  $\alpha/\beta$  is irrational.

Now, using Lemma 1 and Theorem 2, we have the following corollary.

**Corollary 1.** Let  $G = \{G(s) : s \geq 0\}$  be a semigroup on a Hilbert space  $\mathcal{H}$ , then

$$\mathcal{F}\mathbf{ix}(G) = \bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) = \mathcal{F}\mathbf{ix}(G(\alpha)) \cap \mathcal{F}\mathbf{ix}(G(\beta)),$$

where  $\alpha, \beta \in [0, 1]$ , such that  $\alpha/\beta$  is irrational.

Lemma 1 can be extended to a subfamily  $\mathbf{L}_s = \{L(\mathfrak{s}, 0) : \mathfrak{s} \geq 0\}$  of a periodic evolution family. See the following lemma.

**Lemma 2.** Let  $\mathbf{L}_s = \{L(\mathfrak{s}, 0) : \mathfrak{s} \geq 0\}$  be a subfamily of a periodic evolution family with period  $q \in \mathbb{R}_+$ , then

$$\bigcap_{\mathfrak{s} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)) = \bigcap_{0 \leq \mathfrak{s} \leq q} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)). \quad (9)$$

*Proof.* Since it is obvious that

$$\bigcap_{\mathfrak{s} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)) \subseteq \bigcap_{0 \leq \mathfrak{s} \leq q} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)), \quad (10)$$

we will again prove the reverse inclusion.

Let

$$\mathbf{u} \in \bigcap_{0 \leq \mathfrak{s} \leq q} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)), \quad (11)$$

then  $L(\mathfrak{s}, 0)\mathbf{u} = \mathbf{u}$  for all  $0 \leq \mathfrak{s} \leq q$ .

Now, since any  $\mathfrak{s} \geq 0$  can be written as  $\mathfrak{s} = \mathbf{n}q + \varrho$ , for some  $\mathbf{n} \in \mathbb{Z}_+$  and some  $0 \leq \varrho \leq q$ , we have

$$\begin{aligned} L(\mathfrak{s}, 0)\mathbf{u} &= L(\mathbf{n}q + \varrho, 0)\mathbf{u} \\ &= L(\mathbf{n}q + \varrho, \mathbf{n}q)L(\mathbf{n}q, 0)\mathbf{u} \\ &= L(\varrho, 0)L^{\mathbf{n}}(q, 0)\mathbf{u} \\ &= L(\varrho, 0)\mathbf{u} \\ &= \mathbf{u}. \end{aligned} \quad (12)$$

Hence,

$$\bigcap_{\mathfrak{s} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)) = \bigcap_{0 \leq \mathfrak{s} \leq q} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)). \quad (13)$$

This completes the proof.

The Opial condition holds on every Hilbert space, given as follows.

**Proposition 1** (see [10]). If  $\{\beta_n\}$  is sequence in  $\mathcal{H}$ , converging to a point  $\mathbf{a} \in \mathcal{H}$  in a weak sense, then

$$\liminf_{n \rightarrow \infty} \|\beta_n \mathbf{a}\| \leq \liminf_{n \rightarrow \infty} \|\beta_n\| \|\mathbf{a}\|, \quad \text{for all } \mathbf{a} \in \mathcal{H}. \quad (14)$$

The next theorem is about the strong convergence of a sequence to a point near to the fixed point of the subfamily of an evolution family.

**Theorem 3.** Let  $\mathbf{L}_s = \{L(s, 0) : s \geq 0\}$  be a subfamily of strongly continuous evolution operators on  $\mathcal{E}$  such that

$\mathcal{F}\mathbf{ix}(\mathbf{L}_s) \neq \emptyset$ . Let  $\{\gamma_n\} \in (0, 1)$  and  $\{s_n\} \geq 0$  be two sequences of real numbers with the property that  $\lim_{n \rightarrow \infty} s_n = \infty$ ,  $\lim_{n \rightarrow \infty} (\gamma_n/s_n) = 0$ , (e.g.,  $s_n = (1/n)$  and  $\gamma_n = (1/n^2)$ ).

Then, for a fixed  $\zeta$  in  $\mathcal{E}$ , the sequence

$$\zeta_n = \gamma_n \zeta + (1 - \gamma_n)L(s_n, 0)\zeta_n, \quad \text{where } n \in \mathbb{N}, \quad (15)$$

converges to an element of  $\mathcal{F}\mathbf{ix}(L(s, 0))$  nearest to  $\zeta$  in a strong sense.

*Proof.* Let  $\mathfrak{s}$  be a point in  $\mathcal{F}\mathbf{ix}(\mathbf{L}_s)$  nearest to  $\zeta$ . From

$$\begin{aligned} \|\zeta_n - \mathfrak{s}\| &= \|\gamma_n \zeta + (1 - \gamma_n)L(s_n, 0)\zeta_n - \mathfrak{s}\| \\ &\leq \gamma_n \|\zeta - \mathfrak{s}\| + (1 - \gamma_n) \|L(s_n, 0)\zeta_n - \mathfrak{s}\| \\ &\leq \gamma_n \|\zeta - \mathfrak{s}\| + (1 - \gamma_n) \|\zeta_n - \mathfrak{s}\|, \end{aligned} \quad (16)$$

we find that

$$\|L(s_n, 0)\zeta_n - \mathfrak{s}\| \leq \|\zeta_n - \mathfrak{s}\| \leq \|\zeta - \mathfrak{s}\|, \quad \text{for } n \in \mathbb{N}. \quad (17)$$

Therefore,  $\{\zeta_n\}$  and  $\{L(s_n, 0)\zeta_n\}$  both are bounded. Let  $\{\zeta_{n_i}\}$  be any arbitrary subsequence of  $\{\zeta_n\}$ , then there exists a subsequence of  $\{\zeta_{n_i}\}$  (say  $\{\zeta_{n_{i_j}}\}$ ) which converges to  $x$  in a weak sense. Our claim is that  $x \in \mathcal{F}\mathbf{ix}(\mathbf{L}_s)$ .

For this, put  $\omega_j = \zeta_{n_{i_j}}$ ,  $z_j = \gamma_{n_{i_j}}$ ,  $q = [t/t_j]$ , and  $t_j = r_{n_{i_j}}$ , for  $n \in \mathbb{N}$ . Fix  $r > 0$ . One writes

$$\begin{aligned} \|\omega_j - L(s, 0)x\| &\leq \sum_{k=0}^{q-1} \|L(k+1)t_j, 0\omega_j - L(kt_j, 0)\omega_j\| \\ &\quad + \|L(qt_j, 0)\omega_j - L(qt_j, 0)x\| + \|L(qt_j, 0)x - L(s, 0)x\| \\ &\leq q \|L(t_j, 0)\omega_j - \omega_j\| + \|\omega_j - x\| + \|L(t - qt_j, 0)x - x\| \\ &= qz_j \|L(t_j, 0)\omega_j - \zeta\| + \|\omega_j - x\| + \|L(t - tq, 0)x - x\| \\ &\leq \frac{tz_j}{t_j} \|L(t_j, 0)\omega_j - \zeta\| \\ &\quad + \|\omega_j - x\| + \max\{\|L(t_j, 0)x - x\| : 0 \leq t \leq t_j\}, \end{aligned} \quad (18)$$

for  $j \in \mathbb{N}$ . In above inequality, the first and last terms tend to zero as  $j \rightarrow \infty$ , so

$$\liminf_{j \rightarrow \infty} \|\omega_j - L(s, 0)x\| \leq \liminf_{j \rightarrow \infty} \|\omega_j - x\|. \quad (19)$$

By the Opial condition and Proposition 1, we get  $L(s, 0)x = x$ , and therefore,  $x \in \mathcal{F}\mathbf{ix}(\mathbf{L}_s)$ .

Lastly, we will show that  $\{\omega_j\}$  converges to  $\mathfrak{s}$  in a strong sense. From

$$\begin{aligned} &\langle (\omega_j - L(t_j, 0)\omega_j - (\mathfrak{s} - L(t_j, 0)\mathfrak{s})), \omega_j - \mathfrak{s} \rangle \\ &\geq \|\omega_j - \mathfrak{s}\|^2 - \|L(t_j, 0)\omega_j - L(t_j, 0)\mathfrak{s}\| \cdot \|\omega_j - \mathfrak{s}\|, \\ &z_j \|\omega_j - \mathfrak{s}\|^2 + (1 - z_j) \langle (\omega_j - L(t_j, 0)\omega_j \\ &\quad - (\mathfrak{s} - L(t_j, 0)\mathfrak{s})), \omega_j - \mathfrak{s} \rangle = z_j \langle \zeta - \mathfrak{s}, \omega_j - \mathfrak{s} \rangle, \end{aligned} \quad (20)$$

we conclude that

$$z_j \|\omega_j - \mathfrak{s}\|^2 \leq z_j \langle \zeta - \mathfrak{s}, \omega_j - \mathfrak{s} \rangle. \tag{21}$$

That is,

$$\|\omega_j - \mathfrak{s}\|^2 \leq \langle \zeta - \mathfrak{s}, \omega_j - \mathfrak{s} \rangle. \tag{22}$$

Since  $\zeta$  is nearest to  $\mathfrak{s}$ , we can write  $\langle \zeta - \mathfrak{s}, x - \mathfrak{s} \rangle \leq 0$ ,

$$\begin{aligned} \|\omega_j - \mathfrak{s}\|^2 &\leq \langle \zeta - \mathfrak{s}, \omega_j - \mathfrak{s} \rangle \\ &= \langle \zeta - \mathfrak{s}, \omega_j - x \rangle + \langle \zeta - \mathfrak{s}, x - \mathfrak{s} \rangle \\ &\leq \langle \zeta - \mathfrak{s}, \omega_j - x \rangle, \end{aligned} \tag{23}$$

for  $j \in \mathbb{N}$ . We see that  $\{\omega_j\}$  converges to  $\mathfrak{s}$  in a strong sense. As  $\{\omega_j\} = \{\zeta_{n_j}\}$  is arbitrary, we obtain that  $\{\zeta_n\}$  converges to  $\mathfrak{s}$  in a strong sense.  $\square$

*Remark 2.* Here, we mention that the above result is not applicable for a discontinuous family, see [21].

*Remark 3.* If we put the condition of periodicity of every positive real number on the evolution family, then it becomes a semigroup using Remark 1. So, the results in [21] become a special case of this paper.

### 4. Example and Open Problem

*Example 2.* Let  $\mathcal{H} := L^2([0, \pi], \mathbb{C})$  be the Hilbert space and let  $\mathfrak{C} = \{\mathfrak{C}(\mathfrak{s}) : \mathfrak{s} \geq 0\}$  be a semigroup defined by

$$(\mathfrak{C}(\mathfrak{s})v)(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\mathfrak{s}n^2} c_n(v) \sin nt, \quad t \in [0, \pi], \mathfrak{s} \geq 0, \tag{24}$$

where  $c_n(v) := \int_0^\pi x(s) \sin(ns) ds$ . Clearly, it is a strongly continuous and nonexpansive semigroup on  $\mathcal{H}$ , and it is generated by the linear operator  $A$  given by  $Av = \ddot{v}$  and the maximal domain of  $A$  is the set  $D(A)$  of all  $x \in \mathcal{H}$  such that  $v$  and  $\dot{v}$  are absolutely continuous,  $\dot{v} \in \mathcal{H}$  and  $v(0) = v(\pi) = 0$ .

Now, consider the nonautonomous Cauchy problem

$$\begin{cases} \frac{\partial u(t, \xi)}{\partial t} = h(t) \frac{\partial^2 u(t, \xi)}{\partial \xi^2}, & t > 0, \xi \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, & t \geq 0, \\ u(0, \xi) = z(\xi), \end{cases} \tag{25}$$

where  $z(\cdot) \in \mathcal{H}$ , and the function  $h: \mathbb{R}_+ \rightarrow [1, \infty)$  is nonexpansive on  $\mathbb{R}_+$  and obeys the periodicity condition, i.e.,  $h(t+q) = h(t)$  for all  $t \in \mathbb{R}_+$  for some  $q \geq 1$ .

Let  $H(t) = \int_0^t h(s) ds$ . It is obvious that the solution  $x(\cdot)$  of the above Cauchy problem will satisfy the evolution property:

$$x(t) = L(t, s)x(s), \tag{26}$$

where  $L(t, s) = \mathfrak{C}(H(t) - H(s))$ . See [22, Example 2.9b].

We can find  $\nu \geq 0$  such that the function  $t \mapsto e^{\nu t} \|u(t)\|$  is bounded on  $\mathbb{R}_+$ . In fact, we have

$$\begin{aligned} \int_0^\infty \|L(t, 0)v\|^2 dt &= \frac{2}{\pi} \int_0^\infty \sum_{\nu=1}^\infty c_\nu^2(v) e^{-2\nu^2 H(t)} dt \\ &= \frac{2}{\pi} \sum_{\nu=1}^\infty c_\nu^2(v) \int_0^\infty e^{-2\nu^2 H(t)} dt \\ &= \|v\|_2^2 \int_0^\infty e^{-2\nu^2 H(t)} dt \\ &\leq \|v\|_2^2 \int_0^\infty e^{-2H(t)} dt. \end{aligned} \tag{27}$$

On the other hand,

$$\begin{aligned} \int_0^\infty e^{-2H(t)} dt &= \sum_{j=0}^\infty \int_{jq}^{(j+1)q} e^{-2H(t)} dt \\ &= \sum_{j=0}^\infty \int_0^q e^{-2H(jq+a)} da \\ &= \sum_{j=0}^\infty e^{-2jH(q)} \int_0^q e^{-2H(a)} da \\ &\leq q \sum_{j=0}^\infty e^{-2jH(q)} = \frac{qe^{2H(q)}}{e^{2H(q)} - 1} := C. \end{aligned} \tag{28}$$

Hence,

$$\int_0^\infty \|L(t, 0)v\|^2 dt \leq C \|v\|_2^2. \tag{29}$$

Using Theorem 3.2 in [27], we have  $\omega_0(L) \leq -1/2M$ , where  $M \geq 1$  and  $\omega_0(L)$  is the growth bound of the family  $L$ , and see [27] for further details. This shows that the evolution family is nonexpansive on  $\mathcal{H}$ , so Theorem 3 can be applicable for such a family and can help for the uniqueness and existence of a solution for the above system.

Open problem: we leave open the question whether Lemma 2 and Theorem 3 can be generalized for the whole periodic and then for general evolution families?

### 5. Conclusion

The idea of an evolution family is more general than the semigroups. In [21], Suzuki proved a strong convergence to a fixed point of a nonexpansive semigroup of operators on a Hilbert space. In this paper, we generalized the results to a subfamily of an evolution family which is not a semigroup. These results can open the way for researchers to prove such convergence for the whole evolution family of operators on a Hilbert space.

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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