Research Article

On the Solutions of Three-Dimensional Rational Difference Equation Systems

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In this paper, we are interested in a technique for solving some nonlinear rational systems of difference equations of third order, in three-dimensional case as a special case of the following system: $x_{n+1} = (y_n z_{n-1} / y_n \pm x_{n-2})$, $y_{n+1} = (z_n x_{n-1} / z_n \pm y_{n-2})$, and $z_{n+1} = (x_n y_{n-1} / x_n \pm z_{n-2})$ with initial conditions x_{-2} , x_{-1} , x_0 , y_{-2} , y_{-1} , y_0 , z_{-2} , z_{-1} , and z_0 are nonzero real numbers. Moreover, we study some behavior of the systems such as the boundedness of solutions for such systems. Finally, we present some numerical examples by giving some numerical values for the initial values of each case. Some figures have been given to explain the behavior of the obtained solutions in the case of numerical examples by using the mathematical program MATLAB to confirm the obtained results.

1. Introduction

We believe that difference equations, also referred to as recursive sequence, are a hot topic here as there has been increasing interest in the study of qualitative analysis of difference equations and systems of difference equations. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economics, physics, computer sciences, and so on. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solution (see [1–9] and the references cited therein). Recently, a great effort has been made in studying the qualitative analysis of rational difference equations and rational difference system (see [10–35]).

The study of the nonlinear rational difference equations is quite challenging and rewarding [2, 8]. The results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations [1]. Recently, many researchers have investigated the behavior of the solution of difference

equations. Difference equations arise in the situations in which the discrete values of the independent variable involve. Many practical phenomena are modeled with the help of difference equations [1]. In engineering, difference equations arise in control engineering, digital signal processing, electrical networks, etc. In social sciences, difference equations arise to study the national income of a country and then its variation with time, Cobweb phenomenon in economics, etc.

There are many papers related to the difference equation system, for example, the periodicity of the positive solutions of the rational difference equations system:

$$\begin{aligned}x_{n+1} &= \frac{1}{z_n}, \\y_{n+1} &= \frac{y_n}{x_{n-1}y_{n-1}}, \\z_{n+1} &= \frac{1}{x_{n-1}},\end{aligned}\tag{1}$$

has been obtained by Cinar in [4].

Khan et al. [6] studied the equilibrium points, local asymptotic stability of an equilibrium point, instability of equilibrium points, periodicity behavior of positive solutions, and global character of an equilibrium point of a fourth-order system of rational difference equations of the form:

$$\begin{aligned}x_{n+1} &= \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \\y_{n+1} &= \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}.\end{aligned}\quad (2)$$

Elabbasy et al. [7] has obtained the solution of particular cases of the following general system of difference equations:

$$\begin{aligned}x_{n+1} &= \frac{a_1 + a_2 y_n}{a_3 z_n + a_4 x_{n-1} z_n}, \\y_{n+1} &= \frac{b_1 z_{n-1} + b_2 z_n}{b_3 x_n y_n + b_4 x_n y_{n-1}}, \\z_{n+1} &= \frac{c_1 z_{n-1} + c_2 z_n}{c_3 x_{n-1} y_{n-1} + c_4 x_{n-1} y_n + c_5 x_n y_n}.\end{aligned}\quad (3)$$

In [36], Elsayed et al. dealt with the solutions of the systems of the difference equations:

$$\begin{aligned}x_{n+1} &= \frac{1}{x_{n-p} y_{n-p}}, \\y_{n+1} &= \frac{x_{n-p} y_{n-p}}{x_{n-q} y_{n-q}}, \\x_{n+1} &= \frac{1}{x_{n-p} y_{n-p} z_{n-p}}, \\y_{n+1} &= \frac{x_{n-p} y_{n-p} z_{n-p}}{x_{n-q} y_{n-q} z_{n-q}}, \\z_{n+1} &= \frac{x_{n-q} y_{n-q} z_{n-q}}{x_{n-r} y_{n-r} z_{n-r}}.\end{aligned}\quad (4)$$

Kurbanli [13–15] investigated the behavior of the solutions of the difference equation systems:

$$\begin{aligned}x_{n+1} &= \frac{x_{n-1}}{x_{n-1} y_n - 1}, \\y_{n+1} &= \frac{y_{n-1}}{y_{n-1} x_n - 1}, \\z_{n+1} &= \frac{1}{z_n y_n}, \\x_{n+1} &= \frac{x_{n-1}}{x_{n-1} y_n - 1}, \\y_{n+1} &= \frac{y_{n-1}}{y_{n-1} x_n - 1}, \\z_{n+1} &= \frac{z_{n-1}}{z_{n-1} y_n - 1}, \\x_{n+1} &= \frac{x_{n-1}}{x_{n-1} y_n - 1}, \\y_{n+1} &= \frac{y_{n-1}}{y_{n-1} x_n - 1}, \\z_{n+1} &= \frac{x_n}{z_{n-1} y_n}.\end{aligned}\quad (5)$$

In [21], Yalcinkaya et al. studied the periodic character of the following two systems of difference equations:

$$\begin{aligned}x_{n+1}^{(1)} &= \frac{x_n^{(2)}}{x_n^{(2)} - 1}, x_{n+1}^{(2)} = \frac{x_n^{(3)}}{x_n^{(3)} - 1}, \dots, x_{n+1}^{(k)} = \frac{x_n^{(1)}}{x_n^{(1)} - 1}, \\x_{n+1}^{(1)} &= \frac{x_n^{(k)}}{x_n^{(k)} - 1}, x_{n+1}^{(2)} = \frac{x_n^{(1)}}{x_n^{(1)} - 1}, \dots, x_{n+1}^{(k)} = \frac{x_n^{(k-1)}}{x_n^{(k-1)} - 1},\end{aligned}\quad (6)$$

where the initial values are nonzero real numbers for $x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(k)} \neq 1$.

In [37], Zhang et al. studied the boundedness, the persistence, and global asymptotic stability of the positive solutions of the system of difference equations:

$$\begin{aligned}x_n &= A + \frac{1}{y_{n-p}}, \\y_n &= A + \frac{y_{n-1}}{x_{n-r} y_{n-s}}.\end{aligned}\quad (7)$$

Zkan and Kurbanli [38] have investigated the periodical solutions of the following system of third-order rational difference equations:

$$\begin{aligned}
 x_{n+1} &= \frac{y_{n-2}}{-1 \pm y_{n-2}x_{n-1}y_n}, \\
 y_{n+1} &= \frac{x_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_n}, \\
 z_{n+1} &= \frac{x_{n-2} + y_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_n}.
 \end{aligned} \tag{8}$$

Similar to the references above, this paper is devoted to obtain the form of the solutions of the following third-order systems of rational difference equations:

$$\begin{aligned}
 x_{n+1} &= \frac{y_n z_{n-1}}{y_n \pm x_{n-2}}, \\
 y_{n+1} &= \frac{z_n x_{n-1}}{z_n \pm y_{n-2}}, \\
 z_{n+1} &= \frac{x_n y_{n-1}}{x_n \pm z_{n-2}}.
 \end{aligned} \tag{9}$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}$, and z_0 are nonzero real numbers.

2. The System:

$$\begin{aligned}
 x_{n+1} &= (y_n z_{n-1} / y_n + x_{n-2}), \quad y_{n+1} = (z_n x_{n-1} / z_n + y_{n-2}), \quad \text{and} \quad z_{n+1} = (x_n y_{n-1} / x_n + z_{n-2})
 \end{aligned}$$

We obtain the form of solutions of the following system of difference equations:

$$\begin{aligned}
 x_{n+1} &= \frac{y_n z_{n-1}}{y_n + x_{n-2}}, \\
 y_{n+1} &= \frac{z_n x_{n-1}}{z_n + y_{n-2}}, \\
 z_{n+1} &= \frac{x_n y_{n-1}}{x_n + z_{n-2}}.
 \end{aligned} \tag{10}$$

Theorem 1. *If $\{x_n, y_n, z_n\}$ be the solution of (10), then*

$$\begin{aligned}
 x_{6n-2} &= \frac{x_{-2} y_0^{3n}}{\prod_{i=0}^{n-1} (x_{-2} + (6i) y_0) (x_{-2} + (6i+2) y_0) (x_{-2} + (6i+4) y_0)}, \\
 x_{6n-1} &= \frac{x_{-1} z_0^{3n}}{\prod_{i=0}^{n-1} (y_{-2} + (6i+1) z_0) (y_{-2} + (6i+3) z_0) (y_{-2} + (6i+5) z_0)}, \\
 x_{6n} &= \frac{x_0^{3n+1}}{\prod_{i=0}^{n-1} (z_{-2} + (6i+2) x_0) (z_{-2} + (6i+4) x_0) (z_{-2} + (6i+6) x_0)}, \\
 x_{6n+1} &= \frac{z_{-1} y_0^{3n+1}}{(x_{-2} + y_0) \prod_{i=0}^{n-1} (x_{-2} + (6i+3) y_0) (x_{-2} + (6i+5) y_0) (x_{-2} + (6i+7) y_0)}, \\
 x_{6n+2} &= \frac{z_0^{3n+2}}{(y_{-2} + 2z_0) \prod_{i=0}^{n-1} (y_{-2} + (6i+4) z_0) (y_{-2} + (6i+6) z_0) (y_{-2} + (6i+8) z_0)}, \\
 x_{6n+3} &= \frac{y_{-1} x_0^{3n+2}}{(z_{-2} + x_0) (z_{-2} + 3x_0) \prod_{i=0}^{n-1} (z_{-2} + (6i+5) x_0) (z_{-2} + (6i+7) x_0) (z_{-2} + (6i+9) x_0)}, \\
 y_{6n-2} &= \frac{y_{-2} z_0^{3n}}{\prod_{i=0}^{n-1} (y_{-2} + (6i) z_0) (y_{-2} + (6i+2) z_0) (y_{-2} + (6i+4) z_0)}, \\
 y_{6n-1} &= \frac{y_{-1} x_0^{3n}}{\prod_{i=0}^{n-1} (z_{-2} + (6i+1) x_0) (z_{-2} + (6i+3) x_0) (z_{-2} + (6i+5) x_0)}, \\
 y_{6n} &= \frac{y_0^{3n+1}}{\prod_{i=0}^{n-1} (x_{-2} + (6i+2) y_0) (x_{-2} + (6i+4) y_0) (x_{-2} + (6i+6) y_0)}, \\
 y_{6n+1} &= \frac{z_{-1} z_0^{3n+1}}{(y_{-2} + z_0) \prod_{i=0}^{n-1} (y_{-2} + (6i+3) z_0) (y_{-2} + (6i+5) z_0) (y_{-2} + (6i+7) z_0)}, \\
 y_{6n+2} &= \frac{x_0^{3n+2}}{(z_{-2} + 2x_0) \prod_{i=0}^{n-1} (z_{-2} + (6i+4) x_0) (z_{-2} + (6i+6) x_0) (z_{-2} + (6i+8) x_0)}.
 \end{aligned}$$

$$\begin{aligned}
y_{6n+3} &= \frac{z_{-1}y_0^{3n+2}}{(x_{-2} + y_0)(x_{-2} + 3y_0)\prod_{i=0}^{n-1}(x_{-2} + (6i+5)y_0)(x_{-2} + (6i+7)y_0)(x_{-2} + (6i+9)y_0)}, \\
z_{6n-2} &= \frac{z_{-2}x_0^{3n}}{\prod_{i=0}^{n-1}(z_{-2} + (6i)x_0)(z_{-2} + (6i+2)x_0)(z_{-2} + (6i+4)x_0)}, \\
z_{6n-1} &= \frac{z_{-1}y_0^{3n}}{\prod_{i=0}^{n-1}(x_{-2} + (6i+1)y_0)(x_{-2} + (6i+3)y_0)(x_{-2} + (6i+5)y_0)}, \\
z_{6n} &= \frac{z_0^{3n+1}}{\prod_{i=0}^{n-1}(y_{-2} + (6i+2)z_0)(y_{-2} + (6i+4)z_0)(y_{-2} + (6i+6)z_0)}, \\
z_{6n+1} &= \frac{y_{-1}x_0^{3n+1}}{(z_{-2} + x_0)\prod_{i=0}^{n-1}(z_{-2} + (6i+3)x_0)(z_{-2} + (6i+5)x_0)(z_{-2} + (6i+7)x_0)}, \\
z_{6n+2} &= \frac{y_0^{3n+2}}{(x_{-2} + 2y_0)\prod_{i=0}^{n-1}(x_{-2} + (6i+4)y_0)(x_{-2} + (6i+6)y_0)(x_{-2} + (6i+8)y_0)}, \\
z_{6n+3} &= \frac{x_{-1}z_0^{3n+2}}{(y_{-2} + z_0)(y_{-2} + 3z_0)\prod_{i=0}^{n-1}(y_{-2} + (6i+5)z_0)(y_{-2} + (6i+7)z_0)(y_{-2} + (6i+9)z_0)}.
\end{aligned} \tag{11}$$

Proof. Obviously, results are true for $n = 0$. Suppose that it is also true for $n - 1$, i.e.,

$$\begin{aligned}
x_{6n-8} &= \frac{x_{-2}y_0^{3n-3}}{\prod_{i=0}^{n-2}(x_{-2} + (6i)y_0)(x_{-2} + (6i+2)y_0)(x_{-2} + (6i+4)y_0)}, \\
x_{6n-7} &= \frac{x_{-1}z_0^{3n-3}}{\prod_{i=0}^{n-2}(y_{-2} + (6i+1)z_0)(y_{-2} + (6i+3)z_0)(y_{-2} + (6i+5)z_0)}, \\
x_{6n-6} &= \frac{x_0^{3n-2}}{\prod_{i=0}^{n-2}(z_{-2} + (6i+2)x_0)(z_{-2} + (6i+4)x_0)(z_{-2} + (6i+6)x_0)}, \\
x_{6n-5} &= \frac{z_{-1}y_0^{3n-2}}{(x_{-2} + y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+3)y_0)(x_{-2} + (6i+5)y_0)(x_{-2} + (6i+7)y_0)}, \\
x_{6n-4} &= \frac{z_0^{3n-1}}{(y_{-2} + 2z_0)\prod_{i=0}^{n-2}(y_{-2} + (6i+4)z_0)(y_{-2} + (6i+6)z_0)(y_{-2} + (6i+8)z_0)}, \\
x_{6n-3} &= \frac{y_{-1}x_0^{3n-1}}{(z_{-2} + x_0)(z_{-2} + 3x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+5)x_0)(z_{-2} + (6i+7)x_0)(z_{-2} + (6i+9)x_0)}, \\
y_{6n-8} &= \frac{y_{-2}z_0^{3n-3}}{\prod_{i=0}^{n-2}(y_{-2} + (6i)z_0)(y_{-2} + (6i+2)z_0)(y_{-2} + (6i+4)z_0)}, \\
y_{6n-7} &= \frac{y_{-1}x_0^{3n-3}}{\prod_{i=0}^{n-2}(z_{-2} + (6i+1)x_0)(z_{-2} + (6i+3)x_0)(z_{-2} + (6i+5)x_0)}, \\
y_{6n-6} &= \frac{y_0^{3n-2}}{\prod_{i=0}^{n-2}(x_{-2} + (6i+2)y_0)(x_{-2} + (6i+4)y_0)(x_{-2} + (6i+6)y_0)},
\end{aligned}$$

$$\begin{aligned}
y_{6n-5} &= \frac{x_{-1}z_0^{3n-2}}{(y_{-2} + z_0)\prod_{i=0}^{n-2}(y_{-2} + (6i+3)z_0)(y_{-2} + (6i+5)z_0)(y_{-2} + (6i+7)z_0)}, \\
y_{6n-4} &= \frac{x_0^{3n-1}}{(z_{-2} + 2x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+4)x_0)(z_{-2} + (6i+6)x_0)(z_{-2} + (6i+8)x_0)}, \\
y_{6n-3} &= \frac{z_{-1}y_0^{3n-1}}{(x_{-2} + y_0)(x_{-2} + 3y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+5)y_0)(x_{-2} + (6i+7)y_0)(x_{-2} + (6i+9)y_0)}, \\
z_{6n-8} &= \frac{z_{-2}x_0^{3n-3}}{\prod_{i=0}^{n-2}(z_{-2} + (6i)x_0)(z_{-2} + (6i+2)x_0)(z_{-2} + (6i+4)x_0)}, \\
z_{6n-7} &= \frac{z_{-1}y_0^{3n-3}}{\prod_{i=0}^{n-2}(x_{-2} + (6i+1)y_0)(x_{-2} + (6i+3)y_0)(x_{-2} + (6i+5)y_0)}, \\
z_{6n-6} &= \frac{z_0^{3n-2}}{\prod_{i=0}^{n-2}(y_{-2} + (6i+2)z_0)(y_{-2} + (6i+4)z_0)(y_{-2} + (6i+6)z_0)}, \\
z_{6n-5} &= \frac{y_{-1}x_0^{3n-2}}{(z_{-2} + x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+3)x_0)(z_{-2} + (6i+5)x_0)(z_{-2} + (6i+7)x_0)}, \\
z_{6n-4} &= \frac{y_0^{3n-1}}{(x_{-2} + 2y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+4)y_0)(x_{-2} + (6i+6)y_0)(x_{-2} + (6i+8)y_0)}, \\
z_{6n-3} &= \frac{x_{-1}z_0^{3n-1}}{(y_{-2} + z_0)(y_{-2} + 3z_0)\prod_{i=0}^{n-2}(y_{-2} + (6i+5)z_0)(y_{-2} + (6i+7)z_0)(y_{-2} + (6i+9)z_0)}.
\end{aligned} \tag{12}$$

From equation (10), it follows that

$$\begin{aligned}
x_{6n-2} &= \frac{z_{6n-4}y_{6n-3}}{y_{6n-3} + x_{6n-5}} \\
&= \frac{(y_0^{3n-1}/(x_{-2} + 2y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+4)y_0)(x_{-2} + (6i+6)y_0)(x_{-2} + (6i+8)y_0))}{(z_{-1}y_0^{3n-1}/(x_{-2} + y_0)(x_{-2} + 3y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+5)y_0)(x_{-2} + (6i+7)y_0)(x_{-2} + (6i+9)y_0)) +} \\
&\quad (z_{-1}y_0^{3n-2}/(x_{-2} + y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+3)y_0)(x_{-2} + (6i+5)y_0)(x_{-2} + (6i+7)y_0)) \\
&= \frac{(y_0^{3n}/(x_{-2} + 2y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+4)y_0)(x_{-2} + (6i+6)y_0)(x_{-2} + (6i+8)y_0))}{y_0 + (x_{-2} + 3y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+9)y_0)/\prod_{i=0}^{n-2}(x_{-2} + (6i+3)y_0)} \\
&= \frac{(y_0^{3n}/(x_{-2} + 2y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+4)y_0)(x_{-2} + (6i+6)y_0)(x_{-2} + (6i+8)y_0))}{y_0 + (x_{-2} + (6n-3)y_0)} \\
&= \frac{y_0^{3n}}{(x_{-2} + 2y_0)\prod_{i=0}^{n-2}(x_{-2} + (6i+4)y_0)(x_{-2} + (6i+6)y_0)(x_{-2} + (6i+8)y_0)} \left(\frac{1}{x_{-2} + (6n-2)y_0} \right).
\end{aligned} \tag{13}$$

Now, one can see that

$$x_{6n-2} = \frac{x_{-2} y_0^{3n}}{\prod_{i=0}^{n-1} (x_{-2} + (6i)y_0)(x_{-2} + (6i+2)y_0)(x_{-2} + (6i+4)y_0)}. \quad (14)$$

Moreover, from (10), one has

$$\begin{aligned} y_{6n-2} &= \frac{x_{6n-4} z_{6n-3}}{z_{6n-3} + y_{6n-5}} \\ &= \frac{(z_0^{3n-1} / (y_{-2} + 2z_0) \prod_{i=0}^{n-2} (y_{-2} + (6i+4)z_0)(y_{-2} + (6i+6)z_0)(y_{-2} + (6i+8)z_0))}{(x_{-1} z_0^{3n-1} / (y_{-2} + z_0)(y_{-2} + 3z_0) \prod_{i=0}^{n-2} (y_{-2} + (6i+5)z_0)(y_{-2} + (6i+7)z_0)(y_{-2} + (6i+9)z_0)) +} \\ &\quad (x_{-1} z_0^{3n-2} / (y_{-2} + z_0) \prod_{i=0}^{n-2} (y_{-2} + (6i+3)z_0)(y_{-2} + (6i+5)z_0)(y_{-2} + (6i+7)z_0)) \\ &= \frac{(z_0^{3n} / (y_{-2} + 2z_0) \prod_{i=0}^{n-2} (y_{-2} + (6i+4)z_0)(y_{-2} + (6i+6)z_0)(y_{-2} + (6i+8)z_0))}{z_0 + ((y_{-2} + 3z_0) \prod_{i=0}^{n-2} (y_{-2} + (6i+9)z_0) / \prod_{i=0}^{n-2} (y_{-2} + (6i+3)z_0))} \\ &= \frac{(z_0^{3n} / (y_{-2} + 2z_0) \prod_{i=0}^{n-2} (y_{-2} + (6i+4)z_0)(y_{-2} + (6i+6)z_0)(y_{-2} + (6i+8)z_0))}{z_0 + (y_{-2} + (6n-3)z_0)} \\ &= \frac{(z_0^{3n} / (y_{-2} + 2z_0) \prod_{i=0}^{n-2} (y_{-2} + (6i+4)z_0)(y_{-2} + (6i+6)z_0)(y_{-2} + (6i+8)z_0))}{(y_{-2} + (6n-2)z_0)}. \end{aligned} \quad (15)$$

Then,

$$y_{6n-2} = \frac{y_{-2} z_0^{3n}}{(y_{-2} + 2z_0) \prod_{i=0}^{n-1} (y_{-2} + (6i)z_0)(y_{-2} + (6i+2)z_0)(y_{-2} + (6i+4)z_0)}. \quad (16)$$

Finally, from equation (10),

$$\begin{aligned}
 z_{6n-2} &= \frac{y_{6n-4}x_{6n-3}}{x_{6n-3} + z_{6n-5}} \\
 &= \frac{(x_0^{3n-1}/(z_{-2} + 2x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+4)x_0)(z_{-2} + (6i+6)x_0)(z_{-2} + (6i+8)x_0))}{(y_{-1}x_0^{3n-1}/(z_{-2} + x_0)(z_{-2} + 3x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+5)x_0)(z_{-2} + (6i+7)x_0)(z_{-2} + (6i+9)x_0))} \\
 &= \frac{(y_{-1}x_0^{3n-2}/(z_{-2} + x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+3)x_0)(z_{-2} + (6i+5)x_0)(z_{-2} + (6i+7)x_0))}{(x_0^{3n}/(z_{-2} + 2x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+4)x_0)(z_{-2} + (6i+6)x_0)(z_{-2} + (6i+8)x_0))} \\
 &= \frac{x_0 + ((z_{-2} + 3x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+9)x_0)/\prod_{i=0}^{n-2}(z_{-2} + (6i+3)x_0))}{x_0 + ((z_{-2} + 3x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+9)x_0)/\prod_{i=0}^{n-2}(z_{-2} + (6i+3)x_0))} \\
 &= \frac{(x_0^{3n}/(z_{-2} + 2x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+4)x_0)(z_{-2} + (6i+6)x_0)(z_{-2} + (6i+8)x_0))}{x_0 + ((z_{-2} + 3x_0)\prod_{i=0}^{n-2}(z_{-2} + (6i+9)x_0)/\prod_{i=0}^{n-2}(z_{-2} + (6i+3)x_0))}.
 \end{aligned} \tag{17}$$

Thus,

$$z_{3n-2} = \frac{z_{-2}x_0^{3n}}{\prod_{i=0}^{n-1}(z_{-2} + (6i)x_0)(z_{-2} + (6i+2)x_0)(z_{-2} + (6i+4)x_0)}.$$

In a similar way, other relations can also be proved. $\square$

Lemma 1. If $\{x_n, y_n, z_n\}$ is the +ve solution of (10), then it is bounded as well as converge to zero.

Proof. From (10),

$$\begin{aligned}
 x_{n+1} &= \frac{y_n z_{n-1}}{y_n + x_{n-2}} \leq z_{n-1}, \\
 y_{n+1} &= \frac{z_n x_{n-1}}{z_n + y_{n-2}} \leq x_{n-1}, \\
 z_{n+1} &= \frac{x_n y_{n-1}}{x_n + z_{n-2}} \leq y_{n-1},
 \end{aligned} \tag{19}$$

we see that

$$\begin{aligned}
 x_{n+4} &\leq z_{n+2}, z_{n+2} \leq y_n, y_n \leq x_{n-2}, \Rightarrow x_{n+4} < x_{n-2}, \\
 y_{n+4} &\leq x_{n+2}, x_{n+2} \leq z_n, z_n \leq y_{n-2}, \Rightarrow y_{n+4} < y_{n-2}, \\
 z_{n+4} &\leq y_{n+2}, y_{n+2} \leq x_n, x_n \leq z_{n-2}, \Rightarrow z_{n+4} < z_{n-2}.
 \end{aligned} \tag{20}$$

Therefore, subsequences $\{x_{6n-2}\}_{n=0}^{\infty}, \dots, \{x_{6n+3}\}_{n=0}^{\infty}$ are nonincreasing and so are bounded from above by $M = \max\{x_{-2}, x_{-1}, x_0, x_1, x_2, x_3\}$. Similarly, the subsequences $\{y_{6n-2}\}_{n=0}^{\infty}, \dots, \{y_{6n+3}\}_{n=0}^{\infty}$ are nonincreasing and hence bounded above by $M = \max\{y_{-2}, y_{-1}, y_0, y_1, y_2, y_3\}$ and $\{z_{6n-2}\}_{n=0}^{\infty}, \dots, \{z_{6n+3}\}_{n=0}^{\infty}$ are nonincreasing and hence

bounded from above by $M = \max\{z_{-2}, z_{-1}, z_0, z_1, z_2, z_3\}$. $\square$

Example 1. Figure 1 represents the dynamics of (10) with $x_{-2} = 0.18, x_{-1} = 0.15, x_0 = 0.13, y_{-2} = 0.26, y_{-1} = 0.17, y_0 = -0.13, z_{-2} = 0.14, z_{-1} = 0.19$, and $z_0 = 0.15$.

3. The System:

$$x_{n+1} = (y_n z_{n-1}/y_n + x_{n-2}), y_{n+1} = (z_n x_{n-1}/z_n + y_{n-2}), \text{ and } z_{n+1} = (x_n y_{n-1}/x_n - z_{n-2})$$

Here, we will discuss solutions of the following system:

$$\begin{aligned}
 x_{n+1} &= \frac{y_n z_{n-1}}{y_n + x_{n-2}}, \\
 y_{n+1} &= \frac{z_n x_{n-1}}{z_n + y_{n-2}}, \\
 z_{n+1} &= \frac{x_n y_{n-1}}{x_n - z_{n-2}},
 \end{aligned} \tag{21}$$

where $n \in \mathbb{N}_0$ and nonzero initial conditions such that $x_{-2} \neq \pm y_0, x_{-2} \neq 2y_0, y_{-2} \neq \pm z_0, y_{-2} \neq -2z_0, z_{-2} \neq x_0, z_{-2} \neq 2x_0$, and $z_{-2} \neq 3x_0$.

Theorem 2. If $\{x_n, y_n, z_n\}$ are solutions of (21), then solutions of (21) are represented by the following formulas for $n = 0, 1, \dots$:

FIGURE 1: Difference system (10).

$$\begin{aligned}
 x_{6n-2} &= \frac{y_0^{3n}}{x_{-2}^{2n-1}(x_{-2} - 2y_0)^n}, \\
 x_{6n-1} &= \frac{x_{-1}z_0^{3n}}{(y_{-2} + z_0)^{2n}(y_{-2} - z_0)^n}, \\
 x_{6n} &= \frac{(-1)^n x_0^{3n+1}}{z_{-2}^n(z_{-2} - 2x_0)^{2n}}, \\
 x_{6n+1} &= \frac{(-1)^n z_{-1}y_0^{3n+1}}{(x_{-2} + y_0)^{n+1}(x_{-2} - y_0)^{2n}}, \\
 x_{6n+2} &= \frac{(-1)^n z_0^{3n+2}}{y_{-2}^{2n}(y_{-2} + 2z_0)^{n+1}}, \\
 x_{6n+3} &= \frac{y_{-1}x_0^{3n+2}}{(z_{-2} - x_0)^{2n+1}(z_{-2} - 3x_0)^{n+1}}, \\
 y_{6n-2} &= \frac{(-1)^n z_0^{3n}}{y_{-2}^{2n-1}(y_{-2} + 2z_0)^n}, \\
 y_{6n-1} &= \frac{y_{-1}x_0^{3n}}{(x_0 - z_{-2})^{2n}(z_{-2} - 3x_0)^n}, \\
 y_{6n} &= \frac{y_0^{3n+1}}{x_{-2}^{2n}(x_{-2} - 2y_0)^n}, \\
 y_{6n+1} &= \frac{x_{-1}z_0^{3n+1}}{(y_{-2} + z_0)^{2n+1}(y_{-2} - z_0)^n}, \\
 y_{6n+2} &= \frac{(-1)^{n+1} x_0^{3n+2}}{z_{-2}^n(z_{-2} - 2x_0)^{2n+1}}, \\
 y_{6n+3} &= \frac{(-1)^{n+1} z_{-1}y_0^{3n+2}}{(x_{-2} - y_0)^{2n+1}(x_{-2} + y_0)^{n+1}}, \\
 z_{6n-2} &= \frac{(-1)^n x_0^{3n}}{z_{-2}^{n-1}(z_{-2} - 2x_0)^{2n}}, \\
 z_{6n-1} &= \frac{(-1)^n z_{-1}y_0^{3n}}{(x_{-2} - y_0)^{2n}(x_{-2} + y_0)^n}, \\
 z_{6n} &= \frac{(-1)^n z_0^{3n+1}}{y_{-2}^{2n}(y_{-2} + 2z_0)^n}, \\
 z_{6n+1} &= \frac{-y_{-1}x_0^{3n+1}}{(z_{-2} - x_0)^{2n+1}(z_{-2} - 3x_0)^n}, \\
 z_{6n+2} &= \frac{-y_0^{3n+2}}{x_{-2}^{2n+1}(x_{-2} - 2y_0)^n}, \\
 z_{6n+3} &= \frac{-x_{-1}z_0^{3n+2}}{(y_{-2} - z_0)^n(y_{-2} + z_0)^{2n+2}}.
 \end{aligned} \tag{22}$$

Proof. Obviously, results are true for $n = 0$. Now, one suppose that, for $n - 1$, it is also true, i.e.,

$$\begin{aligned}
x_{6n-8} &= \frac{y_0^{3n-3}}{x_{-2}^{2n-3} (x_{-2} - 2y_0)^{n-1}}, \\
x_{6n-7} &= \frac{x_{-1}z_0^{3n-3}}{(y_{-2} + z_0)^{2n-2} (y_{-2} - z_0)^{n-1}}, \\
x_{6n-6} &= \frac{(-1)^{n-1} x_0^{3n-2}}{z_{-2}^{n-1} (z_{-2} - 2x_0)^{2n-2}}, \\
x_{6n-5} &= \frac{(-1)^{n-1} z_{-1} y_0^{3n-2}}{(x_{-2} + y_0)^n (x_{-2} - y_0)^{2n-2}}, \\
x_{6n-4} &= \frac{(-1)^{n-1} z_0^{3n-1}}{y_{-2}^{2n-2} (y_{-2} + 2z_0)^n}, \\
x_{6n-3} &= \frac{y_{-1} x_0^{3n-1}}{(z_{-2} - x_0)^{2n-1} (z_{-2} - 3x_0)^n}, \\
y_{6n-8} &= \frac{(-1)^{n-1} z_0^{3n-3}}{y_{-2}^{2n-3} (y_{-2} + 2z_0)^{n-1}}, \\
y_{6n-7} &= \frac{y_{-1} x_0^{3n-3}}{(x_0 - z_{-2})^{2n-2} (z_{-2} - 3x_0)^{n-1}}, \\
y_{6n-6} &= \frac{y_0^{3n-2}}{x_{-2}^{2n-2} (x_{-2} - 2y_0)^{n-1}}, \\
y_{6n-5} &= \frac{x_{-1} z_0^{3n-2}}{(y_{-2} + z_0)^{2n-1} (y_{-2} - z_0)^{n-1}}, \\
y_{6n-4} &= \frac{(-1)^n x_0^{3n-1}}{z_{-2}^{n-1} (z_{-2} - 2x_0)^{2n-1}}, \\
y_{6n-3} &= \frac{(-1)^n z_{-1} y_0^{3n-1}}{(x_{-2} - y_0)^{2n-1} (x_{-2} + y_0)^n}, \\
z_{6n-8} &= \frac{(-1)^{n-1} x_0^{3n-3}}{z_{-2}^{n-2} (z_{-2} - 2x_0)^{2n-2}}, \\
z_{6n-7} &= \frac{(-1)^{n-1} z_{-1} y_0^{3n-3}}{(x_{-2} - y_0)^{2n-2} (x_{-2} + y_0)^{n-1}}, \\
z_{6n-6} &= \frac{(-1)^{n-1} z_0^{3n-2}}{y_{-2}^{2n-2} (y_{-2} + 2z_0)^{n-1}}, \\
z_{6n-5} &= \frac{-y_{-1} x_0^{3n-2}}{(z_{-2} - x_0)^{2n-1} (z_{-2} - 3x_0)^{n-1}}, \\
z_{6n-4} &= \frac{-y_0^{3n-1}}{x_{-2}^{2n-1} (x_{-2} - 2y_0)^{n-1}}, \\
z_{6n-3} &= \frac{-x_{-1} z_0^{3n-1}}{(y_{-2} - z_0)^{n-1} (y_{-2} + z_0)^{2n}}.
\end{aligned} \tag{23}$$

Now, from equation (21), it follows that

$$\begin{aligned}
 x_{6n-2} &= \frac{z_{6n-4}y_{6n-3}}{y_{6n-3} + x_{6n-5}} \\
 &= \frac{(-y_0^{3n-1}/x_{-2}^{2n-1}(x_{-2} - 2y_0)^{n-1})((-1)^n z_{-1}y_0^{3n-1}/(x_{-2} - y_0)^{2n-1}(x_{-2} + y_0)^n)}{((-1)^n z_{-1}y_0^{3n-1}/(x_{-2} - y_0)^{2n-1}(x_{-2} + y_0)^n) + ((-1)^{n-1} z_{-1}y_0^{3n-2}/(x_{-2} + y_0)^n(x_{-2} - y_0)^{2n-2})} \\
 &= \frac{(-y_0^{3n-1}/x_{-2}^{2n-1}(x_{-2} - 2y_0)^{n-1})y_0}{y_0 - (x_{-2} - y_0)} = \frac{y_0^{3n}}{x_{-2}^{2n-1}(x_{-2} - 2y_0)^n}, \\
 y_{6n-1} &= \frac{x_{6n-3}z_{6n-2}}{z_{6n-2} + y_{6n-4}} = \frac{(y_{-1}x_0^{3n-1}/(z_{-2} - x_0)^{2n-1}(z_{-2} - 3x_0)^n)((-1)^n x_0^{3n}/z_{-2}^{n-1}(z_{-2} - 2x_0)^{2n})}{((-1)^n x_0^{3n}/z_{-2}^{n-1}(z_{-2} - 2x_0)^{2n}) + ((-1)^n x_0^{3n-1}/z_{-2}^{n-1}(z_{-2} - 2x_0)^{2n-1})} \\
 &= \frac{(y_{-1}x_0^{3n-1}/(z_{-2} - x_0)^{2n-1}(z_{-2} - 3x_0)^n)x_0}{x_0 + z_{-2} - 2x_0} = \frac{y_{-1}x_0^{3n}}{(z_{-2} - x_0)^{2n}(z_{-2} - 3x_0)^n} \\
 z_{6n} &= \frac{y_{6n-2}x_{6n-1}}{x_{6n-1} - z_{6n-3}} = \frac{((-1)^n z_0^{3n}/y_{-2}^{2n-1}(y_{-2} + 2z_0)^n)(x_{-1}z_0^{3n}/(y_{-2} + z_0)^{2n}(y_{-2} - z_0)^n)}{(x_{-1}z_0^{3n}/(y_{-2} + z_0)^{2n}(y_{-2} - z_0)^n) - (-x_{-1}z_0^{3n-1}/(y_{-2} - z_0)^{n-1}(y_{-2} + z_0)^{2n-1})} \\
 &= \frac{((-1)^n z_0^{3n}/y_{-2}^{2n-1}(y_{-2} + 2z_0)^n)z_0}{z_0 + y_{-2} - z_0} = \frac{(-1)^n z_0^{3n+1}}{y_{-2}^{2n}(y_{-2} + 2z_0)^n}.
 \end{aligned} \tag{24}$$

Also, from equation (21), one can see that

$$\begin{aligned}
 x_{6n+1} &= \frac{z_{6n-1}y_{6n}}{y_{6n} + x_{6n-2}} = \frac{((-1)^n z_{-1}y_0^{3n}/(x_{-2} - y_0)^{2n}(x_{-2} + y_0)^n)(y_0^{3n+1}/x_{-2}^{2n}(x_{-2} - 2y_0)^n)}{(y_0^{3n+1}/x_{-2}^{2n}(x_{-2} - 2y_0)^n) + (y_0^{3n}/x_{-2}^{2n-1}(x_{-2} - 2y_0)^n)} \\
 &= \frac{((-1)^n z_{-1}y_0^{3n}/(x_{-2} - y_0)^{2n}(x_{-2} + y_0)^n)y_0}{y_0 + x_{-2}} = \frac{(-1)^n z_{-1}y_0^{3n+1}}{(x_{-2} - y_0)^{2n}(x_{-2} + y_0)^{n+1}}, \\
 y_{6n+2} &= \frac{x_{6n}z_{6n+1}}{z_{6n+1} + y_{6n-1}} = \frac{((-1)^n x_0^{3n+1}/z_{-2}^n(z_{-2} - 2x_0)^{2n})(-y_{-1}x_0^{3n+1}/(z_{-2} - x_0)^{2n+1}(z_{-2} - 3x_0)^n)}{(-y_{-1}x_0^{3n+1}/(z_{-2} - x_0)^{2n+1}(z_{-2} - 3x_0)^n) + (y_{-1}x_0^{3n}/(x_0 - z_{-2})^{2n}(z_{-2} - 3x_0)^n)} \\
 &= \frac{((-1)^n x_0^{3n+1}/z_{-2}^n(z_{-2} - 2x_0)^{2n})x_0}{x_0 - (z_{-2} - x_0)} = \frac{(-1)^{n+1}x_0^{3n+2}}{z_{-2}^n(z_{-2} - 2x_0)^{2n+1}}, \\
 z_{6n+3} &= \frac{y_{6n+1}x_{6n+2}}{x_{6n+2} - z_{6n}} \\
 &= \frac{(x_{-1}z_0^{3n+1}/(y_{-2} + z_0)^{2n+1}(y_{-2} - z_0)^n)z_0}{z_0 - (y_{-2} + 2z_0)} = \frac{-x_{-1}z_0^{3n+2}}{(y_{-2} + z_0)^{2n+2}(y_{-2} - z_0)^n}.
 \end{aligned} \tag{25}$$

In a similar pattern, one can prove the other relation. $\square$

Example 2. Figure 2 represents the dynamics of (21) with $x_{-2} = 0.8, x_{-1} = 0.5, x_0 = 0.3, y_{-2} = 0.26, y_{-1} = 0.17, y_0 = -0.13, z_{-2} = 0.14, z_{-1} = 0.19$, and $z_0 = 0.15$.

FIGURE 2: Difference system (21).

In a similar pattern, one can prove the following cases.

4. The System: $x_{n+1} = (y_n z_{n-1} / y_n - x_{n-2})$, $y_{n+1} = (z_n x_{n-1} / z_n - y_{n-2})$, and $z_{n+1} = (x_n y_{n-1} / x_n - z_{n-2})$

We discuss the solutions of the following system:

$$\begin{aligned} x_{n+1} &= \frac{y_n z_{n-1}}{y_n - x_{n-2}}, \\ y_{n+1} &= \frac{z_n x_{n-1}}{z_n - y_{n-2}}, \\ z_{n+1} &= \frac{x_n y_{n-1}}{x_n - z_{n-2}}, \end{aligned} \quad (26)$$

where $n \in \mathbb{N}_0$ and nonzero initial conditions such that $x_{-2} \neq y_0$, $y_{-2} \neq z_0$, and $x_0 \neq z_{-2}$.

Theorem 3. If $\{x_n, y_n, z_n\}$ are solutions of (26), then

$$\begin{aligned} x_{6n-2} &= x_{-2} \left(\frac{y_0}{x_{-2}} \right)^{3n}, \\ x_{6n-1} &= x_{-1} \left(\frac{z_0}{z_0 - y_{-2}} \right)^{3n}, \\ x_{6n} &= x_0 \left(\frac{x_0}{z_{-2}} \right)^{3n}, \\ x_{6n+1} &= (-1)^{n+1} z_{-1} \left(\frac{y_0}{x_{-2} - y_0} \right)^{3n+1}, \end{aligned}$$

$$\begin{aligned} x_{6n+2} &= z_0 \left(\frac{z_0}{y_{-2}} \right)^{3n+1}, \\ x_{6n+3} &= y_{-1} \left(\frac{x_0}{x_0 - z_{-2}} \right)^{3n+2}, \\ y_{6n-2} &= y_{-2} \left(\frac{z_0}{y_{-2}} \right)^{3n}, \\ y_{6n-1} &= y_{-1} \left(\frac{x_0}{x_0 - z_{-2}} \right)^{3n}, \\ y_{6n} &= y_0 \left(\frac{y_0}{x_{-2}} \right)^{3n}, \\ y_{6n+1} &= (-1)^{n+1} x_{-1} \left(\frac{z_0}{y_{-2} - z_0} \right)^{3n+1}, \\ y_{6n+2} &= x_0 \left(\frac{x_0}{z_{-2}} \right)^{3n+1}, \\ y_{6n+3} &= (-1)^n z_{-1} \left(\frac{y_0}{x_{-2} - y_0} \right)^{3n+2}, \\ z_{6n-2} &= z_{-2} \left(\frac{x_0}{z_{-2}} \right)^{3n}, \\ z_{6n-1} &= (-1)^n z_{-1} \left(\frac{y_0}{x_{-2} - y_0} \right)^{3n}, \\ z_{6n} &= z_0 \left(\frac{z_0}{y_{-2}} \right)^{3n}, \\ z_{6n+1} &= y_{-1} \left(\frac{x_0}{x_0 - z_{-2}} \right)^{3n+1}, \\ z_{6n+2} &= x_{-2} \left(\frac{y_0}{x_{-2}} \right)^{3n+2}, \\ z_{6n+3} &= (-1)^n x_{-1} \left(\frac{z_0}{y_{-2} - z_0} \right)^{3n+2}. \end{aligned}$$

(27)

Example 3. Figure 3 represents the behavior of (26) if $x_{-2} = 2$, $x_{-1} = 0.81$, $x_0 = 0.49$, $y_{-2} = 0.3$, $y_{-1} = -0.09$, $y_0 = -0.4$, $z_{-2} = 0.32$, $z_{-1} = 0.12$, and $z_0 = 0.2$.

5. The System: $x_{n+1} = (y_n z_{n-1} / y_n + x_{n-2})$, $y_{n+1} = (z_n x_{n-1} / z_n - y_{n-2})$, and $z_{n+1} = (x_n y_{n-1} / x_n + z_{n-2})$

We discuss the solutions of the following system:

FIGURE 3: Difference system (26).

$$\begin{aligned}
 x_{n+1} &= \frac{y_n z_{n-1}}{y_n + x_{n-2}}, \\
 y_{n+1} &= \frac{z_n x_{n-1}}{z_n - y_{n-2}}, \\
 z_{n+1} &= \frac{x_n y_{n-1}}{x_n + z_{n-2}},
 \end{aligned} \tag{28}$$

where $n \in \mathbb{N}_0$ and nonnegative initial conditions such that $x_{-2} \neq \pm y_0$, $x_{-2} \neq -2y_0$, $y_{-2} \neq z_0$, $y_{-2} \neq 2z_0$, $y_{-2} \neq 3z_0$, $z_{-2} \neq \pm x_0$, and $z_{-2} \neq 2x_0$.

Theorem 4. If $\{x_n, y_n, z_n\}$ are solutions of (28), then

$$\begin{aligned}
 x_{6n-2} &= \frac{(-1)^n y_0^{3n}}{x_{-2}^{2n-1} (x_{-2} + 2y_0)^n}, \\
 x_{6n-1} &= \frac{x_{-1} z_0^{3n}}{(y_{-2} - z_0)^{2n} (y_{-2} - 3z_0)^n}, \\
 x_{6n} &= \frac{(-1)^n x_0^{3n+1}}{z_{-2}^{2n} (2x_0 - z_{-2})^n}, \\
 x_{6n+1} &= \frac{z_{-1} y_0^{3n+1}}{(x_{-2} - y_0)^n (x_{-2} + y_0)^{2n+1}}, \\
 x_{6n+2} &= \frac{(-1)^{n+1} z_0^{3n+2}}{y_{-2}^n (y_{-2} - 2z_0)^{2n+1}}, \\
 y_{6n-2} &= \frac{(-1)^n y_0^{3n}}{x_{-2}^{2n-1} (x_{-2} + 2y_0)^n}, \\
 y_{6n-1} &= \frac{z_{-1} y_0^{3n}}{(x_{-2} - y_0)^n (x_{-2} + y_0)^{2n}}, \\
 y_{6n} &= \frac{(-1)^n z_0^{3n+1}}{y_{-2}^n (y_{-2} - 2z_0)^{2n}}, \\
 y_{6n+1} &= \frac{(-1)^n y_{-1} x_0^{3n+1}}{(x_0 - z_{-2})^{2n} (x_0 + z_{-2})^{n+1}}, \\
 y_{6n+2} &= \frac{(-1)^n y_0^{3n+2}}{x_{-2}^{2n} (x_{-2} + 2y_0)^{n+1}}, \\
 z_{6n-2} &= \frac{(-1)^n x_0^{3n}}{z_{-2}^{2n-1} (2x_0 - z_{-2})^n}, \\
 z_{6n-1} &= \frac{z_{-1} y_0^{3n}}{(x_{-2} - y_0)^n (x_{-2} + y_0)^{2n}}, \\
 z_{6n} &= \frac{(-1)^n z_0^{3n+1}}{y_{-2}^n (y_{-2} - 2z_0)^{2n}}, \\
 z_{6n+1} &= \frac{(-1)^n y_{-1} x_0^{3n+1}}{(x_0 - z_{-2})^{2n} (x_0 + z_{-2})^{n+1}}, \\
 z_{6n+2} &= \frac{(-1)^n y_0^{3n+2}}{x_{-2}^{2n} (x_{-2} + 2y_0)^{n+1}}, \\
 z_{6n+3} &= \frac{x_{-1} z_0^{3n+2}}{(y_{-2} - 3z_0)^{n+1} (y_{-2} - z_0)^{2n+1}}.
 \end{aligned} \tag{29}$$

Example 4. Figure 4 represents the dynamics of (28) with $x_{-2} = 0.8$, $x_{-1} = 0.25$, $x_0 = -0.31$, $y_{-2} = 0.2$, $y_{-1} = 0.7$, $y_0 = -0.13$, $z_{-2} = 0.4$, $z_{-1} = 0.9$, and $z_0 = 0.71$.

6. The System: $x_{n+1} = (y_n z_{n-1} / y_n - x_{n-2})$, $y_{n+1} = (z_n x_{n-1} / z_n + y_{n-2})$, and $z_{n+1} = (x_n y_{n-1} / x_n + z_{n-2})$

We will obtain the solutions of the following system:

FIGURE 4: Difference system (28).

$$\begin{aligned}
 x_{n+1} &= \frac{y_n z_{n-1}}{y_n - x_{n-2}}, \\
 y_{n+1} &= \frac{z_n x_{n-1}}{z_n + y_{n-2}}, \\
 z_{n+1} &= \frac{x_n y_{n-1}}{x_n + z_{n-2}},
 \end{aligned} \tag{30}$$

where $n \in \mathbb{N}_0$ and nonzero initial conditions such that $x_{-2} \neq y_0$, $x_{-2} \neq 2y_0$, $x_{-2} \neq 3y_0$, $y_{-2} \neq \pm z_0$, $y_{-2} \neq 2z_0$, $z_{-2} \neq \pm x_0$, and $z_{-2} \neq -2x_0$.

Theorem 5. If $\{x_n, y_n, z_n\}$ are solutions of (30), then

$$\begin{aligned}
 x_{6n-2} &= \frac{(-1)^n y_0^{3n}}{x_{-2}^{n-1} (x_{-2} - 2y_0)^{2n}}, \\
 x_{6n-1} &= \frac{(-1)^n x_{-1} z_0^{3n}}{(y_{-2} + z_0)^n (y_{-2} - z_0)^{2n}}, \\
 x_{6n} &= \frac{(-1)^n x_0^{3n+1}}{z_{-2}^{2n} (z_{-2} + 2x_0)^n}, \\
 x_{6n+1} &= \frac{-z_{-1} y_0^{3n+1}}{(x_{-2} - 3y_0)^n (x_{-2} - y_0)^{2n+1}}, \\
 x_{6n+2} &= \frac{-z_0^{3n+2}}{y_{-2}^{2n+1} (y_{-2} - 2z_0)^n}, \\
 x_{6n+3} &= \frac{(-1)^{n+1} y_{-1} x_0^{3n+2}}{(x_0 - z_{-2})^n (x_0 + z_{-2})^{2n+2}}, \\
 y_{6n-2} &= \frac{z_0^{3n}}{y_{-2}^{2n-1} (y_{-2} - 2z_0)^n}, \\
 y_{6n-1} &= \frac{(-1)^n y_{-1} x_0^{3n}}{(x_0 - z_{-2})^n (x_0 + z_{-2})^{2n}}, \\
 y_{6n} &= \frac{(-1)^n y_0^{3n+1}}{x_{-2}^n (x_{-2} - 2y_0)^{2n}}, \\
 y_{6n+1} &= \frac{(-1)^n x_{-1} z_0^{3n+1}}{(y_{-2} - z_0)^{2n} (y_{-2} + z_0)^{n+1}}, \\
 y_{6n+2} &= \frac{(-1)^n x_0^{3n+2}}{z_{-2}^{2n} (z_{-2} + 2x_0)^{n+1}}, \\
 y_{6n+3} &= \frac{z_{-1} y_0^{3n+2}}{(x_{-2} - 3y_0)^{n+1} (x_{-2} - y_0)^{2n+1}}, \\
 z_{6n-2} &= \frac{(-1)^n x_0^{3n}}{z_{-2}^{2n-1} (z_{-2} + 2x_0)^n}, \\
 z_{6n-1} &= \frac{z_{-1} y_0^{3n}}{(x_{-2} - y_0)^{2n} (x_{-2} - 3y_0)^n}, \\
 z_{6n} &= \frac{z_0^{3n+1}}{y_{-2}^{2n} (y_{-2} - 2z_0)^n}, \\
 z_{6n+1} &= \frac{(-1)^n y_{-1} x_0^{3n+1}}{(x_0 - z_{-2})^n (x_0 + z_{-2})^{2n+1}}, \\
 z_{6n+2} &= \frac{(-1)^{n+1} y_0^{3n+2}}{x_{-2}^n (x_{-2} - 2y_0)^{2n+1}}, \\
 z_{6n+3} &= \frac{(-1)^{n+1} x_{-1} z_0^{3n+2}}{(y_{-2} - z_0)^{2n+1} (y_{-2} + z_0)^{n+1}}.
 \end{aligned} \tag{31}$$

Example 5. Figure 5 represents the dynamics of (30) with $x_{-2} = -0.2$, $x_{-1} = 0.05$, $x_0 = -0.39$, $y_{-2} = 2$, $y_{-1} = 1.7$, $y_0 = -0.3$, $z_{-2} = 1.1$, $z_{-1} = 8$, and $z_0 = -0.16$.

FIGURE 5: Difference system (30).

7. Conclusion

This paper discussed boundedness and the form expressions of the solutions of some special cases of difference equation systems $x_{n+1} = (y_n z_{n-1}/y_n \pm x_{n-2})$, $y_{n+1} = (z_n x_{n-1}/z_n \pm y_{n-2})$, and $z_{n+1} = (x_n y_{n-1}/x_n \pm z_{n-2})$, after the introduction in Section 1, we investigated in Section 2, the boundedness behavior of equation (10), and we have obtained the solutions of this system. In Section 3, we obtained the form of the solutions of the system $x_{n+1} = (y_n z_{n-1}/y_n + x_{n-2})$, $y_{n+1} = (z_n x_{n-1}/z_n + y_{n-2})$, and $z_{n+1} = (x_n y_{n-1}/x_n - z_{n-2})$. Also, in Section 4, we gave the solution form of the system $x_{n+1} = (y_n z_{n-1}/y_n - x_{n-2})$, $y_{n+1} = (z_n x_{n-1}/z_n - y_{n-2})$, and $z_{n+1} = (x_n y_{n-1}/x_n - z_{n-2})$. In Section 5, we dealt with the solution of system (28). Finally, in Section 6, we have the expressions of the form of the solutions of the system $x_{n+1} = (y_n z_{n-1}/y_n - x_{n-2})$, $y_{n+1} = (z_n x_{n-1}/z_n + y_{n-2})$, and $z_{n+1} = (x_n y_{n-1}/x_n + z_{n-2})$, and in each case, we presented some numerical examples of each case and drew it by using MATLAB.

Data Availability

All the data utilized in this article have been included, and the sources where it was adopted were cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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