Research Article

Properties and Applications of a New Extended Gamma Function Involving Confluent Hypergeometric Function

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In this paper, a new confluent hypergeometric gamma function and an associated confluent hypergeometric Pochhammer symbol are introduced. We discuss some properties, for instance, their different integral representations, derivative formulas, and generating function relations. Different special cases are also considered.

1. Introduction

The classical gamma function was first found by Swiss mathematician Leonhard Euler (1707–1783) in his objective to generalize the factorial to nonintegral values. The classical gamma function and its related Pochhammer symbol are characterized as follows:

and

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0, \quad (1)$$

$$(z)_\rho = \begin{cases} \frac{\Gamma(z+\rho)}{\Gamma(z)}, \\ z(z+1)\dots(z+n-1), (\rho = n \in \mathbb{N}_0), & z \in \mathbb{C}, \\ 1, & (\rho = 0, z \in \mathbb{C} \setminus \{0\}). \end{cases} \quad (2)$$

The gamma function belongs to category of the special transcendental functions. It has applications in different fields, e.g., definite integration, asymptotic series,

hypergeometric series, etc. In the last three decades, various extended forms of classical gamma functions are introduced by different researchers and mathematicians. Among those,

some of the extended gamma functions and their related Pochhammer symbols are defined below.

Chaudhry and Zubair [1] introduced an extended gamma function, defined by

$$\Gamma_{\kappa}(z) = \int_0^{\infty} t^{z-1} e^{-t-k/t} dt, \quad \Re(\kappa) > 0, \Re(z) > 0. \quad (3)$$

Srivastava et al. [2] proposed the following extended form of Pochhammer symbol:

$$(z; \kappa)_{\rho} = \begin{cases} \frac{\Gamma_{\kappa}(z + \rho)}{\Gamma_{\kappa}(z)}, & \Re(\kappa) > 0, z, \rho \in \mathbb{C}, \\ (z)_{\rho}, & \kappa = 0, z, \rho \in \mathbb{C} \setminus \{0\}. \end{cases} \quad (4)$$

Srivastava et al. [3] introduced an extended gamma function involving modified Bessel function and its associated Pochhammer symbol, defined as follows:

$$\Gamma_{\nu}(z; \kappa) = \sqrt{\frac{2\kappa}{\pi}} \int_0^{\infty} t^{z-3/2} e^{-t} K_{\nu+1/2}\left(\frac{\kappa}{t}\right) dt, \quad (5)$$

and

$$(z; \kappa, \nu)_{\rho} = \begin{cases} \frac{\Gamma_{\nu}(z + \rho; \kappa)}{\Gamma_{\nu}(z; \kappa)}, & \Re(\kappa) > 0, \Re(\nu) > 0, z, \rho \in \mathbb{C}, \\ (z; \kappa)_{\rho}, & \nu = 0, z, \rho \in \mathbb{C} \setminus \{0\}, \end{cases} \quad (6)$$

where $\Re(\kappa) > 0$, $\Re(z) > 0$ and $K_{\nu}(\cdot)$ is the modified Bessel function of order ν defined by

$$K_{\nu}(y) = \sum_{i=0}^{\infty} \frac{(-1)^i (y/2)^{n+2i}}{i! \Gamma(n+i+1)}. \quad (7)$$

Safdar et al. [4] defined an extension of classical gamma function and its associated Pochhammer symbol involving Mittag-Leffler function, defined by

$$\Gamma_s(z, \sigma) = \Gamma_s^{\sigma}(z) = \int_0^{\infty} t^{z-1} e^{-t} E_{\sigma}\left(\frac{-s}{t}\right) dt, \quad (8)$$

and

$$(z; s, \sigma)_{\rho} = \begin{cases} \frac{\Gamma_s(z + \rho; \sigma)}{\Gamma_s(z; \sigma)}, & \Re(s) > 0, \Re(\sigma) > 0, z, \rho \in \mathbb{C}, \\ (z; s)_{\rho}, & \sigma = 1, z, \rho \in \mathbb{C} \setminus \{0\}, \end{cases} \quad (9)$$

where $\Re(\rho) > 0$, $\Re(z) > 0$, and $E_{\sigma}(z)$ is the Mittag-Leffler function defined by

$$E_{\sigma}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\sigma + 1)}. \quad (10)$$

Srivastava et al. [5] present the familiar incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$, and they introduce the incomplete Pochhammer symbols that led us to a natural generalization and decomposition of a class of hypergeometric and other related functions which are potentially useful in closed-form representations of definite and semi-infinite integrals of various special functions. Applications of these functions can be found in communication theory, probability distributions, and groundwater pumping modelling.

Şahin et al. [6] introduce a new generalization of the Pochhammer symbol by means of the generalization of extended gamma function as follows:

$$\Gamma_{p,q}^{\kappa,\mu}(z) = \int_0^{\infty} t^{z-1} \exp\left(\frac{-t^{\kappa}}{p} - \frac{q}{t^{\mu}}\right) dt. \quad (11)$$

Using the generalization of Pochhammer symbol, they give a generalization of the extended hypergeometric functions of one or several variables. Also, they obtain various integral representations, derivative formulas, and certain properties of these functions.

The rest of the paper is organized as follows. In Section 2, we define a confluent hypergeometric gamma (CHG) function and derive its closed form in terms of Meijer's G -function, which is built in function of Computational Package Mathematica. In Section 3, a confluent hypergeometric Pochhammer (CHP) symbol is defined and some of its associate properties are also derived. In Section 4, we define an extension of generalized hypergeometric gamma (GHG) function and derive some of its associate properties. In Section 5, we obtain families of generating function relations. In Section 6, we present comparison of confluent hypergeometric gamma (CHG) function with classical and extended gamma functions for certain numerical values using Table 1. Comparison of integral and closed form of confluent hypergeometric gamma (CHG) function is shown in Table 2. Concluding remarks are given in Section 7.

2. A Confluent Hypergeometric Gamma Function

We propose further generalization of extended gamma function (3) by inducing a confluent hypergeometric function [7] in the integrand of the integral. In Theorem 1, we have derived the closed-form representation of the proposed CHG function using the inverse Mellin transform technique [8] in the form of Meijer's G -function.

TABLE 1: Comparison between gamma, extended gamma (3), and CHG (12) functions.

z	$\Gamma(z)$	$\Gamma_p(z)$	$\Gamma_p(z; m, b, \alpha)$
0.5	1.77245	0.09769	0.02519
1	1	0.13152	0.02863
1.5	0.88627	0.19042	0.03455
2	1	0.29568	0.04426
2.5	1.32934	0.49078	0.06013
3	2	0.86756	0.08654
3.5	3.32335	1.62685	0.13177
4	6	3.22362	0.21197
4.5	11.6317	6.72464	0.35966
5	24	14.7164	0.64269

TABLE 2: Comparison between integral form of CHG function and its closed form.

z	Integral form	Closed form
0.5	0.02519	0.02519
1	0.02863	0.02863
1.5	0.03455	0.03455
2	0.04426	0.04426
2.5	0.06013	0.06013
3	0.08654	0.08654
3.5	0.13177	0.13177
4	0.21197	0.21197
4.5	0.35966	0.35966
5	0.64269	0.64269

Theorem 1. One has the following closed form for $\Gamma_p(z; m, b, \alpha)$.

$$\Gamma_p(z; m, b, \alpha) = \int_0^\infty t^{z-1} e^{-t-p/t} {}_1F_1(m; b, -\alpha t) dt, \quad (12)$$

$$= \frac{\Gamma(b)}{\Gamma(m)} \sum_{n=0}^{\infty} \frac{p^n}{n!} \left[G_{3,3}^{2,2} \left(\alpha \middle| \begin{matrix} 1-m, 1-z, 1-z+n \\ 0, 1-z, 1-b \end{matrix} \right) - p^z G_{4,2}^{2,2} \left(p\alpha \middle| \begin{matrix} 1-m, 1-z \\ 0, 1-z, 1-b, -z-n \end{matrix} \right) \right], \quad (13)$$

where $\Re(p) > 0$, $|\arg(\alpha t)| < \pi/2$, $\Re(z) \neq 0, -1, -2, \dots$ and $\Re(b) \neq 0, -1, -2, \dots$

Proof. From [Rainville [7], p. 102, Theorem 36], (12) becomes

$$\begin{aligned} \Gamma_p(z; m, b, \alpha) &= \int_0^\infty t^{z-1} e^{-t-p/t} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(\alpha t)^s \Gamma(-s) \Gamma(m+s)}{\Gamma(b+s)} ds dt \\ &= \frac{\Gamma(b)}{\Gamma(m)} 2(p)^{z/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{\sqrt{p}}{\alpha} \right)^{-s} \frac{\Gamma(-s) \Gamma(\lambda+s)}{(b+s)} k_{z+s}(2\sqrt{p}) ds, \end{aligned} \quad (14)$$

where k_ν is Macdonald function [9], defined by

$$k_\nu = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi}, \quad (15)$$

and $I_\nu(x)$ is a modified Bessel function of first kind [10], defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)}. \quad (16)$$

Using (15) and (16) and the residue theorem [11], one has

$$\Gamma_p(z; m, b, \alpha) = \frac{\Gamma(b)}{\Gamma(m)} \sum_{n=0}^{\infty} \frac{(p)^n}{n!} \times \left(G_{3,3}^{2,2} \left(\alpha \left| \begin{matrix} 1, z, b \\ m, z, z-n \end{matrix} \right. \right) - p^z G_{4,2}^{2,2} \left(\frac{1}{p\alpha} \left| \begin{matrix} 1, z, b, z+n+1 \\ m, z \end{matrix} \right. \right) \right), \quad (17)$$

which can be reexpressed as

$$\Gamma_p(z; m, b, \alpha) = \frac{\Gamma(b)}{\Gamma(m)} \sum_{n=0}^{\infty} \frac{p^n}{n!} \left(G_{3,3}^{2,2} \left(\alpha \left| \begin{matrix} 1-z, 1-m, -z+n+1 \\ 0, 1-z, 1-b \end{matrix} \right. \right) - p^z G_{4,2}^{2,2} \left(p\alpha \left| \begin{matrix} 1-z, 1-m \\ 0, 1-z, 1-b, -z-n \end{matrix} \right. \right) \right), \quad (18)$$

where $\Re(p) > 0$ and $\Re(b), \Re(m), \Re(z) \neq 0, -1, -2, \dots$ $\square$

Remark 1. It is also important to mention here that in most of the extended cases, no closed form is provided but only new parameter in its function is inducted in the integrand of the integral of classical gamma function or its any extended gamma forms. Moreover, the proposed CHG function may find applications in statistics and a few associations with other special functions and polynomials.

Remark 2. Now, let $m = b$ and $\alpha = 0$; then, (12) reduces to extended gamma function defined by Zubair and Chaudhry [12], which is defined as

$$\Gamma_p(z) = \int_0^{\infty} t^{z-1} e^{-t-p/t} dt. \quad (19)$$

If we consider $p = 0$, then (19) becomes the classical gamma function (1).

2.1. Incomplete Confluent Hypergeometric Gamma Functions. Incomplete CHG functions are defined by

$$\gamma_p(z, y; m, b, \alpha) = \int_0^y t^{z-1} e^{-t-p/t} {}_1F_1(m; b, -\alpha t) dt, \quad (20)$$

and

$$\Gamma_p(z, y; m, b, \alpha) = \int_y^0 t^{z-1} e^{-t-p/t} {}_1F_1(m; b, -\alpha t) dt. \quad (21)$$

Now if we consider $m = b$ and $\alpha = 0$, then (20) and (21) are reduced to incomplete gamma function defined by Chaudhry and Zubair [12].

3. A Confluent Hypergeometric Pochhammer Symbol

The classical Pochhammer symbol is defined by

$$(z)_\rho = \begin{cases} \frac{\Gamma(z+\rho)}{\Gamma(z)}, \\ z(z+1) \dots (z+n-1), (\rho = n \in \mathbb{N}_0), & z \in \mathbb{C}, \\ 1, & (\rho = 0, z \in \mathbb{C} \setminus \{0\}). \end{cases} \quad (22)$$

Pochhammer symbol introduced by Srivastava et al. [2] is given by

$$(z; \kappa)_\rho = \begin{cases} \frac{\Gamma_\kappa(z+\rho)}{\Gamma_\kappa(z)}, & \Re(\kappa) > 0, z, \rho \in \mathbb{C}, \\ (z)_\rho, & \kappa = 0, z, \rho \in \mathbb{C} \setminus \{0\}. \end{cases} \quad (23)$$

We propose a CHP symbol in following form:

$$(z; m, b, \alpha, p)_\rho = \begin{cases} \frac{\Gamma_p(z+\rho; m, b, \alpha)}{\Gamma_p(z)}, & \Re(m, b, \alpha) > 0, \Re(p) > 0, z, \rho \in \mathbb{C}, \\ (z; m, b, \alpha)_\rho, & p = 0, z, \rho \in \mathbb{C} \setminus \{0\}, \end{cases} \quad (24)$$

where $\Gamma_p(z; m, b, \alpha)$ is defined in (12).

3.1. Some Results Associated with the Confluent Hypergeometric Pochhammer Symbol. Here, we present an integral representation of CHP symbol $(z; m, b, \alpha, p)_\rho$. Using Equation (12), we define an integral representation of $\Gamma_p(z + \rho; m, b, \alpha)$ as

$$\Gamma_p(z + \rho; m, b, \alpha) = \int_0^\infty t^{z+\rho-1} e^{-t-p/t} {}_1F_1(m; b, -\alpha t) dt. \quad (25)$$

Combining (25) with (24), one gets an integral representation of $(z; m, b, \alpha, p)_\rho$ in the following form:

$$(z; m, b, \alpha, p)_\rho = \frac{1}{\Gamma(z)} \int_0^\infty t^{z+\rho-1} e^{-t-p/t} {}_1F_1(m, b, -\alpha t) dt. \quad (26)$$

Remark 3. Let $m = b, \alpha = 0$, and $p = \kappa$; we get the following Pochhammer symbol defined by Srivastava et al. [2]:

$$(z; \kappa, 0)_\rho = \frac{1}{\Gamma(z)} \int_0^\infty t^{z+\rho-1} e^{-t-\kappa/t} dt. \quad (27)$$

Theorem 2. One obtains the following result for confluent hypergeometric gamma (CHG) function (12).

$$\Gamma_p(z; m, b, \alpha) = 2 \frac{\Gamma(b)}{\Gamma(m)\Gamma(b-m)} p^{z/2} \int_0^1 \mu^{m-1} (1-\mu)^{b-m-1} (1+\alpha\mu)^{-z/2} \times K_{-z} \left(2 \frac{\sqrt{1/\mu+\alpha}}{\sqrt{1/p\mu}} \right) d\mu. \quad (28)$$

Proof. Using integral form of confluent hypergeometric function, (12) becomes

$$\Gamma_p(z; m, b, \alpha) = \frac{\Gamma(b)}{\Gamma(m)\Gamma(b-m)} \int_0^\infty t^{z-1} e^{-t-p/t} \int_0^1 u^{\lambda-1} (1-u)^{b-\lambda-1} e^{-\alpha t u} du dt. \quad (29)$$

Apply a one-one transformation but without including the boundaries. Let $v = ut, \mu = t$ in the above equality and the Jacobian is $J = 1/\mu$; then, one gets

$$\Gamma_p(z; m, b, \alpha) = \frac{\Gamma(b)}{\Gamma(m)\Gamma(b-m)} \int_0^\infty \int_0^1 v^{z-1} e^{(-v/\mu - \alpha v - p\mu/v)} \mu^{m-z-1} \times (1-\mu)^{b-m-1} d\mu dv. \quad (30)$$

Interchanging the order of integration due to uniform convergence of the integrals and using some basic calculus, one obtains (28). $\square$

Theorem 3. For product of two confluent hypergeometric gamma functions, one has the following result:

$$\begin{aligned} \Gamma_p(z; m, b, \alpha) \Gamma_p(n; m, b, \alpha) &= 4 \int_0^{\pi/2} \int_0^\infty \gamma^{2(z+n)-1} \cos^{2z-1} \theta \sin^{2n-1} \theta \times \exp \left(-\gamma^2 - \frac{p}{\gamma^2 \cos^2 \theta \sin^2 \theta} \right) \\ &\quad \times {}_1F_1(m; b; -\alpha \gamma^2 \cos^2 \theta) \times {}_1F_1(m; b; -\alpha \gamma^2 \sin^2 \theta) d\gamma d\theta. \end{aligned} \quad (31)$$

Proof. Substituting $t = \sigma^2$ in (12), one gets

$$\Gamma_p(z; m, b, \alpha) = 2 \int_0^\infty \sigma^{2z-1} e^{-\sigma^2 - p/\sigma^2} {}_1F_1(m; b, -\alpha\sigma^2) d\sigma. \quad (32)$$

Therefore,

$$\Gamma_p(z; m, b, \alpha) \Gamma_p(n; m, b, \alpha) = 4 \int_0^\infty \int_0^\infty \sigma^{2m-1} \omega^{2n-1} \sigma^{2z-1} e^{-\sigma^2 - p/\sigma^2} \times e^{-\omega^2 - p/\omega^2} {}_1F_1(m; b, -\alpha\sigma^2) {}_1F_1(m; b, -\alpha\omega^2) d\sigma d\omega. \quad (33)$$

Letting $\sigma = \gamma \cos \theta$, $\omega = \gamma \sin \theta$ in (33) gives (31). $\square$

Remark 4. $\alpha = 0$ in (33) gives the product of extended gamma function (see [13]).

Remark 5. By putting $p = 0$, $\alpha = 0$ in (33), one obtains the standard relation between gamma and beta function.

Theorem 4. For confluent hypergeometric gamma (CHG) function (12), one has the following recurrence relationship:

$$(m-1)\Gamma_p(z; m, b, \alpha) = (m-b)\Gamma_p(z; m-1, b, \alpha) + (b-1)\Gamma_p(z; m-1, b-1, \alpha). \quad (34)$$

Proof. The confluent hypergeometric functions ${}_1F_1(\xi-1; \tau; z)$, ${}_1F_1(\xi+1; \tau; z)$, ${}_1F_1(\xi; \tau-1; z)$, and ${}_1F_1(\xi; \tau+1; z)$ in which one parameter is increased or decreased by unity are said to be contiguous to ${}_1F_1(\xi; \tau; z)$. The following relations exist between ${}_1F_1$ and two of its contiguous functions [14].

$$\begin{aligned} \tau {}_1F_1(\xi; \tau; z) &= \tau {}_1F_1(\xi-1; \tau; z) \\ &\quad + z {}_1F_1(\xi; \tau+1; z), \end{aligned} \quad (35)$$

$$\begin{aligned} (1+\xi-\tau) {}_1F_1(\xi; \tau; z) &= \xi {}_1F_1(\xi+1; \tau; z) \\ &\quad - (\tau-1) {}_1F_1(\xi; \tau-1; z). \end{aligned} \quad (36)$$

Substituting the function ${}_1F_1$ appearing in (36) and (12) and replacing ξ by m , τ by b , and z by $-\alpha t$, one obtains (34). $\square$

Theorem 5. Let $z, \rho, \mu \in \mathbb{C}$. Then,

$$(z; m, b, \alpha, p)_{\rho+\mu} = (z)_\rho (z + \rho; m, b, \alpha, p)_\mu. \quad (37)$$

Proof. From (22)–(24), we have

$$\begin{aligned} (z; m, b, \alpha, p)_{\rho+\mu} &= \frac{\Gamma_p(z + \rho + \mu; m, b, \alpha)}{\Gamma(z)} \\ &= \frac{\Gamma(z + \rho)}{\Gamma(z + \rho)} \frac{\Gamma_p(z + \rho + \mu; m, b, \alpha)}{\Gamma(z)} \\ &= (z)_\rho (z + \rho; m, b, \alpha, p). \end{aligned} \quad (38)$$

$\square$

Remark 6. From equation (37), one can write

$$(z; m, b, \alpha, p)_{r+n} = (z)_r (z + r; m, b, \alpha, p)_n, \quad (z \in \mathbb{C}, r, n \in \mathbb{N}_0). \quad (39)$$

Now, we can derive the following properties of CHP symbol $(z; m, b, \alpha, p)_n$ through familiar properties of classical and extended Pochhammer symbols $(z)_n$ and $(z; \kappa)_n$ (see, for instance, [2, 7, 15]).

Corollary 1. For $l, r, k, n \in \mathbb{N}_0$ and $R \in \mathbb{N}$, we can derive following properties:

$$\begin{aligned}
(z; m, b, \alpha, p)_{n+r+l} &= (z)_n (z+n)_r (z+n+r; m, b, \alpha, p)_l, \\
(z; m, b, \alpha, p)_{n-r+l} &= \frac{(-1)^r (z)_n}{(1-z-n)_r} (z+n-r; m, b, \alpha, p)_l, \\
(z; m, b, \alpha, p)_{2r+l} &= 2^{2r} \left(\frac{z}{2}\right)_r \left(\frac{z+1}{2}\right)_r (z+2r; m, b, \alpha)_l, \\
(z; m, b, \alpha, p)_{Rr+l} &= R^{Rr} \left(\frac{z}{R}\right)_r \left(\frac{z+1}{R}\right)_r \cdots \left(\frac{z+R-1}{R}\right)_r (z+Rr; m, b, \alpha, p)_l, \\
(z+n; m, b, \alpha, p)_{n+l} &= (z+n)_n (z+2n; m, b, \alpha, p)_l \\
&= \frac{(z)_{2n}}{(z)_n} (z+2n; m, b, \alpha, p)_l, \\
(z+r; m, b, \alpha, p)_{n+l} &= \frac{(z)_n (z+n)_r}{(z)_r} (z+n+r; m, b, \alpha, p)_l, \\
(z+kr; m, b, \alpha, p)_{kn+l} &= \frac{(z)_{kn+kr}}{(z)_{kr}} (z+kn+kr; m, b, \alpha, p)_l, \\
(z-n; m, b, \alpha, p)_{n+l} &= (-1)^n (1-z)_n (z; m, b, \alpha, p)_l, \\
(z-n; m, b, \alpha, p)_{n+l} &= \frac{(1-z)_r (z)_n}{(1-z-n)_r} (z+n-r; m, b, \alpha, p)_l, \\
(z-kr; m, b, \alpha, p)_{kn+l} &= (-1)^{kr} (z)_{kn-kr} (1-z)_{kr} (z+kn-kr; m, b, \alpha, p)_l, \\
(z+r; m, b, \alpha, p)_{n-r+l} &= \frac{(z)_n}{(z)_r} (z+n; m, b, \alpha, p)_l, \\
(z-r; m, b, \alpha, p)_{n-r+l} &= \frac{(-1)^r (1-z)_r (z)_n}{(1-z-n)_{2r}} (z+n-2r; m, b, \alpha, p)_l, \\
&\text{and} \\
(-z; m, b, \alpha, p)_{n+l} &= (-1)^r (z-n+1)_n (-z+n; m, b, \alpha, p)_l.
\end{aligned} \tag{41}$$

4. A Generalized Hypergeometric Gamma Function

We define a GHG function ${}_rF_q(\cdot)$ in the following form:

$${}_rF_q[(\eta_1; m, b, \alpha, p)\eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; z] = \sum_{n=0}^{\infty} \frac{(\eta_1; m, b, \alpha, p)_n (\eta_2)_n \cdots (\eta_r)_n z^n}{(\beta'_1)_n \cdots (\beta'_q)_n n!}, \tag{42}$$

where $\eta_i \in \mathbb{C}$ for $i=1, 2, \dots, r$, $\beta'_j \in \mathbb{C}$ for $j=1, 2, \dots, q$, and $\beta'_j \neq 0, -1, -2, \dots$.

New extensions of Gauss hypergeometric and confluent hypergeometric functions, respectively, are defined by

$$\begin{aligned}
{}_2F_1[(\eta_1; m, b, \alpha, p), \beta'; \gamma; z] &= \sum_{n=0}^{\infty} \frac{(\eta_1; m, b, \alpha, p)_n (\beta')_n z^n}{(\gamma)_n n!}, \\
{}_1F_1[(\eta_1; m, b, \alpha, p); \gamma; z] &= \sum_{n=0}^{\infty} \frac{(\eta_1; m, b, \alpha, p)_n z^n}{(\gamma)_n n!}.
\end{aligned} \tag{43}$$

Remark 7. Let us assume $m=b$, $\alpha=0$, and $p=\kappa$ in (42); one will get the following generalized hypergeometric function defined by Srivastava et al. [2]:

$$\begin{aligned}
{}_rF_q[(\eta_1; \kappa), \eta_2, \dots, \delta_r; \beta_1, \dots, \beta_q; z] \\
= \sum_{n=0}^{\infty} \frac{(\eta_1; \kappa)_n (\eta_2)_n \cdots (\eta_r)_n z^n}{(\beta_1)_n \cdots (\beta_q)_n n!},
\end{aligned} \tag{44}$$

where $\eta_i \in \mathbb{C}$ for $i=1, 2, \dots, r$, $\beta'_j \in \mathbb{C}$ for $j=1, 2, \dots, q$, and $\beta'_j \neq 0, -1, -2, \dots$.

Remark 8. If we assume $\kappa = 0$ in (44), we then obtain the classical generalized hypergeometric function [7]:

$${}_rF_q[(\eta_1), \eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; z] = \sum_{n=0}^{\infty} \frac{(\eta_1)_n (\eta_2)_n \dots (\eta_r)_n}{(\beta'_1)_n \dots (\beta'_q)_n} z^n, \quad (45)$$

where $\eta_i \in \mathbb{C}$ for $i = 1, 2, \dots, r$, $\beta'_j \in \mathbb{C}$ for $j = 1, 2, \dots, q$, and $\beta'_j \neq 0, -1, -2, \dots$.

Now in the following theorems, different integral representations of GHG function (42) are derived.

Theorem 6. If $\eta_i \in \mathbb{C}$ for $i = 1, 2, \dots, r$, $\beta'_j \in \mathbb{C}$ for $j = 1, 2, \dots, q$, and $\beta'_j \neq 0, -1, -2, \dots$, then

$${}_rF_q[(\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; z] = \frac{1}{\Gamma(\eta_1)} \int_0^{\infty} t^{\eta_1+p-1} e^{-t} t^{-\frac{p}{t}} {}_1F_1(m; b, -\alpha t) {}_{r-1}F_q[\eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; zt] dt. \quad (46)$$

Proof. Applying integral representation of extended CHP symbol (26) in the definition of extended hypergeometric function (42), we have

$${}_rF_q[(\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; z] = \frac{1}{\Gamma(\eta_1)} \sum_{n=0}^{\infty} \int_0^{\infty} t^{\eta_1+p-1} e^{-t} t^{-\frac{p}{t}} {}_1F_1(m; b, -\alpha t) \frac{(\eta_2)_n \dots (\eta_r)_n}{(\beta'_1)_n \dots (\beta'_q)_n} \frac{z^n}{n!} dt. \quad (47)$$

Exchanging integration and summation order and then by applying generalized hypergeometric function (45), we obtained the desired result. $\square$

Theorem 7. If $\Re(\gamma) > \Re(\beta') > 0$; $\Re(m, b, \alpha) \geq 0$, then the beta-type integral associated with the GHG function is defined by

$$\begin{aligned} & {}_rF_q[(\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_{r-1}, \beta'; \beta'_1, \dots, \beta'_{q-1}, \gamma; z] \\ &= \frac{1}{B(\beta', \gamma - \beta')} \int_0^1 t^{\beta'-1} (1-t)^{\gamma-\beta'-1} {}_{r-1}F_{q-1}[\delta_2, \dots, \delta_r; \beta'_1, \dots, \beta'_{q-1}; zt] dt. \end{aligned} \quad (48)$$

Proof. From (42), we have

$$\begin{aligned} & {}_rF_q[(\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; z] \\ &= \sum_{n=0}^{\infty} \frac{(\eta_1; m, b, \alpha, p)_n (\eta_2)_n \dots (\eta_{r-1})_n (\beta')_n}{(\eta'_1)_n \dots (\beta'_{q-1})_n (\gamma)_n} \frac{z^n}{n!}. \end{aligned} \quad (49)$$

We know that

$$\begin{aligned} \frac{(\beta')_n}{(\gamma)_n} &= \frac{B(\beta' + n, \gamma - \beta')}{B(\beta', \gamma - \beta')} \\ &= \frac{1}{B(\beta', \gamma - \beta')} \int_0^1 t^{\beta'+n-1} (1-t)^{\gamma-\beta'-1} dt. \end{aligned} \quad (50)$$

Combining (49) and (45) solves the theorem. $\square$

Theorem 8. If $\Re(m, b, \alpha) \geq 0$, $n \in \mathbb{N}_0$, then the n^{th} order derivative of GHG function (42) is given by

$$\begin{aligned} & \frac{d^n}{dz^n} \{ {}_rF_q[(\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; z] \} \\ &= \frac{(\eta_1)_n (\eta_2)_n \dots (\eta_r)_n}{(\beta'_1)_n \dots (\beta'_q)_n} \\ & \quad \times {}_rF_q[(\eta_1 + n; m, b, \alpha, p), \eta_2 + n, \dots, \eta_r + n; \beta'_1, \dots, \beta'_q; z]. \end{aligned} \quad (51)$$

Proof. The proof of (51) is minimal for $n \in \mathbb{N}_0$. Consider if $n \neq 0$, then

$$\begin{aligned} & \frac{d}{dz} \{ {}_rF_q[(\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; z] \} \\ &= \sum_{n=1}^{\infty} \frac{(\eta_1; m, b, \alpha, p)_n (\eta_2)_n \dots (\eta_r)_n}{(\beta'_1)_n \dots (\beta'_q)_n} \frac{z^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{(\eta_1; m, b, \alpha, p)_{n+1} (\eta_2)_{n+1} \dots (\eta_r)_{n+1}}{(\beta'_1)_{n+1} \dots (\beta'_q)_{n+1}} \frac{z^n}{n!}. \end{aligned} \quad (52)$$

In light of (37) and applying $(x)_{n+1} = x(x+1)_n$ in (54), it results in

$$\begin{aligned} & \frac{d}{dz} \left\{ {}_rF_q \left[(\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \beta'_1, \dots, \beta'_q; z \right] \right\} \\ &= \frac{(\eta_1)(\eta_2) \cdots (\eta_r)}{(\beta'_1) \cdots (\beta'_q)} {}_rF_q \left[(\eta_1; m, b, \alpha, p), \eta_2 + 1, \dots, \eta_r + 1; \beta'_1 + 1, \dots, \beta'_q + 1; z \right] \}. \end{aligned} \quad (53)$$

Differentiating n times proves the theorem. $\square$

confluent and Gauss hypergeometric functions become, respectively,

Corollary 2. *If in (40), we consider the cases $r=q=1$ and $r=2, q=1$, then the n th derivative of the new extended*

$$\begin{aligned} & \frac{d^n}{dz^n} \left\{ {}_1F_1 [(\eta; m, b, \alpha, p); \gamma; z] \right\} = \frac{(\eta)_n}{(\gamma)_n} {}_1F_1 [(\eta + n; m, b, \alpha, p); \gamma + n; z], \\ & \frac{d^n}{dz^n} \left\{ {}_2F_1 [(\eta; m, b, \alpha, p); \beta'; \gamma; z] \right\} = \frac{(\eta)_n (\beta')_n}{(\gamma)_n} \\ & \quad \times {}_2F_1 [(\eta + n; m, b, \alpha, p); \beta' + n; \gamma + n; z]. \end{aligned} \quad (54)$$

5. Families of Generating Function Relations

If $R = 0$, then the $\Delta(R, z)$ array will be null (see [16–18]).

Let $\Delta(R, z)$ indicate the below R parameters array.

$$\frac{z}{R}, \frac{z}{R+1}, \dots, \frac{z+R-1}{R}, \quad (z \in \mathbb{C}; R \in \mathbb{N}). \quad (55)$$

Theorem 9. *The following confluent hypergeometric gamma (CHG) generating function is defined by*

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(z)_n}{n!} {}_{r+R}F_q \left[\begin{matrix} \Delta(R, z+n), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} \right] z^n \\ &= (1-x)^{-z} {}_{r+R}F_q \left[\begin{matrix} \Delta(R, z), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} \right] \frac{z}{(1-x)^R}, \end{aligned} \quad (56)$$

$$(|x| < 1, z \in \mathbb{C}, R \in \mathbb{N}),$$

provided every element of (56) exists.

to generalize hypergeometric function extension (GHF) (42) results in required assertion (56) of the Theorem 5. $\square$

Proof. Using the identity

$$\sum_{n=0}^{\infty} \frac{(z)_n}{n!} z^n = (1-z)^{-z}, \quad (|z| < 1; z \in \mathbb{C}) \quad (57)$$

Theorem 10. The below generating function exists:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(z)_n}{n!} {}_{r+R}F_q \left[\begin{matrix} \Delta(R, -n), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \right] x^n \\ &= (1-x)^{-z} {}_{r+R}F_q \left[\begin{matrix} \Delta(R, z), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \left(-\frac{x}{(1-x)} \right)^R \right], \quad (|x| < 1, z \in \mathbb{C}, R \in \mathbb{N}), \end{aligned} \quad (58)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(z)_n}{n!} {}_{r+2R}F_q \left[\begin{matrix} \Delta(R, -n), \Delta(R, z+n), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \right] x^n \\ &= (1-x)^{-z} {}_{r+2R}F_q \left[\begin{matrix} \Delta(2R, z), (\eta_1; m, b, \alpha, p), \delta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \left(-\frac{4x}{(1-x)^2} \right)^R \right], \quad (|x| < 1, z \in \mathbb{C}, R \in \mathbb{N}), \end{aligned} \quad (59)$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(z)_n}{n!} {}_{r+R}F_q \left[\begin{matrix} \Delta(R, -n), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \right] x^n \\ &= (1-x)^{-z} {}_{r+2R}F_q \left[\begin{matrix} \Delta(2R, z), (\delta_1; m, b, \alpha, p), \delta_2, \dots, \delta_r; \\ \Delta(R, 1-z-n), \beta'_1, \dots, \beta'_q; \end{matrix} zx^R \right], \quad (|x| < 1, z \in \mathbb{C}, R \in \mathbb{N}), \end{aligned} \quad (60)$$

provided every element of (58)–(60) exists.

Corollary 3. The following generating function relations exist.

Proof. The proof of (58)–(60) is similar to that of Theorem 6. $\square$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(z)_n}{n!} {}_{r+1}F_q \left[\begin{matrix} (z+n), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \right] x^n \\ &= (1-x)^{-z} {}_{r+1}F_q \left[\begin{matrix} z, (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} \frac{z}{(1-x)} \right], \quad (|x| < 1, z \in \mathbb{C}), \end{aligned} \quad (61)$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(z)_n}{n!} {}_{r+1}F_q \left[\begin{matrix} -n, (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \right] x^n \\ &= (1-x)^{-z} {}_{r+1}F_q \left[\begin{matrix} z, ((\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \left(-\frac{x}{(1-x)} \right)) \right], \quad (|x| < 1, z \in \mathbb{C}), \end{aligned} \quad (62)$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(z)_n}{n!} {}_{r+2}F_q \left[\begin{matrix} -n, (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \right] x^n \\
&= (1-x)^{-z} {}_{r+2}F_q \left[\begin{matrix} \Delta(2, z), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} z \left(\frac{4x}{(1-x)^2} \right) \right], \\
& (|x| < 1, z \in \mathbb{C}, N \in \mathbb{N}),
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(z)_n}{n!} {}_{r+1}F_q \left[\begin{matrix} -n, (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ 1-z-n, \beta'_1, \dots, \beta'_q; \end{matrix} z \right] x^n \\
&= (1-x)^{-z} {}_rF_q \left[\begin{matrix} \Delta(2, z), (\eta_1; m, b, \alpha, p), \eta_2, \dots, \eta_r; \\ \beta'_1, \dots, \beta'_q; \end{matrix} zx \right], \\
& (|x| < 1, z \in \mathbb{C}),
\end{aligned} \tag{64}$$

provided that each number of (61)–(64) exists.

Proof. Assuming $R = 1$ in Theorems 9 and 10, respectively, results in required assertions (61)–(64) of Corollary 3. $\square$

6. Numerical Comparisons

In Table 1, we compared classical gamma, extended gamma (3), and CHG (12) functions for different numerical values of z in Computational Package Mathematica. We fixed values of $m = 1.9, b = 2.3, \alpha = 1.1, p = 2.1$. For different numerical values of z , extended gamma function (3) gives lesser values than gamma function while CHG (12) gives the least values. We also compared integral and closed forms of CHG functions in Table 2. We observed that results for both integral and closed forms are the same, but for the closed form, numerical values are obtained more quickly compared to the integral form.

7. Conclusion

In this paper, further generalization of the extended gamma function of Chaudhry and Zubair [12] is introduced. An associated confluent hypergeometric Pochhammer (CHP) symbol is also defined. Using confluent hypergeometric gamma (CHG) function, further generalizations of hypergeometric functions are also discussed. The new extended gamma function involving confluent hypergeometric was found to be useful in a variety of heat conduction problems. Additionally, we introduced a few families of generating functions for generalized hypergeometric function. Some special cases are also derived. We may develop new extensions of various special functions by applying confluent hypergeometric Pochhammer (CHP) symbol as shown in (24).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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