Research Article

Fractional Versions of Hadamard-Type Inequalities for Strongly Exponentially $(\alpha, h - m)$ -Convex Functions

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In this article, we prove some fractional versions of Hadamard-type inequalities for strongly exponentially $(\alpha, h - m)$ -convex functions via generalized Riemann–Liouville fractional integrals. The outcomes of this paper provide inequalities of strongly convex, strongly m -convex, strongly s -convex, strongly (α, m) -convex, strongly (s, m) -convex, strongly $(h - m)$ -convex, strongly $(\alpha, h - m)$ -convex, strongly exponentially convex, strongly exponentially m -convex, strongly exponentially s -convex, strongly exponentially (s, m) -convex, strongly exponentially $(h - m)$ -convex, and exponentially $(\alpha, h - m)$ -convex functions. The error estimations are also studied by applying two fractional integral identities.

1. Introduction

Fractional calculus is the study of derivatives and integrals of any arbitrary real or complex order. It is the generalization of ordinary calculus in which operations are mainly focused on integers. Its history starts when Leibniz and l'Hospital discussed the meaning of fractional order in 1695. This is the first discussion of fractional calculus. Many mathematicians devoted their efforts to make the foundation of fractional calculus. At that time, it was considered only in mathematics

but now it has several applications in Science and Engineering, signal processing, mathematical biology, and rheology. In mathematics, many fractional integral operators have been introduced by researchers, see [1, 2]. Using these fractional operators, extensive inequalities are established for different types of convexity, see [3–5] and reference therein. The convex function is defined as follows:

A function $f: I \rightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, is called the convex function if the following inequality holds:

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y), \quad t \in [0, 1] \text{ and } x, y \in I. \quad (1)$$

Hadamard inequality is geometrical interpretation of the convex function, and it is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(v) dv \leq \frac{f(a) + f(b)}{2}. \quad (2)$$

This inequality gives estimates of the mean value of a convex function. Recently, many mathematicians

investigated different versions of Hadamard inequality and discussed its basic properties with corresponding fractional integral operators (see [6–9] and reference therein). Our aim is to establish Hadamard inequalities for the strongly exponentially $(\alpha, h - m)$ -convex function via generalized Riemann–Liouville fractional integrals. Also, we have obtained error estimations for this convexity by using two fractional integral identities. Now, we

recall the definition of the strongly exponentially $(\alpha, h - m)$ -convex function.

Definition 1 (see [10]). Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$, and let $h: J \rightarrow \mathbb{R}$ be a nonnegative function. A

function $f: [0, b] \rightarrow \mathbb{R}$ is called a strongly exponentially $(\alpha, h - m)$ -convex function if f is nonnegative and for all x and $y \in [0, b]$, $t \in (0, 1)$, $\eta \in \mathbb{R}$, and $(\alpha, m) \in (0, 1]^2$, with modulus $\lambda \geq 0$, one has

$$f(tx + m(1-t)y) \leq h(t^\alpha) \frac{f(x)}{e^{\eta x}} + mh(1-t^\alpha) \frac{f(y)}{e^{\eta y}} - \frac{m\lambda}{e^{(x+y)\eta}} h(t^\alpha) h(1-t^\alpha) |y-x|^2. \quad (3)$$

The above definition provides some kinds of exponential convexities as follows:

Remark 1

- (i) If we substitute $\alpha = 1, \lambda = 0$ and $h(t) = t^s$, then the exponentially (s, m) -convex function in the second sense introduced by Qiang et al. in [11] can be obtained
- (ii) If we substitute $\alpha = 1, \lambda = 0$, $h(t) = t^s$, and $m = 1$, then the exponentially s -convex function introduced by Mehreen et al. in [12] can be obtained
- (iii) If we substitute $\alpha = 1, \lambda = 0$, $h(t) = t$, and $m = 1$, then the exponentially convex function introduced by Awan et al. in [13] can be obtained

The classical Riemann–Liouville fractional integrals are given as follows:

Definition 2 (see [14]). Let $f \in L_1[a, b]$. Then, left-sided and right-sided Riemann–Liouville fractional integrals of a function f of the order $\xi \in \mathbb{C}$ and $\Re(\xi) > 0$ are given by

$$\begin{aligned} I_{a^+}^\xi f(x) &= \frac{1}{\Gamma(\xi)} \int_a^x (x-t)^{\xi-1} f(t) dt, \quad x > a, \\ I_{b^-}^\xi f(x) &= \frac{1}{\Gamma(\xi)} \int_x^b (t-x)^{\xi-1} f(t) dt, \quad x < b, \end{aligned} \quad (4)$$

where $\Re(\xi)$ denotes the real part of ξ and $\Gamma(\xi) = \int_0^\infty e^{-z} z^{\xi-1} dz$.

Following two theorems are the fractional versions of Hadamard inequalities via Riemann–Liouville fractional integrals.

Theorem 1 (see [15]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following fractional integral inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\xi+1)}{2(b-a)^\xi} \left[I_{a^+}^\xi f(b) + I_{b^-}^\xi f(a) \right] \leq \frac{f(a) + f(b)}{2}, \quad (5)$$

with $\xi > 0$.

Theorem 2 (see [16]). Under the assumptions of Theorem 1, the following fractional integral inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\xi-1} \Gamma(\xi+1)}{(b-a)^\xi} \left[I_{(a+b/2)^+}^\xi f(b) + I_{(a+b/2)^-}^\xi f(a) \right] \leq \frac{f(a) + f(b)}{2}, \quad (6)$$

with $\xi > 0$.

Following theorem is the error estimation of inequality (5).

Theorem 3 (see [15]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following fractional integral inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\xi+1)}{2(b-a)^\xi} \left[I_{a^+}^\xi f(b) + I_{b^-}^\xi f(a) \right] \right| \\ & \leq \frac{b-a}{2(\xi+1)} \left(1 - \frac{1}{2^\xi} \right) \left[|f'(a)| + |f'(b)| \right]. \end{aligned} \quad (7)$$

The k -analogue of the Riemann–Liouville fractional integral is defined as follows:

Definition 3 (see [17]). Let $f \in L_1[a, b]$. Then, k -fractional Riemann–Liouville integrals of order ξ , where $\Re(\xi) > 0$ and $k > 0$, are defined as

$$\begin{aligned} {}_k I_{a^+}^\xi f(x) &= \frac{1}{k\Gamma_k(\xi)} \int_a^x (x-t)^{(\xi/k)-1} f(t) dt, \quad x > a, \\ {}_k I_{b^-}^\xi f(x) &= \frac{1}{k\Gamma_k(\xi)} \int_x^b (t-x)^{(\xi/k)-1} f(t) dt, \quad x < b, \end{aligned} \quad (8)$$

where $\Gamma_k(\cdot)$ is defined by [18]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{a^+}^\xi f(b) + {}_k I_{b^-}^\xi f(a) \right] \leq \frac{f(a) + f(b)}{2}. \quad (10)$$

Theorem 5 (see [20]). Under the assumptions of Theorem 4, the following inequality for k -fractional integral holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(b-a)^{(\xi/k)}} \left[{}_k I_{(a+b/2)^+}^\xi f(b) + {}_k I_{(a+b/2)^-}^\xi f(a) \right] \leq \frac{f(a) + f(b)}{2}. \quad (11)$$

The error estimation of inequality (10) is given in the following theorem.

Theorem 6 (see [19]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for k -fractional integral holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{a^+}^\xi f(b) + {}_k I_{b^-}^\xi f(a) \right] \right| \\ & \leq \frac{b-a}{2((\xi/k)+1)} \left(1 - \frac{1}{2^{(\xi/k)}} \right) \left[|f'(a)| + |f'(b)| \right]. \end{aligned} \quad (12)$$

Now, we recall generalized Riemann–Liouville fractional integrals by a monotonically increasing function.

Definition 4 (see [21]). Let $f \in L_1[a, b]$. Also let ψ be an increasing and positive monotone function on $(a, b]$; further, ψ has a continuous derivative ψ' on (a, b) . Therefore, left as well as right fractional integral operators of order ξ where $\Re(\xi) > 0$ of f with respect to ψ on $[a, b]$ are defined by

$$\begin{aligned} I_{a^+}^{\xi, \psi} f(x) &= \frac{1}{\Gamma(\xi)} \int_a^x \psi'(t) \psi(x) - \psi(t)^{\xi-1} f(t) dt, \quad x > a, \\ I_{b^-}^{\xi, \psi} f(x) &= \frac{1}{\Gamma(\xi)} \int_x^b \psi'(t) \psi(t) - \psi(x)^{\xi-1} f(t) dt, \quad x < b. \end{aligned} \quad (13)$$

The k -analogue of generalized Riemann–Liouville fractional integrals is defined as follows.

$$\Gamma_k(\xi) = \int_0^\infty t^{\xi-1} e^{-(t^k/k)} dt, \quad \Re(\xi) > 0. \quad (9)$$

Two k -fractional versions of Hadamard inequality are given in next two theorems.

Theorem 4 (see [19]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$. If f is a convex function on $[a, b]$, then the following inequality for k -fractional integral holds:

Definition 5 (see [22]). Let $f \in L_1[a, b]$. Also let ψ be an increasing and positive monotone function on $(a, b]$; further, ψ has a continuous derivative ψ' on (a, b) . Therefore, left as well as right k -fractional integral operators of order ξ where $\Re(\xi) > 0$ of f with respect to ψ on $[a, b]$ are defined by

$${}_k I_{a^+}^{\xi, \psi} f(x) = \frac{1}{k\Gamma_k(\xi)} \int_a^x \psi'(t) \psi(x) - \psi(t)^{(\xi/k)-1} f(t) dt, \quad x > a, \quad (14)$$

$${}_k I_{b^-}^{\xi, \psi} f(x) = \frac{1}{k\Gamma_k(\xi)} \int_x^b \psi'(t) \psi(t) - \psi(x)^{(\xi/k)-1} f(t) dt, \quad x < b. \quad (15)$$

For more details of fractional integrals, see [14, 23, 24]. We will utilize the following well-known hypergeometric, Beta, and incomplete Beta functions in our results [25]:

$$\begin{aligned} {}_2F_1[a, b; c; z] &= \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \\ & \quad \cdot (1-zt)^{-a} dt, \quad c > b > 0 \text{ and } |z| < 1, \end{aligned}$$

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

$$B(x, y; z) = \int_0^z t^{x-1} (1-t)^{y-1} dt. \quad (16)$$

In Section 2, we established Hadamard inequality for strongly exponentially $(\alpha, h - m)$ -convex functions via generalized Riemann–Liouville fractional integrals. The special cases of these inequalities are associated with previously published papers. In Section 3, error estimations of fractional Hadamard inequality for strongly exponentially $(\alpha, h - m)$ are obtained with the help of two fractional integral identities. The outcomes of this article are connected with already established results given in [15, 16, 19, 20, 26–37].

2. Main Results

This section is concerned with two fractional versions of Hadamard inequalities for strongly exponentially $(\alpha, h - m)$ -convex functions. One of them is given in the following theorem.

Theorem 7 Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < mb$ and $f \in L_1[a, b]$. Also, suppose that f is the strongly exponentially $(\alpha, h - m)$ -convex function on $[a, b]$ with modulus $\lambda \geq 0$. Then, for $k > 0$ and $(\alpha, m) \in (0, 1]^2$, the following k -fractional integral inequality holds for operators given in (14) and (15):

$$\begin{aligned} & f\left(\frac{a+mb}{2}\right) + \frac{mg_1(\eta)\lambda h(1/2^\alpha)h(2^\alpha - 1/2^\alpha)}{(\xi+k)(\xi+2k)} \left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m} - mb\right)^2 \right. \\ & \left. + 2k\xi(b-a)\left(\frac{a}{m} - mb\right) \right] \leq \frac{\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \\ & \cdot \left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + g_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)m^{(\xi/k)+1} I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \leq \frac{\xi}{k} \left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}} + \frac{mh(2^\alpha - 1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}} \right] \int_0^1 h(t^\alpha)t^{(\xi/k)-1} dt \\ & + \frac{m\xi}{k} \left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}} + \frac{mh(2^\alpha - 1/2^\alpha)g_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] \int_0^1 t^{(\xi/k)-1} h(1-t^\alpha) dt \\ & - \frac{m\lambda\xi}{k} \left[\frac{g_2(\eta)h(1/2^\alpha)(b-a)^2}{e^{\eta(a+b)}} + \frac{mg_3(\eta)h(2^\alpha - 1/2^\alpha)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right] \int_0^1 h(t^\alpha)h(1-t^\alpha)t^{(\xi/k)-1} dt, \end{aligned} \quad (17)$$

where $\xi > 0$ and

$$\begin{aligned} g_1(\eta) &= \begin{cases} e^{-\eta(a+b)}, & \text{if } \eta > 0, \\ e^{-\eta(mb+(a/m))}, & \text{if } \eta < 0, \end{cases} \\ g_2(\eta) &= \begin{cases} e^{-\eta mb}, & \text{if } \eta < 0, \\ e^{-\eta a}, & \text{if } \eta > 0, \end{cases} \\ g_3(\eta) &= \begin{cases} e^{-\eta b}, & \text{if } \eta < 0, \\ e^{-\eta a/m}, & \text{if } \eta > 0. \end{cases} \end{aligned} \quad (18)$$

Proof. From strongly exponentially $(\alpha, h - m)$ -convexity of f , we have

$$f\left(\frac{x+my}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right) \frac{f(x)}{e^{\eta x}} + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) \frac{f(y)}{e^{\eta y}} - \frac{m\lambda}{e^{\eta(x+y)}} h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) |y-x|^2. \quad (19)$$

By setting $x = at + m(1-t)b$ and $y = (a/m)(1-t) + bt$, $t \in [0, 1]$, in (19), multiplying the resulting inequality with $t^{(\xi/k)-1}$, and then integrating with respect to t , we get

$$\begin{aligned} \frac{k}{\xi} f\left(\frac{a+mb}{2}\right) &\leq h\left(\frac{1}{2^\alpha}\right) \int_0^1 \frac{f(at+m(1-t)b)}{e^{\eta(at+m(1-t)b)}} t^{(\xi/k)-1} dt + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\times \int_0^1 \frac{f((a/m)(1-t)+bt)}{e^{\eta((a/m)(1-t)+bt)}} t^{(\xi/k)-1} dt - m\lambda h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\times \int_0^1 \frac{(t(b-a) + (1-t)((a/m)-mb))^2}{e^{\eta(t(a+b)+(1-t)(mb+(a/m)))}} t^{(\xi/k)-1} dt. \end{aligned} \quad (20)$$

Let

$$\begin{aligned} g_1(t) &= e^{-\eta(t(a+b)+(1-t)(mb+(a/m)))}, \\ g_1'(t) &= \eta(1-m)\left(\frac{a}{m}-b\right)e^{-\eta(t(a+b)+(1-t)(mb+(a/m)))}. \end{aligned} \quad (21)$$

Now one can see that g_1 will be increasing if $\eta < 0$ and decreasing if $\eta > 0$. Therefore, from inequality (20), we can have

$$\begin{aligned} \frac{k}{\xi} f\left(\frac{a+mb}{2}\right) &\leq h\left(\frac{1}{2^\alpha}\right) \int_0^1 \frac{f(at+m(1-t)b)}{e^{\eta(at+m(1-t)b)}} t^{(\xi/k)-1} dt + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\times \int_0^1 \frac{f((a/m)(1-t)+bt)}{e^{\eta((a/m)(1-t)+bt)}} t^{(\xi/k)-1} dt - \frac{mk g_1(\eta) \lambda}{\xi(\xi+k)(\xi+2k)} h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) [\xi(\xi+k)(b-a)^2 \\ &+ 2k^2\left(\frac{a}{m}-mb\right)^2 + 2k\xi(b-a)\left(\frac{a}{m}-mb\right)]. \end{aligned} \quad (22)$$

By setting $\psi(u) = at + m(1-t)b$ and $\psi(v) = (a/m)(1-t) + bt$ in (22), we get the following inequality:

$$\begin{aligned} \frac{k}{\xi} f\left(\frac{a+mb}{2}\right) &\leq \frac{1}{(mb-a)^{(\xi/k)}} \left[h\left(\frac{1}{2^\alpha}\right) \int_{\psi^{-1}(a)}^{\psi^{-1}(mb)} \frac{f(\psi(u))}{e^{\eta(\psi(u))}} (mb-\psi(u))^{(\xi/k)-1} \psi'(u) du \right. \\ &+ mh\left(\frac{2^\alpha-1}{2^\alpha}\right) \int_{\psi^{-1}(a/m)}^{\psi^{-1}(b)} \frac{f(\psi(v))}{e^{\eta(\psi(v))}} \left(\psi(v)-\frac{a}{m}\right)^{(\xi/k)-1} \psi'(v) dv \left. \right] - \frac{mk g_1(\eta) \lambda}{\xi(\xi+k)(\xi+2k)} \\ &\times h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) \left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m}-mb\right)^2 + 2k\xi(b-a)\left(\frac{a}{m}-mb\right) \right]. \end{aligned} \quad (23)$$

Further, multiplying by (ξ/k) and using Definition 5, we get

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) &\leq \frac{\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[g_2(\eta) h\left(\frac{1}{2^\alpha}\right) {}_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \\ &\quad \left. + g_3(\eta) m^{(\xi/k)+1} h\left(\frac{2^\alpha-1}{2^\alpha}\right) {}_k I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] - \frac{mg_1(\eta)\lambda}{(\xi+k)(\xi+2k)} \\ &\quad \times h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) \left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m} - mb\right)^2 + 2k\xi(b-a)\left(\frac{a}{m} - mb\right) \right]. \end{aligned} \quad (24)$$

The above inequality leads to the first inequality of (17). Again, using strongly exponentially $(\alpha, h-m)$ -convexity of f , for $t \in [0, 1]$, we have

$$\begin{aligned} &g_2(\eta) h\left(\frac{1}{2^\alpha}\right) f(at + m(1-t)b) + mg_3(\eta) h\left(\frac{2^\alpha-1}{2^\alpha}\right) f\left(\frac{a}{m}(1-t) + bt\right) \\ &\leq h(t^\alpha) \left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}} + \frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}} \right] \\ &\quad + mh(1-t^\alpha) \left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}} + \frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] \\ &\quad - m\lambda h(t^\alpha)h(1-t^\alpha) \left[\frac{g_2(\eta)h(1/2^\alpha)(b-a)^2}{e^{\eta(a+b)}} + \frac{mg_3(\eta)h(2^\alpha-1/2^\alpha)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right]. \end{aligned} \quad (25)$$

By integrating (25) over the interval $[0, 1]$ after multiplying with $t^{(\xi/k)-1}$, we get

$$\begin{aligned} &g_2(\eta) h\left(\frac{1}{2^\alpha}\right) \int_0^1 f(ta + m(1-t)b) t^{(\xi/k)-1} dt \\ &\quad + mg_3(\eta) h\left(\frac{2^\alpha-1}{2^\alpha}\right) \int_0^1 f\left(\frac{a}{m}(1-t) + tb\right) t^{(\xi/k)-1} dt \leq \left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}} + \frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}} \right] \\ &\quad \int_0^1 h(t^\alpha) t^{(\xi/k)-1} dt + m \left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}} + \frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] \\ &\quad \int_0^1 t^{(\xi/k)-1} h(1-t^\alpha) dt - m\lambda \left[\frac{g_2(\eta)h(1/2^\alpha)(b-a)^2}{e^{\eta(a+b)}} + \frac{mg_3(\eta)h(2^\alpha-1/2^\alpha)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right] \\ &\quad \int_0^1 h(t^\alpha) h(1-t^\alpha) t^{(\xi/k)-1} dt. \end{aligned} \quad (26)$$

Again using substitutions as considered in (22), the above inequality leads to the second inequality of (17).

In the following remark, we give the connection of inequality (17) with already established results. $\square$

Remark 2

- (i) If we take $\eta = 0, \lambda = 0, \alpha = 1$, and ψ as the identity function in (17), then the inequality stated in Theorem 2.1 in [29] is obtained
- (ii) If we take $\eta = 0, k = 1, h(t) = t, m = 1, \lambda = 0, \alpha = 1$, and ψ as the identity function in (17), then Theorem 1 is obtained
- (iii) If we take $\eta = 0, h(t) = t, m = 1, \alpha = 1, \lambda = 0$, and ψ as the identity function in (17), then refinement of Theorem 1 is obtained
- (iv) If we take $\alpha = 1, \xi = 1, k = 1, h(t) = t, m = 1, \eta = 0, \lambda = 0$, and ψ as the identity function in (17), then Hadamard inequality is obtained
- (v) If we take $\eta = 0, m = 1, \alpha = 1, \lambda = 0$, and $h(t) = t$ in (17), then the inequality stated in Theorem 1 in [26] is obtained
- (vi) If we take $\eta = 0, m = 1, \alpha = 1$, and $h(t) = t$ in (17), then the inequality stated in Theorem 10 in [32] is obtained
- (vii) If we take $\eta = 0, k = 1, m = 1, \alpha = 1, \lambda = 0$, and $h(t) = t$ in (17), then the inequality stated in Theorem 2.1 in [34] is obtained

- (viii) If we take $\alpha = 1, k = 1, h(t) = t, \eta = 0, \lambda = 0$, and ψ as the identity function in (17), then the inequality stated in Theorem 2.1 in [31] is obtained
- (ix) If we take $\alpha = 1, \lambda = 0, h(t) = t^s$, and ψ as the identity function in (17), then the inequality stated in Theorem 2 in [36] is obtained
- (x) If we take $\alpha = 1, \eta = 0, \lambda = 0$, and $h(t) = t^s$ in (17), then the inequality stated in Corollary 1 in [35] is obtained
- (xi) If we take $\eta = 0$ and $k = 1$ in (17), then the inequality stated in Theorem 4 in [37] is obtained
- (xii) If we take $\eta = 0, k = 1$, and $\alpha = 1$ in (17), then the inequality stated in Corollary 1 in [37] is obtained
- (xiii) If we take $\lambda = 0$ in (17), then the inequality stated in Theorem 7 in [38] is obtained

Now, we give inequality (17) for strongly exponentially $(h - m)$ -convex, strongly exponentially (s, m) -convex, strongly exponentially m -convex, and strongly exponentially convex functions.

Corollary 1. *If we take $\alpha = 1$ in (17), then the following inequality holds for strongly exponentially $(h - m)$ -convex functions:*

$$\begin{aligned}
 & f\left(\frac{a+mb}{2}\right) + \frac{mg_1(\eta)\lambda h^2(1/2)}{(\xi+k)(\xi+2k)} \left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m} - mb\right)^2 \right. \\
 & \left. + 2k\xi(b-a)\left(\frac{a}{m} - mb\right) \right] \leq \frac{h(1/2)\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[g_2(\eta)_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \\
 & \left. + g_3(\eta)m^{(\xi/k)+1} I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \leq \frac{h(1/2)\xi}{k} \left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{mg_3(\eta)f(b)}{e^{\eta b}} \right] \\
 & \times \int_0^1 h(t)t^{(\xi/k)-1} dt + \frac{mh(1/2)\xi}{k} \left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] \int_0^1 t^{(\xi/k)-1} h(1-t) dt \\
 & - \frac{m\lambda h(1/2)\xi}{k} \left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{mg_3(\eta)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right] \int_0^1 h(t)h(1-t)t^{(\xi/k)-1} dt.
 \end{aligned} \tag{27}$$

Corollary 2. *If we take $\alpha = 1$ and $h(t) = t^s$ in (17), then the following inequality holds for strongly exponentially (s, m) -convex functions:*

$$\begin{aligned}
& f\left(\frac{a+mb}{2}\right) + \frac{mg_1(\eta)\lambda}{2^{2s}(\xi+k)(\xi+2k)} \left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m} - mb\right)^2 \right. \\
& \quad \left. + 2k\xi(b-a)\left(\frac{a}{m} - mb\right) \right] \leq \frac{\Gamma_k(\xi+k)}{2^s(mb-a)^{(\xi/k)}} \left[g_2(\eta)_k I_{\psi^{-1}(a)^+}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \\
& \quad \left. + g_3(\eta)m^{(\xi/k)+1} I_{\psi^{-1}(b)^-}^{\xi,\psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \leq \frac{\xi}{2^s(\xi+sk)} \left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{mg_3(\eta)f(b)}{e^{\eta b}} \right] \\
& \quad + \frac{m\xi B(1+s, (\xi/k))}{2^s k} \left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] - \frac{m\xi B(1+s, s+(\xi/k))}{2^s k} \\
& \quad \times \left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{mg_3(\eta)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right].
\end{aligned} \tag{28}$$

Corollary 3. If we take $\alpha = 1$ and $h(t) = t$ in (17), then the following inequality holds for strongly exponentially m -convex functions:

$$\begin{aligned}
& f\left(\frac{a+mb}{2}\right) + \frac{mg_1(\eta)\lambda}{4(\xi+k)(\xi+2k)} \left[\xi(\xi+k)(b-a)^2 + 2k^2\left(\frac{a}{m} - mb\right)^2 \right. \\
& \quad \left. + 2k\xi(b-a)\left(\frac{a}{m} - mb\right) \right] \leq \frac{\Gamma_k(\xi+k)}{2(mb-a)^{(\xi/k)}} \left[g_2(\eta)_k I_{\psi^{-1}(a)^+}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \\
& \quad \left. + g_3(\eta)m^{(\xi/k)+1} I_{\psi^{-1}(b)^-}^{\xi,\psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \leq \frac{\xi}{2(\xi+k)} \left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{mg_3(\eta)f(b)}{e^{\eta b}} \right] \\
& \quad + \frac{mk}{2(\xi+k)} \left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] - \frac{m\xi k}{2(\xi+k)(\xi+2k)} \left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{mg_3(\eta)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right].
\end{aligned} \tag{29}$$

Corollary 4. If we take $\alpha = 1$, $m = 1$, and $h(t) = t$ in (17), then the following inequality holds for strongly exponentially convex functions:

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) + \frac{g_1(\eta)k^2\lambda}{2(\xi+k)(\xi+2k)} \\
& \leq \frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}} \left[g_2(\eta)_k I_{\psi^{-1}(a)^+}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(b)) + g_3(\eta)_k I_{\psi^{-1}(b)^-}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(a)) \right], \\
& \leq \frac{\xi}{2(\xi+k)} \left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{mg_3(\eta)f(b)}{e^{\eta b}} \right] + \frac{k}{2(\xi+k)} \left[\frac{g_2(\eta)f(b)}{e^{\eta a}} + \frac{mg_3(\eta)f(a)}{e^{\eta b}} \right] \\
& \quad - \frac{\lambda\xi k(b-a)^2(g_2(\eta) + g_3(\eta))}{2(\xi+k)(\xi+2k)e^{\eta(a+b)}}.
\end{aligned} \tag{30}$$

The next theorem is another version of Hadamard inequality for strongly exponentially $(\alpha, h - m)$ -convex functions.

Theorem 8. Under the assumptions of Theorem 7, the following k -fractional integral inequality holds:

$$\begin{aligned}
 & f\left(\frac{a+mb}{2}\right) + h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\
 & \frac{m\lambda g_1(\eta)}{4(\xi+k)(\xi+2k)} \left[\xi(\xi+k)(b-a)^2 + (\xi^2 + 5k\xi + 8k^2)\left(\frac{a}{m} - mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m} - mb\right) \right] \\
 & \leq \frac{2^{(\xi/k)}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right) {}_kI_{\psi^{-1}(a+mb/2)^+}^{\xi,\psi}(f \circ \psi)(\psi^{-1}(mb)) + g_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)m^{(\xi/k)+1} {}_kI_{\psi^{-1}(a+mb/2)^-}^{\xi,\psi}(f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\
 & \leq \frac{\xi}{k} \left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}} + \frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}} \right] \\
 & \int_0^h h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt + \frac{\xi m}{k} \left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}} + \frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] \\
 & \int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt - \frac{m\lambda\xi}{k} \left[\frac{h(1/2^\alpha)g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{mh((2^\alpha-1)/2^\alpha)g_3(\eta)(b-a/m^2)^2}{e^{\eta(a/m^2+b)}} \right] \\
 & \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt,
 \end{aligned} \tag{31}$$

where $\xi > 0$ and

$$\begin{aligned}
 g_1(\eta) &= \begin{cases} e^{(-\eta/2)(a(1+(1/m))+b(1+m))}, & \text{if } \eta > 0, \\ e^{-\eta(mb+(a/m))}, & \text{if } \eta < 0, \end{cases} \\
 g_2(\eta) &= \begin{cases} e^{-\eta mb}, & \text{if } \eta < 0, \\ e^{-\eta(a+mb/2)}, & \text{if } \eta > 0, \end{cases} \\
 g_3(\eta) &= \begin{cases} e^{-\eta(a+mb/2m)}, & \text{if } \eta < 0, \\ e^{(-\eta a/m)}, & \text{if } \eta > 0. \end{cases}
 \end{aligned} \tag{32}$$

Proof. By setting $x = (at/2) + m(2-t/2)b$ and $y = (a/m)(2-t/2) + (bt/2)$, $t \in [0, 1]$, in (19), multiplying the resulting inequality with $t^{(\xi/k)-1}$, and then integrating with respect to t , we get

$$\begin{aligned}
 & \frac{k}{\xi} f\left(\frac{a+mb}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right) \int_0^1 \frac{f((at/2) + m(1-(t/2))b)}{e^{\eta((at/2)+m(1-(t/2))b)}} t^{(\xi/k)-1} dt \\
 & + mh\left(\frac{2^\alpha-1}{2^\alpha}\right) \int_0^1 \frac{f((a/m)((2-t)/2) + (bt/2))}{e^{\eta((a/m)((2-t)/2)+(bt/2))}} t^{(\xi/k)-1} dt - m\lambda g_1(\eta)h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\
 & \times \int_0^1 \frac{(t(b-a) + (1-t)(a/m) - mb)^2}{e^{\eta((t/2)(a+b)+(1-t/2)(mb+(a/m)))}} t^{(\xi/k)-1} dt.
 \end{aligned} \tag{33}$$

Let

$$g_1(t) = e^{-\eta((t/2)(a+b)+(1-(t/2))(mb+(a/m)))},$$

$$g_1'(t) = \frac{\eta}{2}(1-m)\left(\frac{a}{m}-b\right)e^{-\eta((t/2)(a+b)+(1-(t/2))(mb+(a/m)))}. \quad (34)$$

Now one can see that g_1 will be increasing if $\eta < 0$ and decreasing if $\eta > 0$. Therefore, from inequality (33), we can have

$$\begin{aligned} \frac{k}{\xi} f\left(\frac{a+mb}{2}\right) &\leq h\left(\frac{1}{2^\alpha}\right) \int_0^1 \frac{f((at/2)+m(1-(t/2))b)}{e^{\eta((at/2)+m(1-(t/2))b)}} t^{(\xi/k)-1} dt \\ &+ mh\left(\frac{2^\alpha-1}{2^\alpha}\right) \int_0^1 \frac{f((a/m)((2-t)/2)+(bt/2))}{e^{\eta((a/m)((2-t)/2)+(bt/2))}} t^{((\xi/k)-1)} dt - m\lambda g_1(\eta) h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\times \frac{k}{4\xi(\xi+k)(\xi+2k)} \left[\xi(\xi+k)(b-a)^2 + (\xi^2+5k\xi+8k^2)\left(\frac{a}{m}-mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right) \right]. \end{aligned} \quad (35)$$

By setting $\psi(u) = (at/2) + m(2-t/2)b$ and $\psi(v) = (a/m)(2-t/2) + (bt/2)$ in (35), we get the following inequality:

$$\begin{aligned} \frac{k}{\xi} f\left(\frac{a+mb}{2}\right) &\leq \frac{2^{(\xi/k)}}{(mb-a)^{(\xi/k)}} \\ &\left[h\left(\frac{1}{2^\alpha}\right) \int_{\psi^{-1}(a+mb/2)}^{\psi^{-1}(mb)} \frac{f(\psi(u))}{e^{\eta(\psi(u))}} (mb-\psi(u))^{(\xi/k)-1} \psi'(u) du + m^{(\xi/k)+1} h\left(\frac{2^\alpha-1}{2^\alpha}\right) \int_{\psi^{-1}(a/m)}^{\psi^{-1}(a+mb/2m)} \frac{f(\psi(v))}{e^{\eta(\psi(v))}} \left(\psi(v)-\frac{a}{m}\right)^{(\xi/k)-1} \psi'(v) dv \right] \\ &- m\lambda g_1(\eta) h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\frac{k}{4\xi(\xi+k)(\xi+2k)} \left[\xi(\xi+k)(b-a)^2 + (\xi^2+5k\xi+8k^2)\left(\frac{a}{m}-mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right) \right]. \end{aligned} \quad (36)$$

Further, multiplying above inequality by (ξ/k) and using Definition 5, we get

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) &\leq \frac{2^{(\xi/k)} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \\ &\left[g_2(\eta) h\left(\frac{1}{2^\alpha}\right) {}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(mb)) + g_3(\eta) h\left(\frac{2^\alpha-1}{2^\alpha}\right) m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi,\psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &- h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) \frac{m\lambda g_1(\eta)}{4(\xi+k)(\xi+2k)} \\ &\left[\xi(\xi+k)(b-a)^2 + (\xi^2+5k\xi+8k^2)\left(\frac{a}{m}-mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m}-mb\right) \right]. \end{aligned} \quad (37)$$

The above inequality leads to the first inequality of (31). Again using strongly exponentially $(\alpha, h - m)$ -convexity of f , for $t \in [0, 1]$, we have

$$\begin{aligned} & g_2(\eta)h\left(\frac{1}{2^\alpha}\right)f\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) + mg_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)f\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) \\ & \leq h\left(\frac{t^\alpha}{2^\alpha}\right)\left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}} + \frac{mh(2^\alpha-1/2^\alpha)f(b)}{e^{\eta b}}\right] \\ & \quad + mh\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\left[\frac{h(1/2^\alpha)g_2(\eta)f(b)}{e^{\eta b}} + \frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(a/m^2)}{e^{\eta a/m^2}}\right] \\ & \quad - m\lambda h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)\left[\frac{h(1/2^\alpha)g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{h(2^\alpha-1/2^\alpha)g_3(\eta)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}}\right]. \end{aligned} \quad (38)$$

By integrating (38) over $[0, 1]$ after multiplying with $t^{(\xi/k)-1}$, the following inequality holds:

$$\begin{aligned} & g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 f\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right)t^{(\xi/k)-1}dt \\ & \quad + mg_3(\eta)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 f\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right)t^{(\xi/k)-1}dt \\ & \leq \left[\frac{h(1/2^\alpha)g_2(\eta)f(a)}{e^{\eta a}} + \frac{mh(2^\alpha-1/2^\alpha)g_3(\eta)f(b)}{e^{\eta b}}\right]\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt \\ & \quad + m\left[h(1/2^\alpha)g_2(\eta)f(b)/e^{\eta b} + mh(2^\alpha-1/2^\alpha)g_2(\eta)f(a/m^2)/e^{\eta a/m^2}\right]\int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt \\ & \quad - m\lambda\left[\frac{h(1/2^\alpha)g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{h(2^\alpha-1/2^\alpha)g_3(\eta)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}}\right]\int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{(\xi/k)-1}dt. \end{aligned} \quad (39)$$

Again, using substitutions as considered in (35), the above inequality leads to the second inequality of (31).

In the following remark, we give the connection of inequality (31) with already established results. $\square$

Remark 3

- (i) If we take $\eta = 0$ and $k = 1$ in (31), then the inequality stated in Theorem 5 in [37] is obtained
- (ii) If we take $\eta = 0$ and $\alpha = 1$ in (31), then the inequality stated in Corollary 3 in [37] is obtained
- (iii) If we take $\alpha = 1$, $h(t) = t$, $m = 1$, $\lambda = 0$, $k = 1$, $\eta = 0$, and ψ as the identity function in (31), then Theorem 2 is obtained

- (iv) If we take $\alpha = 1$, $m = 1$, $h(t) = t$, $\lambda = 0$, $\eta = 0$, and ψ as the identity function in inequality (31), then refinement of Theorem 2 is obtained
- (v) If we take $h(t) = t$, $m = 1$, $\lambda = 0$, $k = 1$, $\xi = 1$, $\eta = 0$, $\alpha = 1$, and ψ as the identity function in (31), then the Hadamard inequality is obtained
- (vi) If we take $h(t) = t$, $m = 1$, $\alpha = 1$, and $\eta = 0$ in (31), then the inequality stated in Theorem 11 in [32] is obtained
- (vii) If we take $\alpha = 1$, $h(t) = t$, $k = 1$, $\lambda = 0$, $\eta = 0$, and ψ as the identity function in (31), then the inequality stated in Theorem 2.1 in [30] is obtained
- (viii) If we take $\lambda = 0$, $\alpha = 1$, $h(t) = t^\xi$, and $\eta = 0$ in (31), then the inequality stated in Corollary 3 in [35] is obtained

(ix) If we take $\lambda = 0$ in (31), then the inequality stated in Theorem 8 in [38] is obtained

Now, we give inequality (31) for strongly exponentially $(h-m)$ -convex, strongly exponentially (s, m) -convex, strongly exponentially m -convex, and strongly exponentially convex functions.

Corollary 5. If we take $\alpha = 1$ in (31), then the following inequality holds for strongly exponentially $(h-m)$ -convex functions:

$$\begin{aligned}
 & f\left(\frac{a+mb}{2}\right) + \frac{mh^2(1/2)\lambda g_1(\eta)}{4(\xi+k)(\xi+2k)} \left[\xi(\xi+k)(b-a)^2 + (\xi^2 + 5k\xi + 8k^2)\left(\frac{a}{m} - mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m} - mb\right) \right] \\
 & \leq \frac{h(1/2)2^{(\xi/k)}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[g_2(\eta)_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + g_3(\eta)m^{(\xi/k)+1} I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\
 & \leq \frac{h(1/2)\xi}{k} \left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + m \frac{g_3(\eta)f(b)}{e^{\eta b}} \right] \times \int_0^1 h\left(\frac{t}{2}\right) t^{(\xi/k)-1} dt \\
 & \quad + \frac{h(1/2)\xi m}{k} \left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] \int_0^1 h\left(\frac{2-t}{2}\right) t^{(\xi/k)-1} dt \\
 & \quad - m\lambda h\left(\frac{1}{2}\right) \left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{g_3(\eta)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right] \int_0^1 h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) t^{(\xi/k)-1} dt.
 \end{aligned} \tag{40}$$

Corollary 6. If we take $\alpha = 1$ and $h(t) = t^s$ in (31), then the following inequality holds for strongly exponentially (s, m) -convex functions:

$$\begin{aligned}
 & f\left(\frac{a+mb}{2}\right) + \frac{2^{-2s}m\lambda g_1(\eta)}{4(\xi+2k)(\xi+k)} \left[\xi(\xi+k)(b-a)^2 + (\xi^2 + 5k\xi + 8k^2)\left(\frac{a}{m} - mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m} - mb\right) \right] \\
 & \leq \frac{2^{(\xi/k)-s}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[g_2(\eta)_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + g_3(\eta)m^{(\xi/k)+1} I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\
 & \leq \frac{\xi}{2^{2s}(\xi+k)} \left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{g_3(\eta)f(b)}{e^{\eta b}} \right] + m {}_2F_1\left(-s, \frac{\xi}{k}, \frac{\xi}{k} + 1; \frac{1}{2}\right) \left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] \\
 & \quad - \frac{2^{(\xi/k)-s}m\lambda \xi B(1/2, s+\xi/k, 1+s)}{k} \times \left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{g_3(\eta)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right].
 \end{aligned} \tag{41}$$

Corollary 7. If we take $\alpha = 1$ and $h(t) = t$ in (31), then the following inequality holds for strongly exponentially m -convex functions:

$$\begin{aligned}
& f\left(\frac{a+mb}{2}\right) + \frac{m\lambda g_1(\eta)}{16(\xi+2k)(\xi+k)} \left[\xi(\xi+k)(b-a)^2 + (\xi^2 + 5k\xi + 8k^2)\left(\frac{a}{m} - mb\right)^2 + 2\xi(\xi+3k)(b-a)\left(\frac{a}{m} - mb\right) \right] \\
& \leq \frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[g_2(\eta)_k I_{\psi^{-1}(a+mb/2)^+}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(mb)) + g_3(\eta)m^{(\xi/k)+1} I_{\psi^{-1}(a+mb/2)^-}^{\xi,\psi} (f \circ \psi)(\psi^{-1}\left(\frac{a}{m}\right)) \right] \\
& \leq \frac{\xi}{4(\xi+k)} \left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{mg_3(\eta)f(b)}{e^{\eta b}} \right] + \frac{m(\xi+2k)}{4(\xi+k)} \left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{mg_3(\eta)f(a/m^2)}{e^{\eta a/m^2}} \right] \\
& \quad - \frac{m\lambda\xi(\xi+3k)}{8(\xi+k)(\xi+2k)} \left[\frac{g_2(\eta)(b-a)^2}{e^{\eta(a+b)}} + \frac{g_3(\eta)(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right].
\end{aligned} \tag{42}$$

Corollary 8. If we take $\alpha = 1$, $m = 1$, and $h(t) = t$ in (31), then the following inequality holds for strongly exponentially convex functions:

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) + \frac{\lambda k^2 g_1(\eta)}{2(\xi+2k)(\xi+k)} \\
& \leq \frac{2^{(\xi/k)-1}\Gamma_k(\xi+k)}{(b-a)^{(\xi/k)}} \left[g_2(\eta)_k I_{\psi^{-1}(a+b/2)^+}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(b)) + g_3(\eta)_k I_{\psi^{-1}(a+b/2)^-}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(a)) \right], \\
& \leq \frac{\xi}{4(\xi+k)} \left[\frac{g_2(\eta)f(a)}{e^{\eta a}} + \frac{g_3(\eta)f(b)}{e^{\eta b}} \right] + \frac{(\xi+2k)}{4(\xi+k)} \left[\frac{g_2(\eta)f(b)}{e^{\eta b}} + \frac{g_3(\eta)f(a)}{e^{\eta a}} \right] \\
& \quad - \frac{\lambda(b-a)^2\xi(\xi+3k)(g_2(\eta) + g_3(\eta))}{8(\xi+k)(\xi+2k)e^{\eta(a+b)}}.
\end{aligned} \tag{43}$$

3. Error Estimations of Hadamard Inequalities for Strongly Exponentially $(\alpha, h-m)$ -Convex Functions

This section deals with error bounds of Hadamard inequalities for strongly exponentially $(\alpha, h-m)$ -convex functions using generalized Riemann–Liouville fractional integrals. Estimations obtained here provide refinements of

several well-known inequalities for different types of convexity. The following identity is used to prove the next theorem.

Lemma 1 (see [26]). Let $a < b$ and $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) . Also, suppose that $f' \in L_1[a, b]$. Then, for $k > 0$, the following identity holds for the operators given in (14) and (15):

$$\begin{aligned}
& \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\xi+k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\xi,\psi} (f \circ \psi)(\psi^{-1}(a)) \right] \\
& = \frac{b-a}{2} \int_0^1 \left((1-t)^{(\xi/k)} - t^{(\xi/k)} \right) f'(ta + (1-t)b) dt.
\end{aligned} \tag{44}$$

Theorem 9. Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) such that $f' \in L_1[a, b]$. Also, suppose that $|f'|$ is strongly exponentially $(\alpha, h-m)$ -convex on $[a, b]$. Then, for

$k > 0$ and $\alpha, m \in (0, 1]^2$, the following k -fractional integral inequality holds for the operators given in (14) and (15):

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\xi + k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\
& \leq \frac{b-a}{2} \left[\frac{|f'(a)|}{e^{\eta a}} \left(\int_0^{(1/2)} h(t^\alpha) ((1-t)^{(\xi/k)} - t^{(\xi/k)}) dt + \int_{(1/2)}^1 h(t^\alpha) (t^{(\xi/k)} - (1-t)^{(\xi/k)}) dt \right) \right. \\
& \quad + \frac{m|f'(b/m)|}{e^{\eta b/m}} \left(\int_0^{(1/2)} h(1-t^\alpha) ((1-t)^{(\xi/k)} - t^{(\xi/k)}) dt + \int_{(1/2)}^1 h(1-t^\alpha) (t^{(\xi/k)} - (1-t)^{(\xi/k)}) dt \right) \\
& \quad \left. - \frac{m\lambda(b-a)^2}{e^{\eta(a+b)}} \left(\int_0^{(1/2)} h(t^\alpha) h(1-t^\alpha) ((1-t)^{(\xi/k)} - t^{(\xi/k)}) dt - \int_{(1/2)}^1 h(t^\alpha) h(1-t^\alpha) (t^{(\xi/k)} - (1-t)^{(\xi/k)}) dt \right) \right], \quad (45)
\end{aligned}$$

where $\xi > 0$.

Proof. From Lemma 1, it follows that

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\xi + k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\
& \leq \frac{b-a}{2} \int_0^1 |(1-t)^{(\xi/k)} - t^{(\xi/k)}| |f'(ta + (1-t)b)| dt. \quad (46)
\end{aligned}$$

By using strongly exponentially $(\alpha, h-m)$ -convexity of $|f'|$ and for $t \in [0, 1]$, we have

$$|f'(ta + (1-t)b)| \leq h(t^\alpha) \frac{|f'(a)|}{e^{\eta a}} + mh(1-t^\alpha) \frac{|f'(b/m)|}{e^{\eta b/m}} - \frac{m\lambda h(t^\alpha) h(1-t^\alpha)}{e^{\eta(a+(b/m))}} \left(\frac{b}{m} - a \right)^2. \quad (47)$$

Using (47) in (46), we get

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\xi + k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\
& \leq \frac{b-a}{2} \int_0^1 |(1-t)^{(\xi/k)} - t^{(\xi/k)}| \left[h(t^\alpha) \frac{|f'(a)|}{e^{\eta a}} + mh(1-t^\alpha) \frac{|f'(b/m)|}{e^{\eta b/m}} \right. \\
& \quad \left. - \frac{m\lambda h(t^\alpha) h(1-t^\alpha)}{e^{\eta(a+(b/m))}} \left(\frac{b}{m} - a \right)^2 \right] dt = \frac{b-a}{2} \left[\frac{|f'(a)|}{e^{\eta a}} \left(\int_0^{(1/2)} h(t^\alpha) ((1-t)^{(\xi/k)} - t^{(\xi/k)}) dt \right. \right. \\
& \quad + \int_{(1/2)}^1 h(t^\alpha) (t^{(\xi/k)} - (1-t)^{(\xi/k)}) dt \Big) + \frac{m|f'(b/m)|}{e^{\eta b/m}} \left(\int_0^{(1/2)} h(1-t^\alpha) ((1-t)^{(\xi/k)} - t^{(\xi/k)}) dt \right. \\
& \quad + \int_{(1/2)}^1 h(1-t^\alpha) (t^{(\xi/k)} - (1-t)^{(\xi/k)}) dt \Big) - \frac{m\lambda(b-a)^2}{e^{\eta(a+(b/m))}} \left(\int_0^{(1/2)} h(t^\alpha) h(1-t^\alpha) ((1-t)^{(\xi/k)} - t^{(\xi/k)}) dt \right. \\
& \quad \left. \left. + \int_{(1/2)}^1 h(t^\alpha) h(1-t^\alpha) (t^{(\xi/k)} - (1-t)^{(\xi/k)}) dt \right) \right]. \quad (48)
\end{aligned}$$

In the following remark, we give the connection of inequality (45) with already established results. $\square$

Remark 4

- (i) If we take $\eta = 0$ and $k = 1$ in (45), then the inequality stated in Theorem 6 in [37] is obtained
- (ii) If we take $\eta = 0$ and $\alpha = 1$ in (45), then the inequality stated in Corollary 7 in [37] is obtained
- (iii) If we take $h(t) = t$, $m = 1$, $\alpha = 1$, and $\eta = 0$ in (45), then the inequality stated in Theorem 12 in [32] is obtained
- (iv) If we take $m = 1$, $\alpha = 1$, $\eta = 0$, $\lambda = 0$, and $h(t) = t^s$ in (45), then the inequality stated in Theorem 2 in [26] is obtained
- (v) If we take $m = 1$, $\alpha = 1$, $\lambda = 0$, $h(t) = t$, $\eta = 0$, and ψ as the identity function in (45), then Theorem 6 is obtained
- (vi) If we take $k = 1$, $m = 1$, $\alpha = 1$, $\lambda = 0$, $h(t) = t$, $\eta = 0$, and ψ as the identity function in (45), then Theorem 3 is obtained

(vii) If we take $k = 1$, $\alpha = 1$, $\xi = 1$, $\lambda = 0$, $h(t) = t$, $m = 1$, $\eta = 0$, and ψ as the identity function in (45), then the inequality stated in Theorem 2.2 in [27] is obtained

(viii) If we take $\eta = 0$, $\alpha = 1$, $\lambda = 0$, and $h(t) = t^s$ in (45), then the inequality stated in Corollary 5 in [35] is obtained

(ix) If we take $\alpha = 1$, $h(t) = t$, $m = 1$, and $\eta = 0$ in (45), then the inequality stated in Theorem 12 in [32] is obtained

(x) If we take $\lambda = 0$ in (45), then the inequality stated in Theorem 9 in [38] is obtained

Now, we give inequality (45) for strongly exponentially $(h - m)$ -convex, strongly exponentially m -convex, and strongly exponentially convex functions.

Corollary 9. *If we take $\alpha = 1$ in (45), then the following inequality holds for strongly exponentially $(h - m)$ -convex functions:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\xi + k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\ & \leq \frac{b-a}{2} \left[\frac{|f'(a)|}{e^{\eta a}} \left(\int_0^{(1/2)} h(t) \left((1-t)^{(\xi/k)} - t^{(\xi/k)} \right) dt + \int_{(1/2)}^1 h(t) \left(t^{(\xi/k)} - (1-t)^{(\xi/k)} \right) dt \right) \right. \\ & \quad + \frac{m|f'(b/m)|}{e^{\eta b/m}} \left(\int_0^{(1/2)} h(1-t) \left((1-t)^{(\xi/k)} - t^{(\xi/k)} \right) dt + \int_{(1/2)}^1 h(1-t) \left(t^{(\xi/k)} - (1-t)^{(\xi/k)} \right) dt \right) \\ & \quad \left. - \frac{m\lambda(b-a)^2}{e^{\eta(a+b)}} \left(\int_0^{(1/2)} h(t)h(1-t) \left((1-t)^{(\xi/k)} - t^{(\xi/k)} \right) dt - \int_{(1/2)}^1 h(t)h(1-t) \left(t^{(\xi/k)} - (1-t)^{(\xi/k)} \right) dt \right) \right]. \end{aligned} \quad (49)$$

Corollary 10. *If we take $\alpha = 1$ and $h(t) = t$ in (45), then the following inequality holds for strongly exponentially m -convex functions:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\xi + k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right| \\ & \leq \frac{b-a}{2((\xi/k) + 1)} \left(1 - \frac{1}{2^{(\xi/k)}} \right) \left(\frac{|f'(a)|}{e^{\eta a}} + \frac{m|f'(b/m)|}{e^{\eta b/m}} \right) - \frac{\lambda((b/m) - a)^3 (1 - ((\xi/k) + 4)/2^{((\xi/k)+2)})}{e^{\eta((b/m)+a)} ((\xi/k) + 2)((\xi/k) + 3)}. \end{aligned} \quad (50)$$

Corollary 11. *If we take $\alpha = 1$, $m = 1$, and $h(t) = t$ in (45), then the following inequality holds for strongly exponentially convex functions:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\xi + k)}{2(b-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(b)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] \right|$$

$$\leq \frac{b-a}{2((\xi/k)+1)} \left(1 - \frac{1}{2^{(\xi/k)}} \right) \left(\frac{|f'(a)|}{e^{\eta a}} + \frac{|f'(b)|}{e^{\eta b}} \right) - \frac{\lambda(b-a)^3 (1 - ((\xi/k)+4)/(2^{((\xi/k)+2)}))}{e^{\eta(b+a)} ((\xi/k)+2)((\xi/k)+3)}.$$

The following integral identity is useful to get our next theorem.

Lemma 2 (see [35]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) such that $f' \in L_1[a, b]$. Then, for $k > 0$ and $m \in (0, 1]$, the following integral identity holds for operators given in (14) and (15):

$$\frac{2^{(\xi/k)-1} \Gamma_k(\xi + k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(\frac{a}{m})) \right]$$

$$- \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + mf\left(\frac{a+mb}{2m}\right) \right] = \frac{mb-a}{4} \left[\int_0^1 t^{(\xi/k)} f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) dt - \int_0^1 t^{(\xi/k)} f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) dt \right].$$

Theorem 10. Let $f: [a, b] \rightarrow \mathbb{R}$, $[a, b] \subset [0, b]$, be a differentiable mapping on (a, b) such that $f' \in L_1[a, b]$. Also suppose that $|f'|^q$ is a strongly exponentially

$(\alpha, h-m)$ -convex function on $[a, b]$ for $q \geq 1$. Then, for $k > 0$ and $(\alpha, m) \in (0, 1]^2$, the following fractional integral inequality holds for operators given in (14) and (15):

$$\left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi + k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(\frac{a}{m})) \right] \right.$$

$$\left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + mf\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4((\xi/k)+1)^{1-(1/q)}} \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right. \right.$$

$$\left. + \frac{m|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt - \frac{m\lambda(b-a)^2}{e^{\eta(a+b)}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right)^{(1/q)}$$

$$+ \left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}} \int_0^1 h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt + \frac{m|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right.$$

$$\left. - \frac{m\lambda(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right)^{(1/q)}.$$

Proof. We divide the proof in two cases:

Case 1 (for $q = 1$). Applying Lemma 2 and using strongly exponentially $(\alpha, h-m)$ -convexity of $|f'|$, we have

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4} \left[\int_0^1 t^{(\xi/k)} f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) dt \right. \\
& \left. + \int_0^1 t^{(\xi/k)} f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) dt \right] \leq \frac{mb-a}{4} \left[\left(\frac{|f'(a)|}{e^{\eta a}} + \frac{|f'(b)|}{e^{\eta b}} \right) \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right. \\
& \left. + m \left(\frac{|f'(b)|}{e^{\eta b}} + \frac{|f'(a/m^2)|}{e^{\eta a/m^2}} \right) \int_0^1 h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt - m\lambda \left(\frac{(b-a)^2}{e^{\eta(a+b)}} + \frac{(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right) \right. \\
& \left. \times \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) dt \right].
\end{aligned} \tag{54}$$

Case 2 (for $q > 1$). From Lemma 2 and using power mean inequality, we get

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4} \left(\int_0^1 t^{(\xi/k)} dt \right)^{1-(1/q)} \\
& \left[\left(\int_0^1 t^{(\xi/k)} \left| f'\left(\frac{at}{2} + m\left(\frac{2-t}{2}\right)b\right) \right|^q dt \right)^{(1/q)} + \left(\int_0^1 t^{(\xi/k)} \left| f'\left(\frac{a}{m}\left(\frac{2-t}{2}\right) + \frac{bt}{2}\right) \right|^q dt \right)^{(1/q)} \right] \\
& \leq \frac{mb-a}{4((\xi/k)+1)^{1-(1/q)}} \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt + \frac{m|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right. \right. \\
& \left. \left. - \frac{m\lambda(b-a)^2}{e^{\eta(a+b)}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right)^{(1/q)} + \left(\left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}} \int_0^1 h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right. \right. \right. \\
& \left. \left. \left. + \frac{|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt - \frac{m\lambda(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) t^{(\xi/k)} dt \right)^{(1/q)} \right].
\end{aligned} \tag{55}$$

Hence, equation (53) is obtained.

In the following remark, we give the connection of inequality (53) with already established results. $\square$

Remark 5

- (i) If we take $\lambda = 0$ in (53), then the inequality stated in Theorem 10 in [38] is obtained
- (ii) If we take $\eta = 0, \alpha = 1, \lambda = 0$, and $h(t) = t^\xi$ in (53), then the inequality stated in Corollary 7 in [35] is obtained
- (iii) If we take $h(t) = t, k = 1, \eta = 0, \alpha = 1, \lambda = 0$, and ψ as the identity function in (53), then the inequality stated in Theorem 2.4 in [30] is obtained

(iv) If we take $\eta = 0, h(t) = t, m = 1, \alpha = 1$, and ψ as the identity function in (53), then the inequality stated in Theorem 3.1 in [20] is obtained

(v) If we take $h(t) = t, m = 1, \eta = 0, k = 1, \alpha = 1, \lambda = 0$, and ψ as the identity function in (53), then the inequality stated in Theorem 5 in [16] is obtained

(vi) If we take $q = 1, h(t) = t, m = 1, \eta = 0, k = 1, \xi = 1, \alpha = 1, \lambda = 0$, and ψ as the identity function in (53), then the inequality stated in Theorem 2.2 in [33] is obtained

(vii) If we take $\eta = 0, \alpha = 1, m = 1$, and $h(t) = t$ in (53), then the inequality stated in Theorem 13 in [32] is obtained

(viii) If we take $\eta = 0$ and $k = 1$ in (53), then the inequality stated in Theorem 7 in [37] is obtained

(ix) If we take $\eta = 0$, $k = 1$, and $\alpha = 1$ in (53), then the inequality stated in Corollary 10 in [37] is obtained

Now, we give inequality (53) for strongly exponentially $(h - m)$ -convex, strongly exponentially (s, m) -convex,

strongly exponentially m -convex, and strongly exponentially convex functions.

Corollary 12. *If we take $\alpha = 1$ in (53), then the following inequality holds for strongly exponentially $(h - m)$ -convex functions:*

$$\begin{aligned} & \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi + k)}{(mb - a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\ & \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb - a}{4((\xi/k) + 1)^{1-(1/q)}} \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} \int_0^1 h\left(\frac{t}{2}\right) t^{(\xi/k)} dt \right. \right. \\ & + \frac{m |f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{2-t}{2}\right) t^{(\xi/k)} dt - \frac{m \lambda (b-a)^2}{e^{\eta(a+b)}} \int_0^1 h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) t^{(\xi/k)} dt \Big)^{1/q} \\ & + \left(\frac{m |f'(a/m^2)|^q}{e^{\eta a/m^2}} \int_0^1 h\left(\frac{2-t}{2}\right) t^{(\xi/k)} dt + \frac{|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{t}{2}\right) t^{(\xi/k)} dt - \frac{m \lambda (b - (a/m^2))^2}{e^{\eta((a/m^2)+b)}} \right. \\ & \left. \left. \times \int_0^1 h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) t^{(\xi/k)} dt \right)^{1/q} \right]. \end{aligned} \quad (56)$$

Corollary 13. *If we take $\alpha = 1$ and $h(t) = t^s$ in (53), then the following inequality holds for strongly exponentially (s, m) -convex functions:*

$$\begin{aligned} & \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi + k)}{(mb - a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\ & \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb - a}{4((\xi/k) + 1)(2^s((\xi/k) + s + 1))^{(1/q)}} \left[\left(\frac{((\xi/k) + 1) |f'(a)|^q}{e^{\eta a}} \right. \right. \\ & + \frac{2^s m |f'(b)|^q ((\xi/k) + s + 1) {}_2F_1(-s, 1 + (\xi/k), 2 + (\xi/k); (1/2))}{e^{\eta b}} \\ & - \frac{2^{1+s+(\mu/k)} m \lambda (b-a)^2 B((1/2), 1 + s + (\mu/k), 1 + s)((\xi/k) + 1)((\xi/k) + s + 1)}{e^{\eta(a+b)}} \Big)^{(1/q)} \\ & + \left(\frac{2^s m {}_2F_1(-s, 1 + (\xi/k), 2 + (\xi/k); (1/2))((\xi/k) + s + 1) |f'(a/m^2)|^q}{e^{\eta a/m^2}} + \frac{((\xi/k) + 1) |f'(b)|^q}{e^{\eta b}} \right. \\ & \left. \left. - \frac{2^{1+s+(\mu/k)} m \lambda (b-a)^2 B((1/2), 1 + s + (\mu/k), 1 + s)((\xi/k) + 1)((\xi/k) + s + 1)}{e^{\eta(a+b)}} \right)^{(1/q)} \right]. \end{aligned} \quad (57)$$

Corollary 14. *If we take $\alpha = 1$, $m = 1$, and $h(t) = t$ in (53), then the following inequality holds for strongly exponentially m -convex functions:*

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(b-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{b-a}{4((\xi/k)+1)(2((\xi/k)+2))^{(1/q)}} \left[\left(\frac{((\xi/k)+1)|f'(a)|^q}{e^{\eta a}} + \frac{|f'(b)|^q((\xi/k)+3)}{e^{\eta b}} - \frac{\lambda(b-a)^2((\xi/k)+1)((\xi/k)+4)}{2e^{\eta(a+b)}((\xi/k)+3)} \right)^{(1/q)} \right. \\
& \quad \left. + \left(\frac{((\xi/k)+3)|f'((a/m^2))|^q}{e^{\eta a/m^2}} + \frac{((\xi/k)+1)|f'(b)|^q}{e^{\eta b}} - \frac{\lambda(b-a)^2((\xi/k)+1)((\xi/k)+4)}{e^{\eta(a+b)}((\xi/k)+3)} \right)^{(1/q)} \right].
\end{aligned} \tag{58}$$

Corollary 15. If we take $\alpha = 1$, $k = 1$, $m = 1$, $q = 1$, $\xi = 1$, $h(t) = t$, and ψ as the identity function in (53), then the following inequality is obtained:

$$\left| \frac{1}{(b-a)} \int_0^1 f(\xi) d\xi - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left[\frac{|f'(a)|}{e^{\eta a}} + \frac{|f'(b)|}{e^{\eta b}} - \frac{5\lambda(b-a)^2}{12e^{\eta(a+b)}} \right]. \tag{59}$$

Theorem 11. Let $f: [a, b] \rightarrow \mathbb{R}$, $[a, b] \subset [0, b]$, be a differentiable mapping on (a, b) with $a < b$. Also, suppose that $|f'|^q$ is the strongly exponentially $(\alpha, h-m)$ -convex function

on $[a, b]$ for $q > 1$. Then, for $k > 0$ and $(\alpha, m) \in (0, 1]^2$, the following fractional integral inequality holds for the operators given in (14) and (15):

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4((\xi p/k)+1)^{(1/p)}} \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) dt \right. \right. \\
& \quad \left. \left. + \frac{m|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right) dt - \frac{m\lambda(b-a)^2}{e^{\eta(a+b)}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right) dt \right)^{(1/q)} \right. \\
& \quad \left. + \left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}} \int_0^1 h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right) dt + \frac{|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) dt \right. \right. \\
& \quad \left. \left. - \frac{m\lambda(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right) dt^{(1/q)} \right) \right],
\end{aligned} \tag{60}$$

with $(1/p) + (1/q) = 1$.

Proof. Applying Lemma 2 and using the property of modulus, we get

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \right. \\
& \quad \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4} \left[\int_0^1 t^{(\xi/k)} f'\left(\frac{at}{2} + m\left(1-\frac{t}{2}\right)b\right) dt \right. \\
& \quad \left. + \int_0^1 t^{(\xi/k)} f'\left(\frac{a}{m}\left(1-\frac{t}{2}\right) + \frac{bt}{2}\right) dt \right].
\end{aligned} \tag{61}$$

Now applying Hölder's inequality for integrals, we get

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}(mb) \right) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}\left(\frac{a}{m}\right) \right) \right] \right. \\
& \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4((\xi p/k)+1)^{(1/p)}} \left[\left(\int_0^1 \left| f' \left(\frac{at}{2} + m \left(\frac{2-t}{2} \right) b \right) \right|^q dt \right)^{(1/q)} \right. \\
& \left. + \left(\int_0^1 \left| f' \left(\frac{a}{m} \left(\frac{2-t}{2} \right) + \frac{bt}{2} \right) \right|^q dt \right)^{(1/q)} \right].
\end{aligned} \tag{62}$$

Using strongly exponentially $(\alpha, h-m)$ -convexity of $|f'|^q$, we get

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}(mb) \right) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}\left(\frac{a}{m}\right) \right) \right] \right. \\
& \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4((\xi p/k)+1)^{(1/p)}} \\
& \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) dt + \frac{m|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(1 - \frac{t^\alpha}{2^\alpha}\right) dt - \frac{m\lambda(b-a)^2}{e^{\eta(a+b)}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) dt \right)^{(1/q)} \right. \\
& \left. + \left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}} \int_0^1 h\left(1 - \frac{t^\alpha}{2^\alpha}\right) dt + \frac{|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) dt - \frac{m\lambda(b - (a/m^2))^2}{e^{\eta((a/m^2)+b)}} \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha - t^\alpha}{2^\alpha}\right) dt \right)^{(1/q)} \right].
\end{aligned} \tag{63}$$

In the following remark, we give the connection of inequality (60) with already established results. $\square$

Remark 6

- (i) If we take $\lambda = 0$ in (60), then the inequality stated in Theorem 11 in [38] is obtained
- (ii) If we take $\eta = 0$ and $k = 1$ in (60), then the inequality stated in Theorem 8 in [37] is obtained
- (iii) If we take $\eta = 0$ and $\alpha = 1$ in (60), then the inequality stated in Corollary 12 in [37] is obtained
- (iv) If we take $h(t) = t$, $k = 1$, $\eta = 0$, $\alpha = 1$, $\lambda = 0$, and ψ as the identity function in (60), then the inequality stated in Theorem 2.7 in [30] is obtained
- (v) If we take $\eta = 0$, $\lambda = 0$, $h(t) = t$, $\alpha = 1$, $m = 1$, and ψ as the identity function in (60), then the inequality stated in Theorem 3.2 in [20] is obtained

(vi) If we take $k = 1$, $h(t) = t$, $m = 1$, $\eta = 0$, $\xi = 1$, $\alpha = 1$, $\lambda = 0$, and ψ as the identity function in (60), then the inequality stated in Theorem 2.4 in [33] is obtained

(vii) If we take $\eta = 0$, $\alpha = 1$, and $h(t) = t^s$ in (60), then the inequality stated in Corollary 9 in [35] is obtained

(viii) If we take $\alpha = 1$, $m = 1$, $h(t) = t$, and $\eta = 0$ in (60), then the inequality stated in Theorem 14 in [32] is obtained

Now, we give inequality (60) for strongly exponentially $(h-m)$ -convex, strongly exponentially (s, m) -convex, strongly exponentially m -convex, and strongly exponentially convex functions.

Corollary 16. *If we take $\alpha = 1$ in (60), then the following inequality holds for strongly exponentially $(h-m)$ -convex functions:*

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}(mb) \right) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}\left(\frac{a}{m}\right) \right) \right] \right. \\
& \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4((\xi p/k)+1)^{(1/p)}} \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} \int_0^1 h\left(\frac{t}{2}\right) dt \right. \right. \\
& \left. \left. + \frac{m|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{2-t}{2}\right) dt - \frac{m\lambda(b-a)^2}{e^{\eta(a+b)}} \int_0^1 h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) dt \right)^{(1/q)} \right. \\
& \left. + \left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}} \int_0^1 h\left(\frac{2-t}{2}\right) dt + \frac{|f'(b)|^q}{e^{\eta b}} \int_0^1 h\left(\frac{t}{2}\right) dt \right. \right. \\
& \left. \left. - \frac{m\lambda(b-(a/m^2))^2}{e^{\eta((a/m^2)+b)}} \int_0^1 h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) dt \right)^{(1/q)} \right].
\end{aligned} \tag{64}$$

Corollary 17. If we take $\alpha = 1$ and $h(t) = t^s$ in (60), then the following inequality holds for strongly exponentially (s, m) -convex functions:

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}(mb) \right) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}\left(\frac{a}{m}\right) \right) \right] \right. \\
& \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{4(1+s)^{(1/q)}((\xi p/k)+1)^{(1/p)}} \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} \right. \right. \\
& \left. \left. + \frac{2^s m |f'(b)|^q (-1+2^{1+s})}{2^s e^{\eta b}} - \frac{2^{s+1}(1+s)mB((1/2), 1+s, 1+s)\lambda(b-a)^2}{e^{\eta(a+b)}} \right)^{(1/q)} + \left(\frac{m|f'(a/m^2)|^q}{e^{\eta a/m^2}} \right. \right. \\
& \left. \left. + \frac{|f'(b)|^q (-1+2^{1+s})}{2^s e^{\eta b}} - \frac{2^{s+1}m(1+s)B((1/2), 1+s, 1+s)\lambda(b-(a/m^2))^2}{e^{\eta(a+b)}} \right)^{(1/q)} \right].
\end{aligned} \tag{65}$$

Corollary 18. If we take $\alpha = 1$ and $h(t) = t$ in (60), then the following inequality holds for strongly exponentially m -convex functions:

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+mb/2)^+}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}(mb) \right) + m^{(\xi/k)+1} {}_k I_{\psi^{-1}(a+mb/2)^-}^{\xi, \psi} (f \circ \psi) \left(\psi^{-1}\left(\frac{a}{m}\right) \right) \right] \right. \\
& \left. - \frac{1}{2} \left[f\left(\frac{a+mb}{2}\right) + m f\left(\frac{a+mb}{2m}\right) \right] \right| \leq \frac{mb-a}{16} \left(\frac{4}{(\xi p/k)+1} \right)^{(1/p)} \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} + \frac{3m|f'(b)|^q}{e^{\eta b}} \right. \right. \\
& \left. \left. - \frac{2m\lambda(b-a)^2}{3e^{\eta(a+b)}} \right)^{(1/q)} + \left(\frac{3m|f'(a/m^2)|^q}{e^{\eta a/m^2}} + \frac{|f'(b)|^q}{e^{\eta b}} - \frac{2m\lambda(b-(a/m^2))^2}{3e^{\eta(a+b)}} \right)^{(1/q)} \right].
\end{aligned} \tag{66}$$

Corollary 19. If we take $\alpha = 1$, $m = 1$, and $h(t) = t$ in (60), then the following inequality holds for strongly exponentially convex functions:

$$\begin{aligned}
& \left| \frac{2^{(\xi/k)-1} \Gamma_k(\xi+k)}{(mb-a)^{(\xi/k)}} \left[{}_k I_{\psi^{-1}(a+b/2)^+}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(b)) + {}_k I_{\psi^{-1}(a+b/2)^-}^{\xi, \psi} (f \circ \psi)(\psi^{-1}(a)) \right] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{b-a}{16} \left(\frac{4}{(\xi p/k) + 1} \right)^{(1/p)} \left[\left(\frac{|f'(a)|^q}{e^{\eta a}} + \frac{3|f'(b)|^q}{e^{\eta b}} - \frac{2\lambda(b-a)^2}{3e^{\eta(a+b)}} \right)^{(1/q)} + \left(\frac{3|f'(a)|^q}{e^{\eta a}} + \frac{f'(b)}{e^{\eta b}} \right. \right. \\
& \quad \left. \left. - \frac{2\lambda(b-(a/m^2))^2}{3e^{\eta(a+b)}} \right)^{(1/q)} \right]. \tag{67}
\end{aligned}$$

4. Conclusion

In this paper, we have proved fractional versions of the Hadamard inequality and their estimations for strongly exponentially $(\alpha, h-m)$ -convex functions via generalized Riemann–Liouville fractional integrals. The outcomes of this article give refinements and generalizations of fractional integral inequalities for different types of convex functions deducible from the definition of the exponentially $(\alpha, h-m)$ -convex function. [39–41].

Data Availability

No data were used for the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

- [1] U. N. Katugampola, “New approach to a generalized fractional integral,” *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 860–865, 2011.
- [2] T. U. Khan and M. A. Khan, “Generalized conformable fractional operators,” *Journal of Computational and Applied Mathematics*, vol. 346, pp. 378–389, 2019.
- [3] H. Chen and U. N. Katugampola, “Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals,” *Journal of Mathematical Analysis and Applications*, vol. 446, no. 2, pp. 1274–1291, 2017.
- [4] H. Ge-Jile, S. Rashid, S. Rashid, M. Suhail, and Y.-M. Chu, “Some unified bounds for exponentially TGS-convex functions governed by conformable fractional operators,” *AIMS Mathematics*, vol. 5, no. 6, pp. 6108–6123, 2020.
- [5] T. U. Khan and M. A. Khan, “Hermite-Hadamard inequality for new generalized conformable fractional operators,” *AIMS Mathematics*, vol. 6, no. 1, pp. 23–38, 2021.
- [6] G. Farid, A. U. Rehman, S. Bibi, and Y.-M. Chu, “Refinements of two fractional versions of Hadamard inequalities for Caputo fractional derivatives and related results,” *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 1–10, 2021.
- [7] M. Jleli and B. Samet, “On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function,” *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 3, pp. 1252–1260, 2016.
- [8] S. Mehmood and G. Farid, “Fractional integrals inequalities for exponentially m -convex functions,” *Open Journal of Mathematical Sciences*, vol. 4, no. 1, pp. 78–85, 2020.
- [9] M. E. Özdemir, A. A. Merve, and H. Kavurmaci-Önalın, “Hermite-Hadamard type inequalities for s -convex and s -concave functions via fractional integrals,” *Turkish Journal of Science*, vol. 1, no. 1, pp. 28–40, 2016.
- [10] T. Yu, G. Farid, K. Mahreen, C. Y. Jung, and S. H. Shim, “On generalized strongly convex functions and unified integral operators,” 2020.
- [11] X. Qiang, G. Farid, J. Pečarić, and S. B. Akbar, “Generalized fractional integral inequalities for exponentially (s, m) -convex functions,” *Journal of Inequalities and Applications*, vol. 2020, no. 1, p. 70, 2020.
- [12] N. Mehreen and M. Anwar, “Hermite-Hadamard type inequalities for exponentially p -convex functions and exponentially s -convex functions in the second sense with applications,” *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 92, 2019.
- [13] M. U. Awan, M. A. Noor, and K. I. Noor, “Hermite-Hadamard inequalities for exponentially convex functions,” *Applied Mathematics & Information Sciences*, vol. 12, no. 2, pp. 405–409, 2018.
- [14] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives-Theory and Applications* Gordon and Breach, Gordon and Breach Science Publishers, Linghorne, PA, USA, 1993.
- [15] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, “Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities,” *Mathematical and Computer Modelling*, vol. 57, no. 9–10, pp. 2403–2407, 2013.
- [16] M. Z. Sarikaya and H. Yildirim, “On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals,” *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2017.
- [17] S. Mubeen and G. M. Habibullah, “ k -fractional integrals and applications,” *International Journal of Contemporary Mathematical Sciences*, vol. 7, pp. 89–94, 2012.
- [18] S. Mubeen and A. Rehman, “A note on k -gamma function and pochhammer k -symbol,” *Journal of Informatics and Mathematical Sciences*, vol. 6, no. 2, pp. 93–107, 2014.
- [19] G. Farid, A. U. Rehman, and M. Zahra, “On Hadamard-type inequalities for k -fractional integrals,” *Journal of Nonlinear Functional Analysis*, vol. 21, no. 3, pp. 463–478, 2016.
- [20] G. Farid, A. U. Rehman, and M. Zahra, “On Hadamard inequalities for k -fractional integrals,” *Konuralp Journal of Mathematics*, vol. 4, no. 2, pp. 79–86, 2016.

- [21] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands, 2006.
- [22] A. Akkurt, M. E. Yildirim, and H. Yildirim, "On some integral inequalities for (k, h) -Riemann-Liouville fractional integral," *New Trends in Mathematical Science*, vol. 4, no. 2, pp. 138–146, 2016.
- [23] M. Lazarević, M. R. Rapaic, T. B. Sekara et al., *Advanced Topics on Applications of Fractional Calculus on Control Problems, System Stability and Modeling*, WSEAS Press, Belgrade, Serbia, 2014.
- [24] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.
- [25] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Springer Basel AG, Basel, Switzerland, 1988.
- [26] M. U. Awan, S. Talib, Y.-M. Chu, M. A. Noor, and K. I. Noor, "Some new refinements of Hermite-Hadamard-type inequalities involving ψ_k -Riemann-Liouville fractional integrals and applications," *Mathematical Problems in Engineering*, vol. 2020, Article ID 3051920, 10 pages, 2020.
- [27] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.
- [28] G. Farid, S. B. Akbar, V. N. Mishra, and L. K. Mishra, "Riemann-Liouville fractional versions of Hadamard inequality for strongly m -convex functions, submitted," 2020.
- [29] G. Farid, A. U. Rehman, and Q. U. Ain, " k -fractional integral inequalities of Hadamard type for $(h - m)$ -convex functions," *Computational Methods for Differential Equations*, vol. 8, no. 1, pp. 119–140, 2020.
- [30] G. Farid, A. U. Rehman, and B. Tariq, "On Hadamard-type inequalities for m -convex functions via Riemann-Liouville fractional integrals," *Studia Universitatis Babes-Bolyai Matematica*, vol. 62, no. 2, pp. 141–150, 2017.
- [31] G. Farid, A. U. Rehman, B. Tariq, and A. Waheed, "On Hadamard type inequalities for m -convex functions via fractional integrals," *Journal of Inequalities and Special Functions*, vol. 7, no. 4, pp. 150–167, 2016.
- [32] G. Farid, H. Yasmeen, C. Y. Jung, S. H. Shim, and G. Ha, "Refinements and generalization of some fractional integral inequalities via strongly convex functions," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6667226, 18 pages, 2021.
- [33] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004.
- [34] K. Liu, J. Wang, and D. O'Regan, "On the Hermite-Hadamard type inequality for ψ -Riemann-Liouville fractional integrals via convex functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 27, 2019.
- [35] C. Miao, G. Farid, H. Yasmeen, and Y. Bian, "Generalized Hadamard fractional integral inequalities for strongly (s, m) -convex functions," *Journal of Mathematics*, vol. 2021, Article ID 6642289, 19 pages, 2021.
- [36] A. Ur Rehman, G. Farid, C. Yong Jung, and S. Min Kang, " k -fractional integral inequalities of Hadamard type for exponentially (s, m) -convex functions," *AIMS Mathematics*, vol. 6, no. 1, pp. 882–892, 2021.
- [37] G. Farid, Y. C. Kwun, H. Yasmeen, A. Akkurt, and S. M. Kang, "Inequalities for generalized Riemann-Liouville fractional integrals of generalized strongly convex functions, submitted," 2020.
- [38] U.-P. Lv, G. Farid, H. Yasmeen, W. Nazeer, and C. Y. Jung, "Generalization of some fractional versions of Hadamard inequality for exponentially $(\alpha, h-m)$ -convex functions," *AIMS Mathematics*, vol. 6, no. 8, pp. 8978–8999, 2021.
- [39] J. Hadamard, "Etude sur les proprietes des fonctions entieres et en particulier d'une fonction consideree par Riemann," *Journal de Mathématiques Pures et Appliquées*, vol. 58, pp. 171–215, 1893.
- [40] E. Set and A. Gözpinar, "Hermite-Hadamard type inequalities for convex functions via generalized fractional integral operators," *Topological Algebra and Its Applications*, vol. 5, no. 1, pp. 55–62, 2016.
- [41] E. Set, M. A. Noor, M. U. Awan, and A. Gözpinar, "Generalized Hermite-Hadamard type inequalities involving fractional integral operators," *Journal of Inequalities and Applications*, vol. 2017, no. 1, p. 169, 2017.