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Research Article

New Topological Approaches to Generalized Soft Rough Approximations with Medical Applications

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There are many approaches to deal with vagueness and ambiguity including soft sets and rough sets. Feng et al. initiated the concept of possible hybridization of soft sets and rough sets. They introduced the concept of soft rough sets, in which parameterized subsets of a universe set serve as the building blocks for lower and upper approximations of a subset. Topological notions play a vital role in rough sets and soft rough sets. So, the basic objectives of the current work are as follows: first, we find answers to some very important questions, such as how to determine the probability that a subset of the universe is definable. Some more similar questions are answered in rough sets and their extensions. Secondly, we enhance soft rough sets from topological perspective and introduce topological soft rough sets. We explore some of their properties to improve existing techniques. A comparison has been made with some existing studies to show that accuracy measure of proposed technique shows an improvement. Proposed technique has been employed in decision-making problem for diagnosing heart failure. For this two algorithms have been given.

1. Introduction

Mathematical modeling for the vagueness and uncertainty of data has many different methods, for instance, rough set theory [1], fuzzy set theory [2], soft set theory [3], and topology [4]. Pawlak [1] introduced the classical rough sets model in the early eighties to study vagueness of data, which originate from daily life situations. The key of this methodology is an equivalence relation which is constructed from the data of an information system. In general, it is very difficult to find an equivalence relation in such data. Therefore application of this technique is very limited. Therefore authors relaxed the condition of equivalence relation by some more general relations such as similarity relations (reflexive and symmetric) [5, 6], preorder relations (reflexive and transitive) [7], reflexive relations [8], general binary relations [8–14], topological approaches [15, 16], and coverings [17–19].

Soft set theory is another mathematical model to deal with uncertainty, when data is collected from real-life situations. This concept was introduced by Molodtsove [3]. This theory has applications in many fields, for instance, game theory, operations research, integration of Riemann, and measurement theory [3]. Recently, scientists and researchers have shown their inclination to the idea of soft sets to apply it in numerous areas. For more information about this theory and its applications, we refer the reader to the references (soft set theoretical concepts [20, 21], soft sets and soft topological spaces [22–24], soft rough sets and their applications [25–28], and medical applications of soft sets and their extensions [29–35]).

In rough set theory [1], basic requirement is to have an equivalence relation among the elements of the set under consideration. But in daily life situations it is not easy to find such an equivalence relation. Perhaps this limitation is associated with rough set theory due to the lack of parameterization tools. The idea of soft rough sets was initiated and studied by Feng et al. in [24] which are very useful in intelligent systems. The concept of the lower (resp., upper) approximation of this theory is particularly useful to extract knowledge hidden in an information system. Decision-making has a crucial part in our daily life, and this method produces the best alternate among dissimilar selections. Chen et al. [34] proposed the choice values of objects in a soft set and considered how to use this notion to address decision-making problems. In [35], Roy and Maji generalized this method for new decision-making problems. There are several subsequent advances after Maji et al.'s work, such as the uni-int decision-making using soft set theory [36]; Jha et al.'s [37] neutrosophic soft set notion in decision-making problems for stock trending analysis, and medical applications [38].

Feng et al. replaced the classes of the equivalence relation by parameterizing subsets of a subset of the universe to define its approximations. In fact, Feng et al. have succeeded in proving that Pawlak's rough set model is a specialization of the soft rough set as shown by Theorem 4.4 and Theorem 4.5 in [24]. It is worth noting that the concept of full soft sets deserves special attention for both theoretical and practical reasons. Theoretically, some typical properties of Pawlak's rough sets hold for soft rough sets if and only if the underlying soft set in the soft approximation space is full. Pragmatically, it is justifiable to consider full soft sets in real-life applications. In fact, if a soft set is not full, it means that the available parameters are insufficient, and there exists at least one object which cannot be described by any of the parameters in the given soft set. With the help of soft rough approximations, some equivalent characterizations of full soft sets were given in [24]. In this paper, a new technique is given to define lower and upper approximations of a set with the help of topology generated by the given soft set; this is known mathematically as the notion of topological soft rough sets ($T_{SR}-$ sets).

The main contribution in the existing work is to present another model for soft rough sets without any restrictions and satisfy the characteristics of Pawlak's rough sets. In other words, we propose a method for modifying soft rough sets from a topological point of view, so a new link between soft sets and general topology is proposed.

First, we discuss the concept of the topology of all definable sets in rough set theory [1] and in soft rough sets [24]. Accordingly, we able to respond with the next very interesting questions:

What is the probability that a subset of the universe U may be a definable set?

What is the probability that the lower approximation of a nonempty subset of U may be an empty set?

What is the probability that the upper approximation of a proper subset of U may be U?

Secondly, a general topology is generated from the soft set to modify and generalize soft rough sets proposed in [24]. The suggested techniques extend the way for more applications of the general topology in soft rough sets theory. In fact, we use the image of parameters as a subbasis for a unique topology generated by a soft set, denoted by T_{SR}. New generalized soft rough approximations, called "topological soft rough approximations" (briefly, T_{SR}-approximations), are defined. It is shown that accuracy of proposed technique is higher than soft rough sets, due to reduction of boundary region. The importance of proposed approximations is clear from the fact that these not only reduce the boundary region but also satisfy basic properties similar to rough sets. Several comparisons among the present method and the preceding one [34] are obtained. Numerous examples are suggested to exemplify the relations between the topological soft rough sets and soft rough sets.

Finally, some medical applications in the medical diagnosis of heart failure problems [39] are introduced. These applications illustrate the importance of the suggested methods in real-life problems. In fact, we apply a topological reduction for data set comprising the effect of five indications for twenty patients with heart failure disease. Accordingly, we can identify the core factors of the heart failure diagnosis. A comparison has been made between proposed technique and some already existing in the literature which shows the usefulness of proposed technique. Two algorithms are given based on proposed technique. The proposed algorithms are tested on hypothetical data for the purpose of comparison with already existing methods.

2. Basic Concepts

The current section is devoted to present some elementary definitions and consequences that are applied through paper are mentioned.

2.1. Topological Space. A topology [4] of a set U is defined by the collection τ of subsets of U which fulfills the following three axioms:

($T1$) $\phi, U \in \tau$.

($T2$) A finite intersection of subsets of τ is a member in τ.

($T3$) An arbitrary union of subsets of τ is a member in τ.

We call a pair (U, τ) "topological space" or "space" and the members of U "points" of τ, and the subsets of U that belong to τ are said to be "open" sets and the complements of the open sets are called "closed" sets in the space. The collection of all closed sets denotes τ^c.

An interior $\text{int}(A)$ (resp., closurecl(A)) of a subset A is given by a union of all open sets contained in A (resp., intersection of all closed sets that contain A), formally:

$$\begin{aligned} \text{int}(A) &= \cup\{V \in \tau : V \subseteq A\}, \\ \text{cl}(A) &= \cap\{Z \in \tau^c : A \subseteq Z\}. \end{aligned} \tag{1}$$

A class $\mathbf{B} \subseteq \tau$ is said to be a basis for τ if all nonempty open subset of U can be represented as a union of subfamily of $\mathbf{B}$.

Evidently, any topology can have numerous bases, but the basis **B** generates a unique topology τ.

Each union of elements of **B** belongs to τ; therefore a basis of τ entirely decides τ.

A family $\mathbf{S}\subseteq\tau$ is said to be a subbasis for a topological space (U,τ) if the collection of all finite intersections of **S** represents a basis for (U,τ).

For any class **S** of subsets of U, **S** represents a subbasis for a unique basis **B** which generates a unique topology τ on U such that for each $i \in I$

$$\begin{aligned}\mathbf{B} &= \bigcap_i \{S_i \ : \ S_i \in \mathbf{S}\},\\ \tau &= \bigcup_i \{B_i \ : \ B_i \in \mathbf{B}\}.\end{aligned} \tag{2}$$

2.2. Pawlak Rough Set Theory. The current subsection presents some elementary notions pertaining to rough sets given by Pawlak [1].

Definition 1. [1] Consider U is a finite set called universe, and R is an equivalence relation on U; we symbolize U/R to represent the collection of all equivalence classes of R and $[s]_R$ to symbolize an equivalence class in R that contains an element $s \in U$. Then, the pair $A_R = (U,R)$ is said to be Pawlak's approximation space and for any $L\subseteq U$, we propose the lower and upper approximation of L by $\underline{R}(L) = \{s \in U\colon\ [s]_R \subseteq L\}$ and $\overline{R}(L) = \{s \in U\colon\ [s]_R \cap L \neq \phi\}$, respectively. Moreover, L is called a rough set if $\overline{R}(L) \neq \underline{R}(L)$. Otherwise, it is an exact set.

Definition 2. [1] Consider $A_R = (U,R)$ is Pawlak approximation space and $L\subseteq U$. Therefore, the boundary, positive, and negative regions and the accuracy of approximations of $L\subseteq U$ are given, respectively, by

$$\begin{aligned}\mathrm{BND}_R(L) &= \overline{R}(L) - \underline{R}(L),\\ \mathrm{POS}_R(L) &= \underline{R}(L),\\ \mathrm{NEG}_R(L) &= U - \overline{R}(L),\\ \mu_R(L) &= \frac{|\underline{R}(L)|}{|\overline{R}(L)|}, \quad \text{where } \overline{R}(L) \neq \phi.\end{aligned} \tag{3}$$

Properties associated with rough sets can be seen in [1].

It is well known that the set of all definable subsets of the approximation space (U,R) gives rise to a clopen topology τ_c [8]. In this paper first, we will study how this topology is obtained and why in this topology each open set is closed as well.

As $(U/R) = \{[s]_R\colon\ s \in U\}$, now, for each $A, B \in (U/R)$, $A \cap B = \varnothing$ and $U = A \cup B$. Thus (U/R) may act as a basis for a τ topology on U.

Theorem 1. *If the pair (U,R) is Pawlak approximation space, then, $\tau_c = \tau$.*

Proof. Let $A \in \tau_c$. Then A is a definable set, so $A = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_1, A_2, \ldots, A_n \in (U/R)$. Hence $A \in \tau$. That is $\tau_c\subseteq\tau$. Conversely, every $B \in \tau$ is union of some elements of (U/R), which are definable. Since union of definable sets is again definable, B is definable. This means $B \in \tau_c$. So $\tau\subseteq\tau_c$ as required. □

Theorem 1 explains that topology of definable sets in any Pawlak's approximation space is produced by the elements of the set (U/R). In this topology every open set is closed because complement of any subset in the basis (U/R) of this topology is the union of all remaining subsets.

Study of topology constructed by definable sets helps us to answer some very interesting questions such as the following:

What is the probability that a subset of U may be a definable set?

What is the probability that a nonempty subset of U has an empty lower approximation?

What is the probability that a proper nonempty subset of U has upper approximation equal to U?

Answer to the first question is a bit simple and the formula to find the probability that a subset of U may be a definable set is given as follows:

$$\mathscr{P} = \frac{|\tau|}{2^n}, \quad \text{where } |\tau| \text{ is the cardinality of } \tau \text{ and } n \text{ represents a number of elements in } U. \tag{4}$$

Thus, the probability that a subset of the universe U is a rough set is $1 - \mathscr{P}$.

Now, for the answer of the second question first the following result must be considered.

Theorem 2. *If X is a subset of U. Then $\underline{R}(X)$ is an empty set if and only if X does not contain any nonempty element of τ.*

Proof. Let $\underline{R}(X) = \varnothing$. Then, by definition, there does not exist any $x \in X$ such that $[x]_R\subseteq X$. This implies $[x]_R \nsubseteq X$, for each x. Therefore, for each $x \in X$, $[x]_R \nsubseteq X$. That is, no element of τ is contained in X. Conversely, let there exist some $x \in X$, with $x \in \mathscr{U}_x \in \tau$ such that $\mathscr{U}_x\subseteq X$. This indicates $\cup_{x\in X}\mathscr{U}_x\subseteq X$. Then by definition $\underline{R}(X) = \cup_{x\in X}\mathscr{U}_x \neq \varnothing$, which is a contradiction, and therefore, the subset X does not contain any nonempty element of τ. □

Now in any Pawlak approximation space, the probability $\mathscr{P}_{\varnothing}$ that lower approximation of a subset is an empty set can be obtained by the following formula:

$$\mathscr{P}_{\varnothing} = \frac{1 + N_X^{\tau}}{2^n}, \quad \text{where } N_X^{\tau} \text{ represents a number of subsets of } U \text{ which does not contain any nonempty element of } \tau \text{ and } n \text{ is the number of elements in } U. \tag{5}$$

To find the answer to the last question, we may have to consider the following result.

Theorem 3. *Let X be a nonempty subset of U. Then $\overline{R}(X) = U$ if and only if the subset X intersects with every nonempty element of τ.*

Proof. Let $\overline{R}(X) = U$. Then, by definition, there does not exist any $x \in X$ such that $[x]_R \cap X = \varnothing$. Since nonempty elements of τ are union of some classes $[x]_R \in (U/R)$. As $x \in \bigcup [x]_R = U$, X intersects with every $[x]_R$. Consequently, it intersects with every nonempty element of τ. Conversely, let there exist some nonempty $\mathscr{U}_x \in \tau$ containing some $x \in U$ such that $\mathscr{U}_x \cap X = \varnothing$. As $\mathscr{U}_x$ is the union of some elements of U/R, there exists some class $[x]_R \in (U/R)$ such that $[x]_R \cap X = \varnothing$. So $[x]_R \notin \overline{R}(X)$, which results in $\overline{R}(X) \neq U$, a contradiction; hence, X intersects with every nonempty element of τ. □

Further, in any Pawlak approximation space, the probability $\mathscr{P}_U$ that the upper approximation of a subset is U may be obtained by the following formula:

$$\mathscr{P}_U = \frac{N_{NX}^{\tau}}{2^n}, \quad \text{where } N_{NX}^{\tau} \text{ is the number of subsets of } U \text{ which intersect with every nonempty element of } \tau, \text{ and } n \text{ represents the number of elements in } U. \tag{6}$$

Example 1. Consider $U = \{1, 2, 3, 4\}$ is a set and R represents an equivalence relation on U, such that $(U/R) = \{\{1\}, \{2, 3\}, \{4\}\}$; then (U, R) represents Pawlak's approximation space. Now U/R can be a basis for a topology τ on U. Let us write the topology τ generated by U/R as follows: $\tau = \{\varnothing, U, \{1\}, \{2, 3\}, \{4\}, \{1, 2, 3\}, \{1, 4\}, \{2, 3, 4\}\}$. The only definable subsets of U are all elements of τ. Now

$$|\tau| = \text{Number of elements in } \tau = \text{Number of definable subsets of } U. \tag{7}$$

Total number of subsets of $U = 2^4 = 16$.

Thus, the probability that a subset of U is definable is given by

$$\frac{|\tau|}{2^4} = \frac{8}{16} = 0.5. \tag{8}$$

Next $\varnothing$, $\{2\}$, $\{3\}$ are the only subsets of U which do not contain any nonempty element of τ. Therefore, their lower approximation is empty.

$$\mathscr{P}_{\varnothing} = \frac{N_X^{\tau}}{2^n} = \frac{3}{2^4} = \frac{3}{16}. \tag{9}$$

Further the subsets $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$ are the only subsets of U, which intersect with every nonempty element of τ. Therefore their upper approximation is U and then

$$\mathscr{P}_U = \frac{N_{NX}^{\tau}}{2^n} = \frac{3}{2^4} = \frac{3}{16}. \tag{10}$$

2.3. Soft Set Theory and Soft Rough Sets

Definition 3 (see [3]). Consider U to be a set of items and E to be a finite set of certain parameters in relative to the objects in U. Parameters represent attributes or characteristics of U objects. A "soft set" on U is the pair (F, A), where $A \subseteq E$, $P(U)$ symbolize the power set of U, and F represents the map $F: A \longrightarrow P(A)$. On the other hand, a soft set over U is a parameterized collection of subsets of U. For $e \in A$, $F(e)$ represents the set of e-approximate elements of a soft set (F, A). Note that, sometimes a soft set is indicated by F_A and expressed as a set of ordered pairs $F_A = \{(e, F(e)) \colon e \in A\}$.

Definition 4 (see [24]). Consider F_A is a soft set on U. Thus, the pair $A_s = (U, F_A)$ is said to be a soft approximation space. Established on a soft approximation space A_s, we give the "soft A_s lower and soft A_s upper" approximations of $X \subseteq U$, respectively, by

$$\begin{aligned} \underline{S}(X) &= \{u \in U : \exists e \in A, [u \in F(e) \subseteq X]\}, \\ \overline{S}(X) &= \{u \in U : \exists e \in A, [u \in F(e), F(e) \cap X \neq \phi]\}. \end{aligned} \tag{11}$$

Generally, $\underline{S}(X)$ and $\overline{S}(X)$ refer to soft rough approximations of $X \subseteq U$ with respect to A_s. Furthermore, the sets

$$\begin{aligned} \mathrm{POS}_{A_s}(X) &= \underline{S}(X), \\ \mathrm{NEG}_{A_s}(X) &= U - \overline{S}(X), \\ \mathrm{BND}_{A_s}(X) &= \overline{S}(X) - \underline{S}(X), \end{aligned} \tag{12}$$

are named the soft "A_s positive, A_s negative, and A_s boundary" regions of $X \subseteq U$, individually. Evidently, if $\overline{S}(X) = \underline{S}(X) = X$, i.e., $\mathrm{BND}_{A_s}(X) = \phi$, then $X \subseteq U$ is said to be "soft A_s definable" or "soft A_s exact" set; or else X is called a "soft A_s rough" set. Moreover there may be a subset which has the same lower and upper approximations but is not definable. Besides, we suggest the accuracy of approximations by

$$\mu_{A_s}(X) = \begin{cases} \dfrac{|\underline{S}(X)|}{|\overline{S}(X)|}, & \text{when } \overline{S}(X) \neq \phi, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Above definition gives a generalization of Pawlak rough sets theory; naturally it may not satisfy some properties of their properties.

Proposition 1 (see [24]). *If F_A is a soft set on U and $A_s = (U, F_A)$ is a soft approximation space, then, for each $X \subseteq U$:*

$$\begin{aligned} \underline{S}(X) &= \bigcup_{e \in A} \{F(e) : F(e) \subseteq X\}, \\ \overline{S}(X) &= \bigcup_{e \in A} \{F(e) : F(e) \cap X \neq \phi\}. \end{aligned} \tag{14}$$

Properties associated with soft rough sets can be gotten in [34].

Definition 5 (see [24]). Suppose that F_A is a soft set on U and $A_s = (U, F_A)$ a soft approximation space. Then, F_A is said to be a "full soft set" if $F_A = \cup_{e \in A} F(e)$. It is clear that if F_A is a full soft set, then $\forall x \in U, \exists e \in A$ such that $x \in F(e)$.

Proposition 2 (see [24]). *If F_A is a full soft set on U and $A_s = (U, F_A)$ is a soft approximation space, then, the subsequent conditions are true:*

(i) $\underline{S}(U) = \overline{S}(U) = U$

(ii) $X \subseteq \overline{S}(X), \forall X \subseteq U$

(iii) $\overline{S}(\{x\}) \neq \phi, \forall x \in U$

Now, we present and study the idea of the topology of all definable sets generated by soft set F_A. Moreover, we illustrate the condition in which this topology is well defined. Now, again, we can find the answers to the questions which are related to the probability associated with subsets of a set in soft rough sets.

The following example illustrates that the condition "full soft set" in Proposition 2 is necessary to achieve the properties (i)-(iii).

Example 2. Consider $A_s = (U, F_A)$ to be a soft approximation space, such that $U = \{a, b, c, d\}$, and F_A to be a soft set on U where $A = \{e_1, e_2, e_3\}$ and $F_A = \{(e_1, \{a\}), (e_2\{b, c\}), (e_3, \{a, b\})\}$. Then, it is clear that F_A is not full soft set and thus we have $\underline{S}(U) = \overline{S}(U) = \{a, b, c\} \neq U$. Also, if $X = \{d\}$, then $\underline{S}(X) = \overline{S}(X) = \phi \neq X$.

Theorem 4. *If F_A is a full soft set on U and $A_s = (U, F_A)$ is a soft approximation space, then, the collection $\tau_D = \{X \subseteq U \colon \underline{S}(X) = \overline{S}(X)\}$ is a quasi-discrete topology on U.*

Proof. Since F_A is a soft set on U, by using the properties of the soft approximations in [24], we get

($T1$) $\underline{S}(U) = \overline{S}(U)$ and $\underline{S}(\phi) = \overline{S}(\phi)$. Therefore, $U, \phi \in \tau_D$.

($T2$) Let $\{X_i \colon i \in I\}$ be a class of members in τ_D. Then, $\cup_{i \in I} X_i \in \tau_D$.

($T3$) Let $\{X_i \colon i = 1, 2, 3, \ldots, n\}$ be a class of finite members in τ_D. Then, $\cap_{i=1}^{n} X_i \in \tau_D$. Now, we need to prove that τ_D is a quasi-discrete as follows: Let $X \in \tau_D$; then $\underline{S}(X) = \overline{S}(X)$. By taking the complement to both sides, we obtain $[\underline{S}(X)]^c = [\overline{S}(X)]^c$ and this implies $\overline{S}(X^c) = \underline{S}(X^c)$. Thus, $X^c \in \tau_D$.

Remark 1. According to Proposition 2, the condition "full soft set" in the above theorem is necessary to construct the topology τ_D. If the soft set is not full then U may not be a definable set. Moreover there may be a subset which has same lower and upper approximations but is not definable.

For any soft approximation space $A_s = (U, F_A)$, such that F_A be a full soft set, the probability of a subset of U is proposed by

$\mathscr{P} = (|\tau_D|/2^n)$, where $|\tau_D|$ is the cardinality of τ_D and n represents a number of elements in U.

Thus, the probability that a subset of U is a rough set is $1 - \mathscr{P}$.

Now, for the answer of the second question, first the following results must be considered.

Lemma 1. *Suppose that F_A is a full soft set on U. Hence, we get*

(1) $\forall x \in U, \exists B_x \in \tau_D$ *such that* $x \in B_x$ *and* B_x *is called an open set containing* x.

(2) $\forall x \in U, \exists e \in A$ *such that* $x \in F(e)$.

(3) $\forall e \in A, \exists B_x \in \tau_D$ *such that* $F(e) \subseteq B_x$.

Proof. The proof of (1) is obvious and from the definition of $\underline{S}$ and $\overline{S}$ (Definition 4), the proofs of (2) and (3) are straightforward.

Now, for any soft approximation space $A_s = (U, F_A)$, such that F_A is a full soft set, the probability $\mathscr{P}_\varnothing$ that lower approximation of a subset is an empty set can be obtained by the following formula:

$$\mathscr{P}_{\varnothing} = \frac{1 + \Lambda_{\varnothing}^{F_A}}{2^n}, \quad \text{where } \Lambda_{\varnothing}^{F_A} \text{ is the number of subsets of } U \text{ which does not contain any nonempty element of } F_A \text{ and } n \text{ represents a number of elements in } U. \tag{15}$$

In order to find the answer of the last question we may have to consider the following. □

Theorem 5. *Consider $A_s = (U, F_A)$ is a soft approximation space such that F_A is a full soft set and X is a subset of U. Thus, $\overline{S}(X) = U$ if and only if X intersects with every nonempty element of τ_D.*

Proof. Firstly, since F_A is a full soft set, then, $\forall e \in A$, $\cup_{e\in A} F(e) = U$. Let $\overline{S}(X) = U$; then $\overline{S}(X) = \cup_{e\in A} F(e)$ such that $F(e) \cap X \neq \phi$. Therefore, by Lemma 4, $\exists B_x \in \tau_D$ such that $F(e) \subseteq B_x$ and $F(e) \cap X \neq \phi$. Accordingly, $\forall B_x \in \tau_D$, $B_x \cap X \neq \phi$ which means that X intersects with every nonempty element of τ_D. Conversely, let $\exists B_x \in \tau_D$ such that $B_x \cap X = \phi$ and $\overline{S}(X) \neq U$. Thus $\exists e \in A$ such that $F(e) \subseteq B_x$ and $F(e) \cap X = \phi$. Accordingly, $\overline{S}(X) \neq U$ which is a contradiction.

Note that if $\overline{S}(X) = U$, then no need for X to intersect with $F(e)$, $\forall e \in A$ in general as Example 3 illustrates. Clearly, the subsets $\{b, c\}$ and $\{b, c, d\}$ do not intersect with $F(e_1) = \{a\}$ although their soft upper approximation is U.

Further in any soft approximation space, $A_s = (U, F_A)$, such that F_A is a full soft set; the probability $\mathscr{P}_U$ that upper approximation of a subset is U can be obtained by the following formula:

$$\mathscr{P}_U = \frac{\Lambda_X^{\tau_D}}{2^n}, \quad \text{where } \Lambda_X^{\tau_D} \text{ is the number of subsets of } U \text{ which intersect with every nonempty element of } \tau \text{ and } n \text{ is the number of elements in } U. \tag{16}$$

The next example explains the previous discussion. □

Example 3. Consider $A_s = (U, F_A)$ is a soft approximation space, such that $U = \{a, b, c, d\}$ and F_A is a full soft set on U, where $A = \{e_1, e_2, e_3, e_4\}$ and $F_A = \{(e_1, \{a\}), (e_2, \{a, b\}), (e_3, \{c\}), (e_4, \{b, d\})\}$. Then, we obtain the topology of all soft definable subsets by $\tau_D = \{U, \phi, \{c\}, \{a, b, d\}\}$. Therefore, $|\tau_D|$ =number of elements in τ_D = number of all definable subsets of U. Thus, the probability that a subset of U is definable is given by $\mathscr{P} = (|\tau_D|/2^n) = (|\tau_D|/2^n) = (4/2^4) = (4/16) = (1/4) = 0.25$.

Now, the subsets $\{b\}$ and $\{d\}$ are the only subsets of U that do not contain any nonempty element of F_A. Therefore, their lower approximation is empty.

Accordingly, $\mathscr{P}_{\varnothing} = ((1 + \Lambda_{\varnothing}^{F_A})/2^n) = ((1 + 2)/2^4) = (3/16)$. Further, the subsets $\{b, c\}, \{a, b, c\}, \{a, c, d\}$, and $\{b, c, d\}$ are the only subsets of U, which intersect with every nonempty element of τ_D. Therefore, their upper approximation is U and accordingly $\mathscr{P}_U = (\Lambda_X^{\tau_D}/2^n) = (4/2^4) = (4/16) = (1/4) = 0.25$.

3. Topological Soft Rough Approximations of Soft Rough Sets

The current section is devoted to introduction of topological soft rough approximations in view of topological structure. Firstly, it will be seen that soft sets and topological spaces have a very close relationship. The concept of topological soft rough approximations will be presented and their properties will be studied. On the other hand, it will be shown that accuracy of the proposed approach is better than existing techniques. Besides, we will give answers to some important questions about the probability in topological soft rough sets.

Definition 6. Consider F_A is a soft set on U and $K = \cup F(e)$. Thus, we propose the following:

(i) $\mathbf{S}_{F_A} = \{F(e)\colon\ e \in A\}$ may denote a subbasis;

(ii) $\mathbf{B}_{F_A} = \{A \cap B\colon\ (A, B) \in \mathbf{S}_{F_A} \times \mathbf{S}_{F_A}\}$ may denote a basis for the topology T_{SR} defined as the following.

If F_A is a soft set on U and $K = \cup_{F(e)\in \mathbf{S}_{F_A}} F(e)$, then the topology T_{SR} can be defined on K with a basis $\mathbf{B}_{F_A}$. That is, $T_{\mathrm{SR}} = \cup\{B\colon\ B \in \mathbf{B}_{F_A}\}$. This topology may be called topology generated by F_A and we call it "soft rough topology" (in brief, SR topology).

Remark 2. There are three cases of a subbasis $\mathbf{S}_{F_A}$:

(i) If $\mathbf{S}_{F_A}$ is a partition of U, then $\mathbf{S}_{F_A} = \mathbf{B}_{F_A}$ and will be a basis for a quasi-discrete (clopen) topology (in which all open sets are closed).

(ii) If $\mathbf{S}_{F_A}$ is a covering (not partition) of U, then $\mathbf{S}_{F_A} \neq \mathbf{B}_{F_A}$ and $\mathbf{B}_{F_A}$ will be a basis for a general topology.

(iii) If $\mathbf{S}_{F_A}$ is not a covering (not partition) of U, then $\mathbf{S}_{F_A} \neq \mathbf{B}_{F_A}$ and $\mathbf{B}_{F_A}$ will be a basis for a general topology. The following examples explain Remark 2.

Example 4. If $A_s = (U, F_A)$ is a soft approximation space, such that $U = \{a, b, c, d\}$ and F_A is a soft set on U where

$A = \{e_1, e_2, e_3\}$ and $F_A = \{(e_1, \{a, b\}), (e_2, \{c\}), (e_3, \{d\})\}$, then, we get that the subbasis of T_{SR} is $\mathbf{S}_{F_A} = \{\{c\}, \{d\}, \{a, b\}\}$ and the basis is $\mathbf{B}_{F_A} = \{\{c\}, \{d\}, \{a, b\}\}$. Clearly, $\mathbf{S}_{F_A} = \mathbf{B}_{F_A}$. Accordingly, the topology generated by $\mathbf{B}_{F_A}$ is

$$T_{\mathrm{SR}} = \{U, \phi, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}. \quad (17)$$

Obviously, T_{SR} is a quasi-discrete topology.

Example 5. Suppose that $A_s = (U, F_A)$ is a soft approximation space, such that $U = \{a, b, c, d\}$ and F_A is a soft set on U where $A = \{e_1, e_2, e_3\}$ and $F_A = \{(e_1, \{a, b\}), (e_2, \{a, c\}), (e_3, \{d\})\}$. Then, the subbasis of T_{SR} is $\mathbf{S}_{F_A} = \{\{d\}, \{a, b\}, \{a, c\}\}$ and the basis is $\mathbf{B}_{F_A} = \{\{a\}, \{d\}, \{a, b\}, \{a, c\}\}$. Evidently, $\mathbf{S}_{F_A} \neq \mathbf{B}_{F_A}$. Accordingly, the topology T_{SR} generated by $\mathbf{B}_{F_A}$ is

$$T_{\mathrm{SR}} = \{U, \phi, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}. \quad (18)$$

Obviously, T_{SR} is not a quasi-discrete topology.

Example 6. Consider $A_s = (U, F_A)$ is a soft approximation space, such that $U = \{a, b, c, d\}$ and F_A is a soft set on U where $A = \{e_1, e_2, e_3\}$ and $F_A = \{(e_1, \{a\}), (e_2, \{b, c\}), (e_3, \{a, c\})\}$. Then, the subbasis of T_{SR} is $\mathbf{S}_{F_A} = \{\{a\}, \{a, c\}, \{b, c\}\}$ and the basis is $\mathbf{B}_{F_A} = \{\{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Undoubtedly, $\mathbf{S}_{F_A} \neq \mathbf{B}_{F_A}$. Accordingly, the topology T_{SR} generated by $\mathbf{B}_{F_A}$ is

$$T_{\mathrm{SR}} = \{U, \phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}. \quad (19)$$

Obviously, T_{SR} is not a quasi-discrete topology.

Definition 7. Consider $A_s = (U, F_A)$ is a soft approximation space and T_{SR} is the SR topology on U. The triple $A_{T_{\mathrm{SR}}} = (U, F_A, T_{\mathrm{SR}})$ is said to be a "topological soft rough approximation space" (briefly, T_{SR} approximation space).

Definition 8. Consider $A_{T_{\mathrm{SR}}} = (U, F_A, T_{\mathrm{SR}})$ to be a T_{SR} approximation space. Therefore, for each $X \subseteq U$ we suggest the topological soft rough approximations, "T_{SR} lower" and "T_{SR} upper," respectively, by

$$\begin{aligned} \underline{S}_{T_{\mathrm{SR}}}(X) &= \cup\{G \in T_{\mathrm{SR}} : G \subseteq X\}, \\ \overline{S}_{T_{\mathrm{SR}}}(X) &= \cap\{H \in T_{\mathrm{SR}}^c : X \subseteq H\}. \end{aligned} \quad (20)$$

Remark 3

(i) In general, $\underline{S}_{T_{\mathrm{SR}}}(X)$ and $\overline{S}_{T_{\mathrm{SR}}}(X)$ represent the interior and closure of X associated with the topology T_{SR}, respectively.

(ii) If $\mathbf{S}_{F_A}$ is a partition on U, then $\mathbf{S}_{F_A} = \mathbf{B}_{F_A} = (U/R)$ and hence $\underline{S}_{T_{\mathrm{SR}}}(X)$ and $\overline{S}_{T_{\mathrm{SR}}}(X)$ are identical with Pawlak's rough set approximations. Therefore, it can be said that the proposed approach is equivalent to Pawlak's approach only in case $\mathbf{S}_{F_A}$ is a partition of U. Accordingly, we can say that Pawlak's rough set model is a specialization of proposed model. Example 4 illustrated this fact.

(iii) If $\mathbf{S}_{F_A}$ is not a partition on U, then $\mathbf{S}_{F_A} \neq \mathbf{B}_{F_A} \neq (U/R)$ and hence $\underline{S}_{T_{\mathrm{SR}}}(X)$ and $\overline{S}_{T_{\mathrm{SR}}}(X)$ will be different from Pawlak's approximations as illustrated in Example 7.

Example 7. Suppose that $U = \{a, b, c, d\}$ is a set of students reading some languages. Let $A = \{e_1, e_2, e_3\}$ and $A_s = (U, F_A)$ be a soft approximation space, where F_A is a soft set on U. Consider the next information system in Table 1.

Therefore, the equivalence classes are $(U/R) = \{\{b\}, \{d\}, \{a, c\}\}$.

If we consider each attribute of the set $\{\text{English}, \text{French}, \text{German}\}$ represents a parameter as the following: e_1 =English, e_2 =French and e_3 =German, then, we get the following:

The soft set of U is $F_A = \{(e_1, \{a, c, d\}), (e_2, \{b, d\}), (e_3, \{a, c\})\}$, where $A = \{e_1, e_2, e_3\}$. Thus, we get $\mathbf{S}_{F_A} = \{\{a, c\}, \{b, d\}, \{a, c, d\}\}$ and the basis is $\mathbf{B}_{F_A} = \left\{\{d\}, t\{a, c\}n, q\{b, d\}h,_{\{a,c,d\}}\right\}$ Accordingly, the topology T_{SR} generated by $\mathbf{B}_{F_A}$ is

$$T_{\mathrm{SR}} = \{U, \phi, \{d\}, \{a, c\}, \{b, d\}, \{a, c, d\}\}, \quad (21)$$

and the complement of T_{SR} is

$$T_{\mathrm{SR}}^c = \{U, \phi, \{b\}, \{a, c\}, \{b, d\}, \{a, b, c\}\}. \quad (22)$$

Evidently, the T_{SR} approximations of any subsets of U differ than Pawlak's rough approximations.

Definition 9. Suppose that $A_{T_{\mathrm{SR}}} = (U, F_A, T_{\mathrm{SR}})$ is a T_{SR} approximation space. Hence, for every $X \subseteq U$, we express the "T_{SR} positive, T_{SR} negative, and T_{SR} boundary" regions and the "T_{SR} accuracy" of the T_{SR} approximations, respectively, by

$$\begin{aligned} \mathrm{POS}_{T_{\mathrm{SR}}}(X) &= \underline{S}_{T_{\mathrm{SR}}}(X), \\ \mathrm{NEG}_{T_{\mathrm{SR}}}(X) &= U - \overline{S}_{T_{\mathrm{SR}}}(X), \\ \mathrm{BND}_{T_{\mathrm{SR}}}(X) &= \overline{S}_{T_{\mathrm{SR}}}(X) - \underline{S}_{T_{\mathrm{SR}}}(X), \\ \mu_{T_{\mathrm{SR}}}(X) &= \begin{cases} \dfrac{\left|\underline{S}_{T_{\mathrm{SR}}}(X)\right|}{\left|\overline{S}_{T_{\mathrm{SR}}}(X)\right|}, & \text{if } X \neq \phi, \\ 1, & \text{if } X \text{ is a } T_{\mathrm{SR}} - \text{definable}. \end{cases} \end{aligned} \quad (23)$$

Remark 4

(i) It is clear that $0 \leq \mu_{T_{\mathrm{SR}}}(X) \leq 1$, for any $X \subseteq U$.

(ii) If $\overline{S}_{T_{\mathrm{SR}}}(X) = \underline{S}_{T_{\mathrm{SR}}}(X)$, then $\mathrm{BND}_{T_{\mathrm{SR}}}(X) = \phi$ and $\mu_{T_{\mathrm{SR}}}(X) = 1$. Thus $X \subseteq U$ is said to be "T_{SR} definable" or "T_{SR} exact" set; otherwise X is called a "T_{SR} rough" set.

The core objective of the following propositions is to discuss the basic properties of T_{SR} rough approximations $\underline{S}_{T_{\mathrm{SR}}}$ and $\overline{S}_{T_{\mathrm{SR}}}$.

Table 1: The tabular form for a soft set (F, A).

	English	French	German
a	Yes	No	Yes
b	No	Yes	No
c	Yes	No	Yes
d	Yes	Yes	No

According to the characteristics of the interior and closure, we can demonstrate the subsequent results, so we omit the proof.

Proposition 3. *If $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space and $X, Y \subseteq U$, then, the T_{SR} lower and T_{SR} upper approximations operators satisfy the next properties:*

(i) $\underline{S}_{T_{SR}}(\varnothing) = \overline{S}_{T_{SR}}(\varnothing) = \varnothing$.
(ii) $\underline{S}_{T_{SR}}(U) = \overline{S}_{T_{SR}}(U) = U$.
(iii) $\underline{S}_{T_{SR}}(X) \subseteq X \subseteq \overline{S}_{T_{SR}}(X)$.
(iv) If $X \subseteq Y$, *then* $\underline{S}_{T_{SR}}(X) \subseteq \underline{S}_{T_{SR}}(Y)$.
(v) If $X \subseteq Y$, *then* $\overline{S}_{T_{SR}}(X) \subseteq \overline{S}_{T_{SR}}(Y)$.
(vi) $\underline{S}_{T_{SR}}(X^c) = (\overline{S}_{T_{SR}}(X))^c$.
(vii) $\underline{S}_{T_{SR}}(X \cap Y) = \underline{S}_{T_{SR}}(X) \cap \underline{S}_{T_{SR}}(Y)$.
(viii) $\overline{S}_{T_{SR}}(X \cup Y) = \overline{S}_{T_{SR}}(X) \cup \overline{S}_{T_{SR}}(Y)$.
(ix) $\overline{S}_{T_{SR}}(X \cap Y) \subseteq \overline{S}_{T_{SR}}(X) \cap \overline{S}_{T_{SR}}(Y)$.
(x) $\underline{S}_{T_{SR}}(X \cup Y) \supseteq \underline{S}_{T_{SR}}(X) \cup \underline{S}_{T_{SR}}(Y)$.
(xi) If $X \in T_{SR}$, *then* $\underline{S}_{T_{SR}}(X) = X$.
(xii) If $X \in (T_{SR})^c$, *then* $\overline{S}_{T_{SR}}(X) = X$.

Proposition 4. *If $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space and $X \subseteq U$, then:*

(i) $\underline{S}_{T_{SR}}(\underline{S}_{T_{SR}}(X)) = \underline{S}_{T_{SR}}(X)$.
(ii) $\overline{S}_{T_{SR}}(\overline{S}_{T_{SR}}(X)) = \overline{S}_{T_{SR}}(X)$.
(iii) $\underline{S}_{T_{SR}}(X) \subseteq \overline{S}_{T_{SR}}(\underline{S}_{T_{SR}}(X))$.
(iv) $\underline{S}_{T_{SR}}(\overline{S}_{T_{SR}}(X)) \subseteq \overline{S}_{T_{SR}}(X)$.

Remark 5. The inclusion relations in Proposition 4 may be strict, as shown in Example 8.

Example 8. Suppose that F_A is a soft set on U and $A_s = (U, F_A)$ is a soft approximation space, where $U = \{a, b, c, d\}$ and $A = \{e_1, e_2, e_3, e_4, e_5\}$ such that

$F_A = \{(e_1, \{b\}), (e_2, \{a, b\}), (e_3, \{c, d\}), (e_4, \{a, c, d\}), (e_5, \{a, b, c\})\}$. Therefore, we get the following.

The subbasis of T_{SR} is $\mathbf{S}_{F_A} = \{\{b\}, t\{a, b\}n, q\{c, d\}h,_{\{a,c,d\}} x, 7\{a, b, c\}\}$.

The basis of T_{SR} is $\mathbf{B}_{F_A} = \{\{a\}, t\{b\}n, q\{c\}h,_{\{a,b\}} x, 7\{a, c\}C, ; \{c, d\}, \{a, c, d\}, \{a, b, c\}\}$.

Accordingly, we get

$$T_{\mathrm{SR}} = \{U, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c\}\}. \tag{24}$$

The complement of T_{SR} is

$$T^c_{\mathrm{SR}} = \{U, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}\}. \tag{25}$$

Thus, $A_{T_{\mathrm{SR}}} = (U, S, T_{\mathrm{SR}})$ is a T_{SR} approximation space. Now, let $X = \{a, c\}$ and $Y = \{a, b, d\}$. Then, $\underline{S}_{T_{\mathrm{SR}}}(X) = \{a, c\}$ and then $\overline{S}_{T_{\mathrm{SR}}}(\underline{S}_{T_{\mathrm{SR}}}(X)) = \{a, c, d\}$, which means that $\underline{S}_{T_{\mathrm{SR}}}(X) \neq \overline{S}_{T_{\mathrm{SR}}}(\underline{S}_{T_{\mathrm{SR}}}(X))$. Also, $\overline{S}_{T_{\mathrm{SR}}}(Y) = \{a, b, d\}$ and $\underline{S}_{T_{\mathrm{SR}}}(\overline{S}_{T_{\mathrm{SR}}}(Y)) = \{a, b\}$, which means that $\overline{S}_{T_{\mathrm{SR}}}(Y) \neq \underline{S}_{T_{\mathrm{SR}}}(\overline{S}_{T_{\mathrm{SR}}}(Y))$.

The following theorem establishes a relationship between approximations of a set in soft rough sets [24] and topological soft rough sets.

Theorem 6. *If $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space and $X \subseteq U$, then:*

(i) $\underline{S}(X) \subseteq \underline{S}_{T_{SR}}(X)$.
(ii) $\overline{S}_{T_{SR}}(X) \subseteq \overline{S}(X)$.

Proof. We shall verify only the first item and the other likewise. Let $x \in \underline{S}(X)$; then $\exists e \in A$, such that $x \in F(e) \subseteq X$ and $F(e) \in \mathbf{S}_{F_A}$. Accordingly, $F(e) \in T_{\mathrm{SR}}$ such that $x \in F(e)$ and $F(e) \subseteq X$; this implies $F(e) \subseteq \underline{S}_{T_{\mathrm{SR}}}(X)$, and therefore, $x \in \underline{S}_{T_{\mathrm{SR}}}(X)$. Hence, $\underline{S}(X) \subseteq \underline{S}_{T_{\mathrm{SR}}}(X)$. □

Corollary 1. *If $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space and $X \subseteq U$, then:*

(i) $BND_{T_{SR}}(X) \subseteq BND_{A_s}(X)$
(ii) $\mu_{A_s}(X) \leq \mu_{T_{SR}}(X)$

Corollary 2. *If $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space and $X \subseteq U$ and if X is a soft exact set, then it is a T_{SR} exact set.*

Remark 6

(1) According to the above results, it is easy to see that boundary region in case of topological soft rough sets is smaller than boundary for soft rough sets.

Therefore accuracy has been improved in proposed technique.

(2) In general reverse inclusions do not hold in case of Theorem 6. Following example explains this.

Example 9 According to Example 8: Let $X = \{b, c, d\}$. Thus $\underline{S}(X) = \{b, c, d\}$ and $\overline{S}(X) = U$; therefore $\mathrm{BND}_{A_s}(X) = \{a\}$ and $\mu_{A_s}(X) = (3/4)$. But, $\underline{S}_{T_{SR}}(X) = \overline{S}_{T_{SR}}(X) = X$ and thus $\mathrm{BND}_{T_{SR}}(X) = \phi$ and $\mu_{T_{SR}}(X) = 1$. Obviously, X is a T_{SR} exact (definable) set (according to our approach) although it is a soft rough set.

Next we define some very important notions.

Definition 10. Consider $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space and $X \subseteq U$. Thus, we describe the subsequent four elementary sorts of T_{SR} soft rough sets as follows:

The subset X represents

(i) a roughly T_{SR} definable set if $\underline{S}_{T_{SR}}(X) \neq \phi$ and $\overline{S}_{T_{SR}}(X) \neq U$,

(ii) an internally T_{SR} indefinable set if $\underline{S}_{T_{SR}}(X) = \phi$ and $\overline{S}_{T_{SR}}(X) \neq U$,

(iii) an externally T_{SR} indefinable set if $\underline{S}_{T_{SR}}(X) \neq \phi$ and $\overline{S}_{T_{SR}}(X) = U$,

(iv) a totally T_{SR} indefinable set if $\underline{S}_{T_{SR}}(X) = \phi$ and $\overline{S}_{T_{SR}}(X) = U$.

The axiomatic significance of this classification is given as follows:

(i) If the subset X is a roughly T_{SR} definable set, then we can identify for some members of U that they belong to X, and for other members of U that they belong to X_c, by using existing knowledge from the T_{SR} approximation space $A_{T_{SR}}$.

(ii) If the subset X is an internally T_{SR} indefinable set, then we can identify about some members of U that they belong to X^c, but we cannot identify for any member of U that it belongs to X, by using $A_{T_{SR}}$.

(iii) If the subset X is an externally T_{SR} indefinable set, then we can identify for some members of U that they belong to X, but we cannot identify for any member of U that it belongs to X^c, by using $A_{T_{SR}}$.

(iv) If the subset X is a totally T_{SR} indefinable set, then we cannot identify for any member of U whether it belongs to X or X^c, by using $A_{T_{SR}}$.

Theorem 7. *Consider $A_{T_{SR}} = (U, F_A, T_{SR})$ to be a T_{SR} approximation space and $X \subseteq U$. Therefore, we have the following.*

(i) If the subset X is a roughly T_{SR} definable set, then X is roughly soft A_s definable.

(ii) If the subset X is an internally T_{SR} definable set, then X is internally soft A_s indefinable.

(iii) If the subset X is an externally T_{SR} definable set, then X is externally soft A_s indefinable.

(iv) If the subset X is a totally T_{SR} indefinable set, then X is totally soft A_s indefinable.

Proof. Only, the first statement will be proved and the other statements can be made by a similar way. (i) Suppose that the subset X is a roughly soft A_s definable set; then $\underline{S}_{T_{SR}}(X) \neq \phi$ and $\overline{S}_{T_{SR}}(X) \neq U$. Therefore, by using Theorem 6, $\underline{S}(X) \neq \phi$ and $\overline{S}(X) \neq U$ and thus X represents a roughly soft A_s definable set. □

Remark 7

(i) Theorem 7 shows that soft rough approximations of a set given in [24] are different from T_{SR} rough approximations proposed in this paper. Moreover, it clarifies the significance of the proposed approach in defining approximations of sets; for example, let X be a totally soft A_s indefinable set. Then, we get $\underline{S}(X) = \phi$ and $\overline{S}(X) = U$. Thus, we are incapable of identifying what are the elements of U that belong to X or X^c. But, by using T_{SR} rough approximations, it may be $\underline{S}_{T_{SR}}(X) \neq \phi$ and $\overline{S}_{T_{SR}}(X) \neq U$ which means that X can be roughly T_{SR} definable set and accordingly we can determine for some elements of U that they belong to X, and meanwhile, for some elements of U, we can identify that they belong to X^c, by using the existing information from the T_{SR} approximation space (to illustrate this, see Examples 8 and Subsection 4.1).

(ii) The inverse of Theorem 7 does not hold, generally, as demonstrated in Example 9 and Subsection 4.1.

Now, once again, we can find the probability for different types of subsets in topological soft rough sets. Firstly, to answer the first question, we consider the following results.

Theorem 8. *If F_A is a soft set on U, $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space, and $X \subseteq U$, then, the class $\tau_D = \{X \subseteq U\colon\ \underline{S}_{T_{SR}}(X) = \overline{S}_{T_{SR}}(X)\}$ gives rise a topology on U.*

Proof. According to Proposition 3, the proof is clear.

Note that τ_D represents a topology of all definable sets in U. □

Lemma 2. *If $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space, then, $\tau_D \subseteq T_{SR}$.*

Proof:. Let $X \in \tau_D$; then $\underline{S}_{T_{SR}}(X) = \overline{S}_{T_{SR}}(X)$ and this implies $\underline{S}_{T_{SR}}(X) = X$. Accordingly, $X \in T_{SR}$. □

Lemma 3. *Let $A_{T_{SR}} = (U, F_A, T_{SR})$ be a T_{SR} approximation space and S_{F_A} be a partition of U; then $\tau_D = T_{SR}$ and S_{F_A} is a basis for τ_D.*

Proof. Firstly, using Lemma 2, $\tau_D \subseteq T_{SR}$. Now, let $X \in T_{SR}$ which is a quasi-discrete topology; then X is an open and closed subset. Therefore, $\underline{S}_{T_{\mathrm{SR}}}(X) = \overline{S}_{T_{\mathrm{SR}}}(X) = X$ and this implies $X \in \tau_D$. Hence, $T_{\mathrm{SR}} \subseteq \tau_D$.

The subsequent example explains that the condition "S_{F_A} is a partition of U" is necessary condition. □

Example 10. Consider Example 7; we get $\tau_D = \{U, \phi, \{a,c\}, \{b,d\}\}$ but $T_{\mathrm{SR}} = \{U, \phi, \{d\}, t\{a,c\}n, q\{b,d\}h_{\{a,c,d\}}\}$. It is clear that $T_{\mathrm{SR}} \nsubseteq \tau_D$.

Definition 11. Suppose that $A_{T_{\mathrm{SR}}} = (U, F_A, T_{\mathrm{SR}})$ is a T_{SR} approximation space and τ_D is a topology of all definable sets in U. The probability P_D that a subset of U is definable is defined by

$$P_D = \frac{|\tau_D|}{2^n}, \tag{26}$$

where $|\tau_D|$ is the cardinality of τ_D and n represents a number of elements of U.

Therefore, the probability that a subset is a rough set $X \subseteq U$ is $1 - P_D$.

Example 11

(1) Consider Example 4; we have $T_{\mathrm{SR}} = \tau_D$. Thus, the probability P_D that a subset of the universe set is definable is given by

$$P_D = \frac{|\tau_D|}{2^n} = \frac{8}{2^4} = \frac{1}{2} = 0.5. \tag{27}$$

(2) Consider Example 5; we have $\tau_D = \{U, \phi, \{d\}, t\{a,b,c\}\}$. Therefore, the probability P_D that a subset of the universe set is definable is given by

$$P_D = \frac{|\tau_D|}{2^n} = \frac{4}{2^4} = \frac{1}{4} = 0.25. \tag{28}$$

Secondly, to identify the probability that lower approximation of a nonempty subset of U may be an empty set, we propose the next results.

Lemma 4. *If $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space, then, $\mathbf{B}_{F_A} \subseteq T_{SR}$.*

Proof. The proof is straightforward. □

Theorem 9. *Consider $A_{T_{SR}} = (U, F_A, T_{SR})$ to be a $A_{T_{SR}} = (U, F_A, T_{SR})$ approximation space and $X \subseteq U$. $\underline{S}_{T_{SR}}(X) = \phi$ if and only if the subset X does not contain any nonempty element of T_{SR}.*

Proof. Using Definition 8, the proof is clear. □

Theorem 10. *Let $A_{T_{SR}} = (U, F_A, T_{SR})$ be a $A_{T_{SR}} = (U, F_A, T_{SR})$ approximation space and $X \subseteq U$. $\underline{S}_{T_{SR}}(X) = \phi$ if and only if the proper subsets of X are not element in $\mathbf{B}_{F_A}$.*

Proof. Firstly, if $\underline{S}_{T_{\mathrm{SR}}}(X) = \phi$ then $\forall G \subseteq X$, $G \notin T_{\mathrm{SR}}$, and this implies $\forall G \subseteq X$, $G \notin \mathbf{B}_{F_A}$. Conversely, let $\exists G \subseteq X$, $G \in \mathbf{B}_{F_A}$ such that $\underline{S}_{T_{\mathrm{SR}}}(X) = \phi$. Then, $G \in T_{\mathrm{SR}}$ such that $G \subseteq X$ and this implies $G \subseteq \underline{S}_{T_{\mathrm{SR}}}(X)$ which contradicts assumption $\underline{S}_{T_{\mathrm{SR}}}(X) = \phi$. Accordingly, X does not contain any nonempty element of T_{SR}. □

Definition 12. Suppose that $A_{T_{\mathrm{SR}}} = (U, F_A, T_{\mathrm{SR}})$ is a T_{SR} approximation space. The probability $\mathscr{P}_\varnothing$ that the T_{SR} lower approximation of a subset of the universe set is an empty set is defined by

$$\mathscr{P}_\varnothing = \frac{1 + \Lambda_\varnothing^{T_{\mathrm{SR}}}}{2^n}, \quad \text{where } \Lambda_\varnothing^{T_{\mathrm{SR}}} \text{ represents a number of subsets of } U \text{ which does not contain any nonempty element of } T_{\mathrm{SR}} \text{ and } n \text{ is the number of elements in } U. \tag{29}$$

Therefore, the probability that the T_{SR} lower approximation of $X \subseteq U$ is not an empty set is $1 - \mathscr{P}_\varnothing$.

Example 12

(1) Consider Example 4; obviously, the subsets $\{a\}$ and $\{b\}$ are the only subsets which do not contain any nonempty element of T_{SR}. Thus, the probability $\mathscr{P}_\varnothing$ that the T_{SR} lower approximation of a subset of the universe set is an empty set is defined by

$$\mathscr{P}_\varnothing = \frac{1 + \Lambda_\varnothing^{T_{\mathrm{SR}}}}{2^n} = \frac{1+2}{2^4} = \frac{3}{16} = 0.19. \tag{30}$$

(2) Consider Example 5; obviously, the subsets $\{b\}, \{c\}$, and $\{b,c\}$ are the only subsets which do not contain any nonempty element of T_{SR}. Therefore, the probability P_ϕ that the T_{SR} lower approximation of a subset of the universe set is an empty set is defined by

$$P_\phi = \frac{1 + \Lambda_\varnothing^{T_{\mathrm{SR}}}}{2^n} = \frac{1+3}{2^4} = \frac{4}{16} = 0.25. \tag{31}$$

In order to find the probability that T_{SR} upper approximation of any subset is U, we may have to suggest the subsequent result.

Theorem 11. *Consider $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space and $X \subseteq U$. $\overline{S}(X) = U$ if and only if the subset X intersects with every element in T_{SR}.*

Proof. Obvious.

Now, for any T_{SR} approximation space, the probability $\mathscr{P}_U$ that T_{SR} upper approximation of a subset is U can be obtained by the following formula:

$$\mathscr{P}_U = \frac{\Lambda_X^{T_{SR}}}{2^n}, \quad \text{where } \Lambda_X^{T_{SR}} \text{ is the number of subsets of } U \text{ which intersect with every nonempty element of } T_{SR} \text{ and } n \text{ is the number of elements in } U. \tag{32}$$

The next example explains the above discussion. □

Example 13

(1) Consider Example 4; obviously, the subsets $\{a, c, d\}$ and $\{b, c, d\}$ are the only subsets which intersect with every nonempty element of T_{SR}. Thus, the probability $\mathscr{P}_U$ that T_{SR} upper approximation of a subset is U is defined by

$$\mathscr{P}_U = \frac{\Lambda_X^{T_{SR}}}{2^n} = \frac{2}{2^4} = \frac{2}{16} = 0.125. \tag{33}$$

(2) Consider Example 5; obviously, the subsets $\{a, d\}, \{a, b, d\}$, and $\{a, c, d\}$ are the only subsets which intersect with every nonempty element of T_{SR}. Thus, the probability $\mathscr{P}_U$ that T_{SR} upper approximation of a subset is U is defined by

$$\mathscr{P}_U = \frac{\Lambda_X^{T_{SR}}}{2^n} = \frac{3}{2^4} = \frac{3}{16} = 0.19. \tag{34}$$

4. Medical Application in Heart Failure

In the current article, we illustrate the significance of the suggested approach in decision-making problems for medical applications. Consequently, we apply it to the issue of heart failure. We have a data set with the results of five symptoms for twenty patients divided into twelve males $(p_3, p_6, p_8, p_9, p_{11}, \ldots, p_{17}, p_{19})$ and 8 females $(p_1, p_2, p_4, p_5, p_7, p_{10}, p_{18}, p_{20})$. The study was conducted at Om El-Kora Cardiac Center, Hospital of Heart Diseases, Tanta, Egypt. This research involved twenty patients who came to the hospital with various symptoms and underwent a thorough history, physical examination, lab tests, resting ECG, and conventional echo assessment. Finally, the diagnosis of heart failure was verified.

4.1. The Experimental Results. The experimental findings are discussed in this subsection by adding a preparatory analysis performed on five heart disease symptoms for twenty patients, according to Thivagar and Richard [40]. Table 2 shows the data from the information system for twenty patients, addressing the heart failure issue. The columns reflect the signs of heart failure diagnosis (where "Yes" indicates that the patient has symptoms and "No" indicates that the patient has none) [40], with condition attributes, such that e_1 indicates "the breathlessness," e_2 indicates "the orthopnea," e_3 indicates "the paroxysmal nocturnal dyspnea," e_4 indicates "reduced exercise tolerance," and e_5 indicates "the ankle swelling." Attribute D indicates "decision of heart failure." For rows in Table 2, $P = \{p_1, p_2, p_3, \ldots, p_{20}\}$ represents the set of twenty patients. Therefore, the set of all attributes is $A = \{e_1, e_2, e_3, e_4, e_5\} \cup D$ which is represented by columns.

Here 1 and 0 denote "yes" and "no", respectively.

We apply the suggested method in the set of female's patients only and the others similarly.

Accordingly, Table 3 represents the soft set of female's patients, where the set of female's patients is $U = \{p_1, p_2, p_4, p_5, p_7, p_{10}, p_{18}, p_{20}\}$ and the set of attributes is $A = \{e_1, e_2, e_3, e_4, e_5\}$.

Let (F, A) be a soft set over given by Table 3; the basis, generated by (F, A), is given by

$$\mathbf{B}_{F_A} = \{\{p_4\}, \{p_{18}\}, \{p_2, p_{20}\}, \{p_4, p_{18}\}, \{p_4, p_{20}\}, \{p_{18}, p_{20}\}, \{p_7, p_{18}, p_{20}\}, \{p_2, p_4, p_5, p_{20}\}, \{p_2, p_7, p_{18}, p_{20}\}, \{p_1, p_4, p_{18}, p_{20}\}\}. \tag{35}$$

Therefore, the topology generated by this base is

Table 2: Original medical information system.

U/A	e_1	e_2	e_3	e_4	e_5	D
p_1	1	1	0	0	0	Yes
p_2	0	0	0	1	1	No
p_3	0	1	1	0	1	Yes
p_4	1	1	1	1	0	Yes
p_5	0	0	0	1	0	No
p_6	0	0	0	1	1	No
p_7	1	0	0	1	0	No
p_8	1	0	0	1	0	No
p_9	0	1	1	0	1	Yes
p_{10}	0	0	0	0	0	No
p_{11}	1	0	0	1	1	No
p_{12}	0	1	1	0	1	No
p_{13}	1	1	1	1	1	Yes
p_{14}	1	0	1	0	0	Yes
p_{15}	1	0	0	1	1	No
p_{16}	0	0	0	1	1	No
p_{17}	0	1	1	0	0	Yes
p_{18}	1	1	1	0	1	Yes
p_{19}	0	1	1	0	1	Yes
p_{20}	1	1	0	1	1	Yes

Table 3: The tabular form for female's soft set (F, A).

U/A	e_1	e_2	e_3	e_4	e_5	D
p_1	1	1	0	0	0	Yes
p_2	0	0	0	1	1	No
p_4	1	1	1	1	0	Yes
p_5	0	0	0	1	0	No
p_7	1	0	0	1	0	No
p_{10}	0	0	0	0	0	No
p_{18}	1	1	1	0	1	Yes
p_{20}	0	1	1	0	1	Yes

$$\begin{aligned} T_{SR} = \{&U, \phi, \{p_2\}, \{p_4\}, \{p_{18}\}, \{p_2, p_4\}, \{p_2, p_{18}\}, \{p_4, p_7\}, \{p_4, p_{18}\}, \{p_{18}, p_{20}\}, \{p_1, p_4, p_{18}\}, \{p_2, p_4, p_7\}, \{p_2, p_4, p_{18}\}, \\ &\{p_2, p_{18}, p_{20}\}, \{p_4, p_7, p_{18}\}, \{p_4, p_{18}, p_{20}\}, \{p_1, p_2, p_4, p_{18}\}, \{p_1, p_4, p_7, p_{18}\}, \{p_1, p_4, p_{18}, p_{20}\}, \{p_2, p_4, p_5, p_7\}, \\ &\{p_2, p_4, p_{18}, p_{20}\}, \{p_4, p_7, p_{18}, p_{20}\}, \{p_1, p_2, p_4, p_7, p_{18}\}, \{p_1, p_2, p_4, p_{18}, p_{20}\}, \{p_1, p_4, p_7, p_{18}, p_{20}\}, \{p_2, p_4, p_5, p_7, p_{18}\}, \\ &\{p_2, p_4, p_7, p_{18}, p_{20}\}, \{p_1, p_2, p_4, p_5, p_7, p_{18}\}, \{p_1, p_2, p_4, p_7, p_{18}, p_{20}\}, \{p_2, p_4, p_5, p_7, p_{18}, p_{20}\}, \{p_1, p_2, p_4, p_5, p_7, p_{18}, p_{20}\}\}. \end{aligned} \quad (36)$$

The complement of T_{SR} is

$$\begin{aligned} T_{SR}^c = \{&U, \phi, \{p_{10}\}, \{p_1, p_{10}\}, \{p_5, p_{10}\}, \{p_{10}, p_{20}\}, \{p_1, p_5, p_{10}\}, \{p_1, p_{10}, p_{20}\}, \{p_2, p_5, p_{10}\}, \{p_5, p_7, p_{10}\}, \{p_5, p_{10}, p_{20}\}, \\ &\{p_1, p_5, p_{10}\}, \{p_1, p_5, p_7, p_{10}\}, \{p_1, p_{10}, p_{18}, p_{20}\}, \{p_2, p_5, p_7, p_{10}\}, \{p_2, p_5, p_{10}, p_{20}\}, \{p_5, p_7, p_{10}, p_{20}\}, \\ &\{p_1, p_2, p_5, p_7, p_{10}\}, \{p_1, p_2, p_5, p_{10}, p_{20}\}, \{p_1, p_5, p_7, p_{10}, p_{20}\}, \{p_1, p_4, p_5, p_7, p_{10}\}, \{p_1, p_5, p_{10}, p_{18}, p_{20}\}, \\ &\{p_2, p_5, p_7, p_{10}, p_{20}\}, \{p_1, p_2, p_4, p_5, p_7, p_{10}\}, \{p_1, p_2, p_5, p_{10}, p_{18}, p_{20}\}, \{p_1, p_2, p_5, p_7, p_{10}, p_{20}\}, \{p_1, p_4, p_5, p_7, p_{10}, p_{20}\}, \\ &\{p_1, p_5, p_7, p_{10}, p_{18}, p_{20}\}, \{p_1, p_2, p_4, p_5, p_7, p_{10}, p_{20}\}, \{p_1, p_2, p_5, p_7, p_{10}, p_{18}, p_{20}\}, \{p_1, p_4, p_5, p_7, p_{10}, p_{18}, p_{20}\}\}. \end{aligned} \quad (37)$$

Now, we introduce a comparison between the boundary and the accuracy of the approximations using the proposed approximations "T_{SR} approximations" and the previous ones in Table 4.

Table 4: Comparisons among some soft rough approximations and T_{SR} approximations.

X	Soft rough set approach [24]		The suggested approach	
	$BND_{A_S}(X)$	$\mu_{A_S}(X)$	$BND_{T_{SR}}(X)$	$\mu_{T_{SR}}(X)$
$\{p_4\}$	$\{p_1, p_2, p_4, p_5, p_7, p_{18}, p_{20}\}$	0	$\{p_1, p_5, p_{10}\}$	1/4
$\{p_{10}\}$	ϕ	0	$\{p_{10}\}$	0
$\{p_4, p_{10}\}$	$U - \{p_{10}\}$	0	$\{p_4\}$	1/5
$\{p_2, p_4, p_{10}\}$	$U - \{p_{10}\}$	0	$\{p_1, p_5, p_7, p_{10}\}$	1/2
$\{p_2, p_{10}, p_{18}, p_{20}\}$	$\{p_1, p_4, p_5, p_7, p_{18}\}$	3/7	$\{p_1, p_5, p_{10}\}$	1/2
$\{p_2, p_4, p_5, p_7, p_{10}, p_{20}\}$	$\{p_1, p_{18}, p_{20}\}$	4/7	$\{p_1, p_{10}, p_{20}\}$	4/7

(1) Table 4 presents a comparison between the soft rough sets [24] and topological soft rough sets.

(2) For a comparison with the method given in [26], we have the following classes for each parameter:

For e_1 classes are $\{p_1, p_4, p_7, p_{18}\}, \{p_2, p_5, p_{10}, p_{20}\}$
For e_2 classes are $\{p_1, p_4, p_{18}, p_{20}\}, \{p_2, p_5, p_7, p_{10}\}$
For e_3 classes are $\{p_4, p_{18}, p_{20}\}, \{p_1, p_2, p_5, p_7, p_{10}\}$
For e_4 classes are $\{p_2, p_4, p_5, p_7\}, \{p_1, p_{10}, p_{18}, p_{20}\}$
For e_5 classes are $\{p_2, p_{18}, p_{20}\}, \{p_1, , p_4, p_5, p_7, p_{10}\}$

Now, the intersection of all these classes is $\{p_1\}$, $\{p_2\}$, $\{p_4\}$, $\{p_5\}$, $\{p_7\}$, $\{p_{10}\}$, $\{p_{18}\}$, $\{p_{20}\}$. This means intersection of all equivalence relations gives rise to identity relation $R = \{(p_1, p_1), (p_2, p_2), (p_4, p_4), (p_5, p_5), (p_7, p_7), (p_{10}, p_{10}), (p_{18}, p_{18}), (p_{20}, p_{20})\}$. Therefore, every subset will be definable in approximation space (U, R). Therefore, the method discussed in [26] fails to produce distinct lower and upper approximations in the given example.

(3) For a comparison with the method given in [25], we have the map $\varphi: U \longrightarrow P(E)$ as the following $\varphi(p_1) = \{e_1, e_2\}$, $\varphi(p_2) = \{e_4, e_5\}$, $\varphi(p_4) = \{e_1, e_2, e_3, e_4\}$, $\varphi(p_5) = \{e_4\}$, $\varphi(p_7) = \{e_1, e_4\}$, $\varphi(p_{10}) =$, $\varphi(p_{18}) = \{e_1, e_2, e_3, e_5\}$, $\varphi(p_{20}) = \{e_2, e_3, e_5\}$. Clearly these sets are distinct. Therefore, we define the approximation of any subset X of U as per the following:

$$\begin{aligned} \underline{X}_\varphi &= \{x \in X, \varphi(x) \neq \varphi(y) \quad \text{for all } y \in X^c\}, \\ \overline{X}_\varphi &\{x \in X, \varphi(x) = \varphi(y) \quad \text{for some } y \in X\}. \end{aligned} \tag{38}$$

So, all subsets of U will be definable so this method also fails to produce distinct lower and upper approximations in the given example.

Decision-making is essential in the daily lives, and this process yields the best alternative from a variety of options. We give Algorithm 1 in table for a decision-making of an information system in terms of the T_{SR} approximations.

4.2. Reduction of Attributes. A very important purpose of rough sets is the reduction of data by removing redundant attributes in the information system. So, the present subsection is devoted for the reduction of an information system in case of topological rough sets for the data given in Table 3. We extend the notion of "nanotopology," which has proposed by Thivagar and Richard [40], to T_{SR} approximations. We shall apply the nanotopology of T_{SR} approximations to identify the key factors of "heart failure" using topological reduction of attributes in information system of Table 3.

First, let us extend the definition of "nanotopology" into "T_{SR} nanotopology" using "T_{SR} approximations."

Definition 13. Consider $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space, and $X \subseteq U$. Therefore, the class $N_{T_{SR}} = \{U, \phi, \underline{S}_{T_{SR}}(X), \overline{S}_{T_{SR}}(X), BND_{T_{SR}}(X)\}$ is called "T_{SR} nanotopology" which represents a general topology generated by the soft rough set $X \subseteq U$. The basis of this topology is given the class $\beta_{T_{SR}} = \{U, \underline{S}_{T_{SR}}(X), BND_{T_{SR}}(X)\}$.

Definition 14. Consider $A_{T_{SR}} = (U, F_A, T_{SR})$ is a T_{SR} approximation space, and $N_{T_{SR}}$ is a T_{SR} nanotopology with a basis $\beta_{T_{SR}}$. Then,

(i) if $\beta_{T_{SR}-e_k} = \beta_{T_{SR}}$, then the attribute e_k is called "dispensable";

(ii) if $\beta_{T_{SR}-e_k} \neq \beta_{T_{SR}}$, then the attribute e_k is not "dispensable." Therefore, the core of attributes is $CORE = \{e_k\}$ which represents the common part of reduction.

Now, we apply the topological reduction for Table 3 to identify the key factors of "heart failure" as follows: We compute the T_{SR} nanotopology to decision-making for two sets of patients:

$X = \{p_1, p_4, p_{18}, p_{20}\}$ which represents a set of patients that have the disease of heart failure and $Y = \{p_2, p_5, p_7, p_{10}\}$ which represents a set of patients that do not have the disease of heart failure.

We will make a topological reduction for first set X and the second set Y similarly.

Case 1 (patients having the heart failure disease).

According to Table 3, we get $\underline{S}_{T_{SR}}(X) = \{p_1, p_4, p_{18}, p_{20}\}$, $\overline{S}_{T_{SR}}(X) = U - \{p_2\}$, and $BND_{T_{SR}}(X) = \{p_5, p_7, p_{10}\}$. Thus, the basis of T_{SR} nanotopology generated by the above T_{SR} approximations is

$$\beta_{T_{SR}} = \{U, \{p_1, p_4, p_{18}, p_{20}\}, \{p_5, p_7, p_{10}\}\}. \tag{39}$$

Step 1. When the attribute "the breathlessness (e_1)" is removed:

The topology generated by this base is

Step 1: Input the soft set (F, A).
Step 2: Take the class $S_{F_A} = \{F(e)\colon \forall e \in A\}$ as a subbasis for a basis $\mathbf{B}_{F_A}$.
Step 3: Compute the basis $\mathbf{B}_{F_A} = \{A \cap B\colon (A, B) \in \mathbf{S}_{F_A} \times \mathbf{S}_{F_A}\}$ by Definition 6.
Step 4: Generate the topology $T_{SR} = \cup\{B\colon B \in \mathbf{B}_{F_A}\}$ by Definition 6.
Step 5: Investigate the T_{SR} upper approximations, say $\overline{S}_{T_{SR}}(X)$, and T_{SR} lower approximations, say $\underline{S}_{T_{SR}}(X)$, for every $X \subseteq U$, according to Definition 8.
Step 6: Determine the boundary region, say $\mathrm{BND}_{T_{SR}}(X)$, from Step 2, according to Definition 9.
Step 7: Calculate the accuracy of the approximation, say $\mu_{T_{SR}}(X)$, from Step 2, according to Definition 9.
Step 8: Decide, exactly, rough sets and exact sets, using Definition 9.

Algorithm 1: A decision-making via T_{SR} approximations.

$$T_{SR} = \{U, \phi, \{p_2\}, \{p_4\}, \{p_{20}\}, \{p_2, p_4\}, \{p_2, p_{20}\}, \{p_4, p_{20}\}, \{p_{18}, p_{20}\}, \{p_2, p_{18}, p_{20}\}, \{p_4, p_{18}, p_{20}\}, \{p_2, p_4, p_5, p_7\}, \{p_1, p_4, p_{18}, p_{20}\}, \{p_2, p_4, p_{18}, p_{20}\}, \{p_1, p_2, p_4, p_{18}, p_{20}\}, \{p_2, p_4, p_5, p_7, p_{20}\}, \{p_2, p_4, p_5, p_7, p_{18}, p_{20}\}, \{p_1, p_2, p_4, p_7, p_{18}, p_{20}\}\}. \tag{40}$$

The complement of T_{SR} is

$$T^c_{SR} = \{U, \phi, \{p_{10}\}, \{p_1, p_{10}\}, \{p_1, p_{10}, p_{18}\}, \{p_5, p_7, p_{10}\}, \{p_1, p_5, p_7, p_{10}\}, \{p_2, p_5, p_7, p_{10}\}, \{p_1, p_{10}, p_{18}, p_{20}\}, \{p_1, p_2, p_5, p_7, p_{10}\}, \{p_1, p_4, p_5, p_7, p_{10}\}, \{p_1, p_2, p_4, p_5, p_7, p_{10}\}, \{p_1, p_2, p_5, p_7, p_{10}, p_{18}\}, \{p_1, p_4, p_5, p_7, p_{10}, p_{18}\}, \{p_1, p_5, p_7, p_{10}, p_{18}, p_{20}\}, \{p_1, p_2, p_4, p_5, p_7, p_{10}, p_{18}\}, \{p_1, p_2, p_5, p_7, p_{10}, p_{18}, p_{20}\}, \{p_1, p_4, p_5, p_7, p_{10}, p_{18}, p_{20}\}\}. \tag{41}$$

Therefore, T_{SR} approximations of X in this case are $\underline{S}_{T_{SR}}(X) = \{p_1, p_4, p_{18}, p_{20}\}$, $\overline{S}_{T_{SR}}(X) = U - \{p_2\}$, and $\mathrm{BND}_{T_{SR}}(X) = \{p_5, p_7, p_{10}\}$. Thus, the basis of T_{SR} nano-topology generated by the above T_{SR} approximations is

$$\beta_{T_{SR}-e_1} = \{U, \{p_1, p_4, p_{18}, p_{20}\}, \{p_5, p_7, p_{10}\}\} = \beta_{T_{SR}}. \tag{42}$$

Step 2. When the attribute "the orthopnea (e_2)" is removed:
By the same way as in Step 1, we get

$$\beta_{T_{SR}-e_2} = \{U, \{p_4, p_{18}\}, \{p_1, p_5, p_7, p_{10}, p_{20}\}\} \neq \beta_{T_{SR}}. \tag{43}$$

Step 3. When the attribute "the paroxysmal nocturnal dyspnea (e_3)" is removed: By the same way as in Step 1, we get

$$\beta_{T_{SR}-e_3} = \{U, \{p_1, p_4, p_{18}, p_{20}\}, \{p_5, p_7, p_{10}\}\} = \beta_{T_{SR}}. \tag{44}$$

Step 4. When the attribute "reduced exercise tolerance (e_4)" is removed:
By the same way as in Step 1, we get

$$\beta_{T_{SR}-e_4} = \{U, \{p_1, p_4, p_{18}, p_{20}\}, \{p_2, p_5, p_7, p_{10}\}\} \neq \beta_{T_{SR}}. \tag{45}$$

Step 5. When the attribute "the ankle swelling (e_5)" is removed:
By the same way as in Step 1, we get

$$\beta_{T_{SR}-e_5} = \{U, \{p_1, p_4, p_{18}, p_{20}\}, \{p_2, p_5, p_7, p_{10}\}\} \neq \beta_{T_{SR}}. \tag{46}$$

Therefore, we get the attributes $\{e_1, e_3\}$ are dispensable and $\{e_2, e_4, e_5\}$ are not dispensable. Accordingly, the core of attributes is $\mathrm{CORE}(T_{SR}) = \{e_2, e_4, e_5\}$, i.e., "the orthopnea, reduced exercise tolerance, and the ankle swelling" represent the main attributes that have close joining to the disease of the heart failure.

By the same manner, we can make a topological reduction to Table 2 (an information system of all "the heart failure" patients).

At the end of the paper, we give Algorithm 2 in table which can be used to make a topological reduction of attributes for information system in terms of the T_{SR} approximations.

In literature many methods for reduction of parameters for soft sets have been given, for example, [32, 34, 41]. In all these methods only positive parameters are considered which result in a decision parameter as the sum of the values allotted to an alternative. Then on the basis of this decision parameter most suitable alternative is selected. In the present case of decision-making these methods fail. In given case

Step 1: Input a soft set (F, A), by using the two finite sets, a universe U and A a set of attributes (parameters). which represent the data as an information table, rows of which are labeled by attributes (A) and columns by objects, and entries of the table are attribute values.
Step 2: Compute the T_{SR} upper approximations, say $\overline{S}_{T_{SR}}(X)$, and T_{SR} lower approximations, say $\underline{S}_{T_{SR}}(X)$, and T_{SR} boundary, say $\mathrm{BND}_{T_{SR}}(X)$, for the decision set $X \subseteq U$, according to Definition 8.
Step 3: Generate the base $\beta_{T_{SR}}$ of T_{SR} nanotopology generated by T_{SR} approximations in Step 2 using Definition 13.
Step 4: Eliminate an attribute e_k from the attributes (A) and find the T_{SR} upper approximation, T_{SR} lower approximation, and T_{SR} boundary for the decision set X on $A - (e_k)$.
Step 5: Generate the base $\beta_{T_{SR}-e_k}$ of T_{SR} nanotopology generated by T_{SR} approximations given in Step 4 using Definition 13.
Step 6: Recurrence Steps 4 and 5 for every attribute in A.
Step 7: The attributes in A for which $\beta_{T_{SR}-e_k} \neq \beta_{T_{SR}}$ form the CORE(T_{SR}).

Algorithm 2: Topological reduction of attributes via T_{SR} approximations.

parameters are not all positive but a blend of positive and negative, decided by the experts. Moreover decision parameter is attached already.

5. Conclusion

In this article notion of topological soft rough sets is introduced, where topology generated from a soft set plays a vital role. Here notion of soft rough approximations is discussed and some of their properties are given. Their properties have been studied and their relationships with some other methods have been examined. In fact, the proposed approaches fulfill all axioms of Pawlak's rough sets without adding extra restrictions as Propositions 3 and 4 illustrated. The proposed techniques depend basically on general topology and hence they open the way for applications of topology in soft rough sets. Further, we have answered some very important questions, such as how to determine the probability that a subset of the universe is definable in the classical rough sets and their extensions (like the soft sets and topological soft rough sets).

Finally, we have introduced medical applications, in the decision-making of medical diagnosis for heart failure problems [39], to illustrate the importance of current methods and also to compare proposed method and the previous ones. Moreover, we have succeeded in making a topological reduction for the data set covering the result of five symptoms for twenty patients with heart failure disease, and thus we identify the core factors of the heart failure diagnosis. Besides, two algorithms to our method have been obtained.

For future works, it is hoped that presented framework may be useful to study its application in COVID-19 and other diseases.

Data Availability

Not data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

References

[1] Z. A. Pawlak, "Rough sets," *International Journal of Computer & Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.
[2] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
[3] D. A. Molodtsov, "Soft set theory-first results," *Computers and Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.
[4] J. L. Kelly, "General topology," *The University Series in Higher Mathematics*, D. Nostrand Co., Inc., New York, NY, USA, 1955.
[5] E. A. Abo-Tabl, "Rough sets and topological spaces based on similarity," *International Journal of Machine Learning and Cybernetics*, vol. 4, no. 5, pp. 451–458, 2013.
[6] E. A. Abo-Tabl, "A comparison of two kinds of definitions of rough approximations based on a similarity relation," *Information Sciences*, vol. 181, no. 12, pp. 2587–2596, 2011.
[7] K. Kin, J. Yang, and Z. Pei, "Generalized rough sets based on reflexive and transitive relations," *Information Sciences*, vol. 178, no. 21, pp. 4138–4141, 2008.
[8] M. Kondo, "On the structure of generalized rough sets," *Information Sciences*, vol. 176, no. 5, pp. 589–600, 2006.
[9] T. M. Al-shami, "An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application," *Information Sciences*, vol. 569, pp. 110–124, 2021.
[10] T. M. Al-shami, W. Q. Fu, and E. A. Abo-Tabl, "New rough approximations based on E-neighborhoods," *Complexity*, vol. 2021, Article ID 6666853, 6 pages, 2021.
[11] M. I. Ali, B. Davvaz, and M. Shabir, "Some properties of generalized rough sets," *Information Sciences*, vol. 224, pp. 170–179, 2013.
[12] M. K. El-Bably and E. A. Abo-Tabl, "A topological reduction for predicting of a lung cancer disease based on generalized rough sets," *Journal of Intelligent & Fuzzy Systems*, vol. 41, no. 2, pp. 3045–3060, 2021.
[13] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl, "New approach for basic rough set concepts," in *International workshop on Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing. Lecture Notes in Artificial Intelligence*, vol. 3641, pp. 64–73, Springer, Regina, Canada, 2005.
[14] Y. Y. Yao, *Generalized Rough Set Models, Rough Sets In Knowledge Discovery 1*, L. Polkowski and A. Skowron, Eds., pp. 286–318, Physica Verlag, Heidelberg, Germany, 1998.
[15] M. K. El-Bably, K. K. Fleifel, and O. A. Embaby, "Topological approaches to rough approximations based on closure operators," *Granular Computing*, pp. 1–14, 2021.
[16] M. E. A. E. Monsef, O. A. Embaby, and M. K. E. Bably, "Comparison between rough set approximations based on

different topologies," *International Journal of Granular Computing, Rough Sets and Intelligent Systems*, vol. 3, no. 4, pp. 292–305, 2014.

[17] A. S. Nawar, M. K. El-Bably, and A. E. F. El-Atik, "Certain types of coverings based rough sets with application," *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 3, pp. 3085–3098, 2020.

[18] M. E. Abd El-Monsef, A. M. Kozae, and M. K. El-Bably, "On generalizing covering approximation space," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 3, pp. 535–545, 2015.

[19] W.-H. Xu and W.-X. Zhang, "Measuring roughness of generalized rough sets induced by a covering," *Fuzzy Sets and Systems*, vol. 158, no. 22, pp. 2443–2455, 2007.

[20] K. V. Babitha and S. J. John, "Soft topologies generated by soft set relations," *Handbook of Research on Generalized and Hybrid Set Structures and Applications for Soft Computing*, IGI Global, Hershey, PA, USA, pp. 118–126, 2016.

[21] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, "On some new operations in soft set theory," *Computers & Mathematics with Applications*, vol. 57, no. 9, pp. 1547–1553, 2009.

[22] S. Jafari, A. E. F. El-Atik, R. M. Latif, and M. K. El-Bably, "Soft topological spaces induced via soft relations," *WSEAS Transactions on Mathematics*, vol. 20, pp. 1–8, 2021.

[23] I. Zorlutuna, M. Akdag, W. K. Min, and S. Atmaca, "Remarks on soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 3, pp. 171–185, 2012.

[24] F. Feng, X. Liu, V. Leoreanu-Fotea, and Y. B. Jun, "Soft sets and soft rough sets," *Information Sciences*, vol. 181, no. 6, pp. 1125–1137, 2011.

[25] M. Shabir, M. Irfan Ali, and T. Shaheen, "Another approach to soft rough sets," *Knowledge-Based Systems*, vol. 40, pp. 72–80, 2013.

[26] M. Irfan Ali, "A note on soft sets, rough soft sets and fuzzy soft sets," *Applied Soft Computing*, vol. 11, no. 4, pp. 3329–3332, 2011.

[27] Z. Li and T. Xie, "The relationship among soft sets, soft rough sets and topologies," *Soft Computing*, vol. 18, no. 4, pp. 717–728, 2014.

[28] M. K. El-Bably and A. E. F. A. El Atik, "Soft β-rough sets and their application to determine COVID-19," *Turkish Journal of Mathematics*, vol. 45, no. 3, pp. 1133–1148, 2021.

[29] M. El Sayed, A. Q. Al Qubati, A. Gawad, and K. El-Bably, "Soft pre-rough sets and its applications in decision making," *Mathematical Biosciences and Engineering*, vol. 17, no. 5, pp. 6045–6063, 2020.

[30] A. M. Khalil, S.-G. Li, Y. Lin, H.-X. Li, and S.-G. Ma, "A new expert system in prediction of lung cancer disease based on fuzzy soft sets," *Soft Computing*, vol. 24, no. 18, pp. 14179–14207, 2020.

[31] S. Yuksel, T. Dizman, G. Yildizdan, and U. Sert, "Application of soft sets to diagnose the prostate cancer risk," *Journal of Inequalities and Applications*, vol. 2013, no. 1, pp. 1–11, 2013.

[32] P. K. Maji, R. Roy, and R. Biswas, "An application of soft sets in decision making problem," *Computers and Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[33] Y. Liu, K. Qin, and L. Martínez, "Improving decision making approaches based on fuzzy soft sets and rough soft sets," *Applied Soft Computing*, vol. 65, pp. 320–332, 2018.

[34] D. Chen, E. C. C. Tsang, D. S. Yeung, and X. Wang, "The parameterization reduction of soft sets and its applications," *Computers and Mathematics with Applications*, vol. 49, no. 5-6, pp. 757–763, 2005.

[35] A. R. Roy and P. K. Maji, "A fuzzy soft set theoretic approach to decision making problems," *Journal of Computational and Applied Mathematics*, vol. 203, no. 2, pp. 412–418, 2007.

[36] N. Cagman and S. Enginoglu, "Soft set theory and *uni-int* decision making," *European Journal of Operational Research*, vol. 207, no. 2, pp. 848–855, 2010.

[37] S. Jha, R. Kumar, L. H. Son et al., "Neutrosophic soft set decision making for stock trending analysis," *Evolving Systems*, vol. 10, no. 4, pp. 621–627, 2019.

[38] M. El Sayed, M. A. El Safty, and M. K. El-Bably, "Topological approach for decision-making of COVID-19 infection via a nano-topology model," *AIMS Mathematics*, vol. 6, no. 7, pp. 7872–7894, 2021.

[39] K. Dickstein, A. Cohen-Solal, G. Filippatos et al., "Developed in collaboration with the heart failure association of the ESC (HFA) and endorsed by the European society of intensive care medicine (ESICM)," *European Journal of Heart Failure*, vol. 10, no. 10, pp. 933–989, 2008.

[40] M. L. Thivagar and C. Richard, "On nano forms of weakly open sets," *International Journal of Mathematics and Statistics Invention*, vol. 1, no. 1, pp. 31–37, 2013.

[41] M. I. Ali, "Another view on reduction of parameters in soft sets," *Applied Soft Computing*, vol. 12, no. 6, pp. 1814–1821, 2012.