Research Article

Latest Inversion-Free Iterative Scheme for Solving a Pair of Nonlinear Matrix Equations

Sourav Shil and Hemant Kumar Nashine

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, TN, India

Correspondence should be addressed to Hemant Kumar Nashine; hemant.nashine@vit.ac.in

Received 14 June 2021; Revised 30 July 2021; Accepted 11 August 2021; Published 29 August 2021

Academic Editor: Ali Jaballah

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In this work, the following system of nonlinear matrix equations is considered, $X_1 + A^* X_1^{-1} A + B^* X_2^{-1} B = I$ and $X_2 + C^* X_2^{-1} C + D^* X_1^{-1} D = I$, where A, B, C , and D are arbitrary $n \times n$ matrices and I is the identity matrix of order n . Some conditions for the existence of a positive-definite solution as well as the convergence analysis of the newly developed algorithm for finding the maximal positive-definite solution and its convergence rate are discussed. Four examples are also provided herein to support our results.

1. Introduction and Preliminaries

Consider the system of matrix equations of the form

$$\Omega_1(X) + A^* P_1(X) A + B^* P_2(X) B = Q, \quad (1)$$

$$\Omega_2(X) + C^* Q_1(X) C + D^* Q_2(X) D = Q, \quad (2)$$

where Q is an $n \times n$ Hermitian positive definite matrix (HPD, for short), A, B, C , and D are complex matrices of order $n \times n$, $\Omega_1(X)$, $\Omega_2(X)$, $P_1(X)$, $P_2(X)$, $Q_1(X)$, and $Q_2(X)$ are mappings from the set of positive definite matrices to itself, and A^* is the conjugate transpose of A .

We can see that the above equations incorporate a few linear as well as nonlinear matrix equations (NMEs, in short). Over the last few decades, many researchers studied systems (1) and (2) with different types of P_i, Q_i , and Ω_i , $i = 1, 2$.

For (1), $\Omega_1(X) = X^s$, $P_1(X) = X^{-p_1}$, and $P_2(X) = X^{-p_2}$, $p_1, p_2 \in (0, 1]$ and $s \in \mathbb{I}^+$ [1]

For (1), $\Omega_1(X) = X$, $P_1(X) = X^{-1}$, $P_2(X) = X^{-1}$, and $Q = I$ [2, 3]

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For (1), $\Omega_1(X) = X^s$, $P_1(X) = X^{-p_1}$, and $P_2(X) = X^{-p_2}$, $s, p_1, p_2 \geq 1$ [7]

For (1) and (2), $\Omega_1(X) = \Omega_2(X) = X$, $Q_1(X) = X^{-p_1}$, $P_1(X) = X^{-p_2}$, and $Q_2(X) = Q_2(X) = O$, $Q = I$ and $p_1, p_2 \in \mathbb{I}^+$ [8]

For (1) and (2), $\Omega_1(X) = \Omega_2(X) = X$, $Q_1(X) = \pm X^{-p_1}$, $P_1(X) = \pm X^{-p_2}$, and $Q_2(X) = Q_2(X) = O$, $Q = I$ and $p_1, p_2 \in \mathbb{I}^+$ [9, 10]

For (1) and (2) $\Omega_1(X) = \Omega_2(X) = X$, $P_1(X) = X^{-p_1}$, $Q_1(X) = X^{-p_2}$, and $Q_2(X) = Q_2(X) = O$, $Q = I$ and $p_1, p_2 \in (0, 1]$ [11]

For different types of applications of the Riccati equation, one can check [4, 12, 13].

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{I}_+$ denotes the set of positive numbers, and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

$H(n)$ (resp. $K(n)$ and $P(n)$) denotes the set of all $n \times n$ Hermitian (resp. positive semidefinite and positive definite) matrices over $\mathbb{C}$ and $M(n)$ stands for the set of all $n \times n$ matrices over $\mathbb{C}$.

For a matrix $D \in H(n)$, $s_1(D) \geq s_2(D) \geq \dots \geq s_n(D)$ denotes its singular values and $\|D\|$ denotes the sum of these values, i.e., the trace norm of D . The Frobenius norm will be denoted by $\|D\|_F = (\sum_{i,j=1}^n |D_{ij}|^2)^{1/2}$. For $P, Q \in H(n)$, $P \geq Q$ (resp. $P > Q$) indicates that $P - Q$ is positive semidefinite (resp. positive definite). O and I stand for the zero and unit matrix in $H(n)$, respectively.

2. Conditions in Support of the Existence of a Positive Definite Solution

We begin with two useful lemmas.

Lemma 1 (see [14]). *If P and $Q > 0$ are Hermitian matrices having the same order, then $2P - PQP \leq Q^{-1}$.*

Lemma 2 (see [15]). *If $U \geq V > O$, then $U^p \geq V^p > O$, for all $p \in (0, 1]$.*

Now, let us consider the system of NMEs of the form

$$\begin{aligned} X_1 + A^* X_1^{-1} A + B^* X_2^{-1} B &= I, \\ X_2 + C^* X_2^{-1} C + D^* X_1^{-1} D &= I. \end{aligned} \quad (3)$$

Taking $Y = X_1^{-1}$ and $Z = X_2^{-1}$, the system of nonlinear matrix equation (3) is equivalent to

$$\begin{aligned} Y^{-1} + A^* Y A + B^* Z B &= I, \\ Z^{-1} + C^* Z C + D^* Y D &= I. \end{aligned} \quad (4)$$

Theorem 1. *Let (Y, Z) be a positive-definite solution (PDS, for short) of (4). Then, $I \leq Y$ and $I \leq Z$.*

Proof. Suppose that system (4) has a PDS $(Y, Z) > (O, O)$. Then, from Lemma 2,

$$\begin{aligned} A^* Y A + B^* Z B &\geq O, \\ C^* Z C + D^* Y D &\geq O, \end{aligned} \quad (5)$$

which gives

$$\begin{aligned} Y^{-1} &= I - A^* Y A - B^* Z B \leq I, \\ Z^{-1} &= I - C^* Z C - D^* Y D \leq I. \end{aligned} \quad (6)$$

This, in turn, implies that $I \leq Y$ and $I \leq Z$. $\square$

Lemma 3 (see Theorem 2.1 in [16]). *Let U and V be positive operators acting on a Hilbert space H in such a way that $P_1 I \geq U \geq p_1 I > O$, $P_2 I \geq V \geq p_2 I > O$, and $V \geq U > O$. Then,*

$$\begin{aligned} U^s &\leq \left(\frac{P_1}{p_1} \right)^{s-1} V^s, \\ U^s &\leq \left(\frac{P_2}{p_2} \right)^{s-1} V^s, \end{aligned} \quad (7)$$

hold for any $s \geq 1$.

Remark 1. In the case that $V > U > O$ in Lemma 3, we have $U^s < (P_1/p_1)^{s-1} V^s$ and $U^s < (P_2/p_2)^{s-1} V^s$.

Theorem 2. *Suppose that system (4) has a PDS (Y, Z) . Then,*

- (i) $A^* A + B^* B < I$ and $C^* C + D^* D < I$
- (ii) $Y^{-1} > AA^*$, $Y^{-1} > DD^*$, $Z^{-1} > CC^*$, and $Z^{-1} > BB^*$
- (iii) $Y < (1/s_n(A)^2)I$ if A is nonsingular, $Y < (1/s_n(D)^2)I$ if D is nonsingular, $Z < (1/s_n(C)^2)I$ if C is nonsingular, and $Z < (1/s_n(B)^2)I$ if B is nonsingular

Proof. To prove (i), since (Y, Z) is a PDS of the NMEs (4), we have $Y \geq I$ and $Z \geq I$. Together with (4), this gives

$$A^* A + B^* B \leq A^* Y A + B^* Z B = I - Y^{-1} < I, \quad (8)$$

and

$$C^* C + D^* D \leq C^* Z C + D^* Y D = I - Z^{-1} < I. \quad (9)$$

To prove (ii), from (6), we have

$$\begin{aligned} A^* Y A &< I, \\ A^* Y^{1/2} Y^{1/2} A &< I, \\ Y^{1/2} A A^* Y^{1/2} &< I, \\ A A^* &< Y^{-1}, \\ B^* Z B &< I, \\ B B^* &< Z^{-1}. \end{aligned} \quad (10)$$

Similarly, by (9), we can obtain $CC^* < Z^{-1}$ and $DD^* < Y^{-1}$.

To prove (iii), if A is nonsingular, then, by $AA^* < Y^{-1}$, we obtain

$$(AA^*)^{-1} > Y. \quad (11)$$

We have

$$\frac{1}{s_1^2(A)} I \leq (AA^*)^{-1} \leq \frac{1}{s_n^2(A)} I. \quad (12)$$

Since $s_n^2(A)I \leq AA^* \leq s_1^2(A)I$, where $s_1(A)$ and $s_n(A)$ are the maximal and the minimal singular value of the matrix A , respectively, by (11) and (12) and Remark 1, we obtain $Y < (AA^*)^{-1} < (1/s_n(A)^2)I$.

For the case of nonsingular matrices D, C , and B , we can obtain similarly. $\square$

Theorem 3. *System (4) attains a PDS (Y, Z) iff A, B, C, D admit the subsequent factorizations:*

$$\begin{aligned} A &= (W^* W)^{-(1/2)} K_1, \\ B &= (U^* U)^{-(1/2)} K_2, \\ C &= (U^* U)^{-(1/2)} L_1, \\ D &= (W^* W)^{-(1/2)} L_2, \end{aligned} \quad (13)$$

where the matrices W and U are nonsingular and the columns $\begin{pmatrix} W^{-*} \\ K_1 \\ K_2 \end{pmatrix}$ and $\begin{pmatrix} U^{-*} \\ L_1 \\ L_2 \end{pmatrix}$ are orthonormal. In this case, $(Y, Z) = (W^*W, U^*U)$ is a solution of (4).

Proof. If the system of NMEs (4) has a PDS (Y, Z) , then $(Y, Z) = (W^*W, U^*U)$ for some nonsingular matrices W, U . Rewrite the equation (4) as

$$\begin{cases} (W^*W)^{-1} + A^*(W^*W)A + B^*(U^*U)B = I, \\ (U^*U)^{-1} + C^*(U^*U)C + D^*(W^*W)D = I, \end{cases} \quad (14)$$

or

$$\begin{cases} (W^{-*})^*W^{-*} + [(W^*W)^{1/2}A]^* (W^*W)^{1/2}A + [(U^*U)^{1/2}B]^* (U^*U)^{1/2}B = I, \\ (U^{-*})^*U^{-*} + [(U^*U)^{1/2}C]^* (U^*U)^{1/2}C + [(W^*W)^{1/2}D]^* (W^*W)^{1/2}D = I, \end{cases} \quad (15)$$

or equivalently

$$\begin{aligned} \begin{pmatrix} W^{-*} \\ (W^*W)^{1/2}A \\ (U^*U)^{1/2}B \end{pmatrix}^* \begin{pmatrix} W^{-*} \\ (W^*W)^{1/2}A \\ (U^*U)^{1/2}B \end{pmatrix} &= I, \\ \begin{pmatrix} U^{-*} \\ (U^*U)^{1/2}C \\ (W^*W)^{1/2}D \end{pmatrix}^* \begin{pmatrix} U^{-*} \\ (U^*U)^{1/2}C \\ (W^*W)^{1/2}D \end{pmatrix} &= I. \end{aligned} \quad (16)$$

Let

$$\begin{aligned} (W^*W)^{1/2}A &= K_1, \\ (U^*U)^{1/2}B &= K_2, \\ (U^*U)^{1/2}C &= L_1, \\ (W^*W)^{1/2}D &= L_2. \end{aligned} \quad (17)$$

Then,

$$\begin{aligned} A &= (W^*W)^{-(1/2)}K_1, \\ B &= (U^*U)^{-(1/2)}K_2, \\ C &= (U^*U)^{-(1/2)}L_1, \\ D &= (W^*W)^{-(1/2)}L_2. \end{aligned} \quad (18)$$

Now, (16) implies that $\begin{pmatrix} W^{-*} \\ K_1 \\ K_2 \end{pmatrix}$ and $\begin{pmatrix} U^{-*} \\ L_1 \\ L_2 \end{pmatrix}$ are orthonormal. Conversely, assume that A, B, C , and D have decomposition (13). Set $(Y, Z) = (W^*W, U^*U)$. Then,

$$\begin{aligned} Y^{-1} + A^*YA + B^*ZB &= (W^*W)^{-1} + \left[(W^*W)^{-(1/2)}K_1 \right]^* (W^*W)(W^*W)^{-(1/2)}K_1 \\ &\quad + \left[(U^*U)^{-(1/2)}K_2 \right]^* (U^*U)(U^*U)^{-(1/2)}K_2 \\ &= W^{-1}W^{-*} + K_1^*K_1 + K_2^*K_2 = I, \\ Z^{-1} + C^*ZC + D^*YD &= (U^*U)^{-1} + \left[(U^*U)^{-(1/2)}L_1 \right]^* (U^*U)(U^*U)^{-(1/2)}L_1 \\ &\quad + \left[(W^*W)^{-(1/2)}L_2 \right]^* (W^*W)(W^*W)^{-(1/2)}L_2 \\ &= U^{-1}U^{-*} + L_1^*L_1 + L_2^*L_2 = I, \end{aligned} \quad (19)$$

which implies $(Y, Z) = (W^*W, U^*U)$ is a PDS of (4). $\square$

Example 1. We can find W, U, K_1, K_2, L_1 , and L_2 (see Theorem 7.2.7, page 440, in [17]) for the following numerical experiment (2) which satisfies all the conditions of Theorem 3:

$$\begin{aligned}
 W &= \begin{bmatrix} 1.0355 & 0.0341 & 0.0307 \\ 0.0341 & 1.0448 & 0.0359 \\ 0.0307 & 0.0359 & 1.0335 \end{bmatrix}, \\
 U &= \begin{bmatrix} 1.0314 & -0.0004 & 0.0020 \\ -0.0004 & 1.0276 & 0.0003 \\ 0.0020 & 0.0003 & 1.0489 \end{bmatrix}, \\
 K_1 &= \begin{bmatrix} 0.0446 & 0.1012 & 0.0750 \\ 0.0667 & 0.1494 & 0.0809 \\ 0.1614 & 0.1460 & 0.1117 \end{bmatrix}, \\
 K_2 &= \begin{bmatrix} 0.1087 & 0.0182 & 0.0888 \\ 0.0770 & 0.1376 & 0.1556 \\ 0.1089 & 0.0686 & 0.0387 \end{bmatrix}, \\
 L_1 &= \begin{bmatrix} 0.2442 & 0.0011 & 0.0017 \\ -0.0059 & 0.2283 & -0.0207 \\ 0.0122 & 0.0224 & 0.2800 \end{bmatrix}, \\
 L_2 &= \begin{bmatrix} -0.0104 & 0.0009 & -0.0024 \\ -0.0014 & -0.0171 & -0.0143 \\ 0.0043 & 0.0110 & -0.1093 \end{bmatrix}.
 \end{aligned} \tag{20}$$

Also, it is easy to derive

$$\begin{aligned}
 s_1(A) &= 0.2981, \\
 s_n(A) &= 0.0146, \\
 s_1(B) &= 0.2691, \\
 s_n(B) &= 0.048, \\
 s_1(C) &= 0.2692, \\
 s_n(C) &= 0.2227, \\
 s_1(D) &= 0.1065, \\
 s_n(D) &= 0.0101.
 \end{aligned} \tag{21}$$

It meets all of the requirements of Theorem 2.

3. Construction of Iteration Schemes

This section contains a new iteration scheme for the NMEs (4) in the context of [2, 11].

By pre- and postmultiplying the first and the second equation of (4), respectively, by Y and Z , we obtain

$$\begin{aligned}
 Y - Y(I - A^*YA - B^*ZB)Y &= O, \\
 Z - Z(I - C^*ZC - D^*YD)Z &= O.
 \end{aligned} \tag{22}$$

After some simple calculation, we have

$$\begin{aligned}
 Y &= 2Y - Y(I - A^*YA - B^*ZB)Y, \\
 Z &= 2Z - Z(I - C^*ZC - D^*YD)Z.
 \end{aligned} \tag{23}$$

Apparently, (Y, Z) is a solution of (23) if (Y, Z) solves equation (4). Conversely, if (Y, Z) is a nonsingular solution of equation (23), (Y, Z) solves equation (4), too.

Thus, to attain a HPDS of (4), we need to solve (23). From (23), the iterative scheme is as follows.

$$\begin{cases} \hat{Y}_n = I - A^*Y_nA - B^*Z_nB \\ Y_{n+1} = (2I - Y_n\hat{Y}_n)Y_n, \\ \hat{Z}_n = I - C^*Z_nC - D^*Y_{n+1}D \\ Z_{n+1} = (2I - Z_n\hat{Z}_n)Z_n. \end{cases} \tag{24}$$

4. Convergence Analysis

This section contains the proof that the sequences $\{(Y_n, Z_n)\}$ generated by Algorithm 1, where $Y_0 = I$ and $Z_0 = I$ are initial conditions and converge to the minimal positive-definite solution of equation (4).

Theorem 4. Suppose that the system of NMEs (4) has a PDS and the sequence $\{(Y_n, Z_n)\}$ is generated by Algorithm 1 with initial values $Y_0 = I$ and $Z_0 = I$. Let the pair $(\hat{Y}, \hat{Z})$ be the minimal PDS of (4). Then, the sequence $\{(Y_n, Z_n)\}$ is well defined:

$$\begin{aligned}
 Y_0 &\leq Y_1 \leq Y_2 \leq \dots \leq Y_n \leq \dots \leq \hat{Y}, \\
 Z_0 &\leq Z_1 \leq Z_2 \leq \dots < Z_n \leq \dots \leq \hat{Z}, \\
 \lim_{n \rightarrow \infty} Y_n &= \hat{Y}, \\
 \lim_{n \rightarrow \infty} Z_n &= \hat{Z}.
 \end{aligned} \tag{25}$$

Proof. Let (Y, Z) be any PDS of the system of NMEs (4). We will prove that $O < Y_n \leq Y_{n+1} \leq Y$ and $O < Z_n \leq Z_{n+1} \leq Z$ for all $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$ using mathematical induction.

For $n = 0$, we have $Y_0 = I > O$ and $Z_0 = I > O$. Then, by (24), we have

$$\begin{aligned}
 Y_1 &= 2I - I + A^*A + B^*B = I + A^*A + B^*B \geq I = Y_0, \\
 Z_1 &= 2I - I + C^*C + D^*Y_1D = I + C^*C + D^*Y_1D \geq I = Z_0.
 \end{aligned} \tag{26}$$

Following Theorem 4, $I - A^*A - B^*B > O$. Using this condition in Lemma 1 together with NMEs (4),

$$\begin{aligned}
 Y_1 &= 2I - I(I - A^*A - B^*B)I \\
 &\leq (I - A^*A - B^*B)^{-1} \\
 &\leq (I - A^*YA - B^*ZB)^{-1} \\
 &= Y,
 \end{aligned} \tag{27}$$

Step 1: input $A, B, C, D \in \mathbb{C}^{n \times n}$.
 Take $Y_0 = I$ and $Z_0 = I$ for initialization and tolerance error $\epsilon \geq 0$. Set $n := 0$.
 Step 2: find out Y_{n+1} and Z_{n+1} by the following iterative scheme:
 Step 3: stop if $\|Y_{n+1} - Y_n\|_F + \|Z_{n+1} - Z_n\|_F \leq \epsilon$. Otherwise, $n := n + 1$, and go to Step 2.

ALGORITHM 1: An inversion-free iterative algorithm for equation (4).

since $(Y, Z) \geq (I, I)$. That is, $Y_1 \leq Y$. Using Lemma 2, this implies that $I - C^*C - D^*Y_1D \geq I - C^*ZC - D^*YD = Z^{-1} > O$. Using this fact in Lemma 1 together with NMEs (4), we obtain

$$\begin{aligned} Z_1 &= 2I - I(I - C^*C - D^*Y_1D)I \\ &\leq (I - C^*C - D^*Y_1D)^{-1} \\ &\leq (I - C^*ZC - D^*YD)^{-1} \\ &= Z. \end{aligned} \quad (28)$$

Hence, $O < Y_n \leq Y_{n+1} \leq Y$ and $O < Z_n \leq Z_{n+1} \leq Z$ holds true for $n = 0$.

Assume that $O < Y_n \leq Y_{n+1} \leq Y$ and $O < Z_n \leq Z_{n+1} \leq Z$ hold for $n = k \geq 0$. Since $I - A^*Y_{k+1}A - B^*Z_{k+1}B \geq I - A^*YA - B^*ZB = Y^{-1} > O$, by equation (24), Lemma 1, and the fact $Y_{k+1} > O$, we obtain

$$\begin{aligned} Y_{k+2} &= 2Y_{k+1} - Y_{k+1}(I - A^*Y_{k+1}A - B^*Z_{k+1}B)Y_{k+1} \\ &\leq (I - A^*Y_{k+1}A - B^*Z_{k+1}B)^{-1} \\ &\leq (I - A^*YA + B^*ZB)^{-1} \\ &= Y, \end{aligned} \quad (29)$$

that is, $Y_{k+2} \leq Y$. Then, by this condition and Lemma 2, we have

$I - C^*Z_{k+1}C + D^*Y_{k+2}D \geq I - C^*ZC - D^*YD = Z^{-1} > O$. Using this fact in Lemma 1 together with the NMEs (22) and the fact $Z_{k+1} > O$, we have

$$\begin{aligned} Z_{k+2} &= 2Z_{k+1} - Z_{k+1}(I - C^*Z_{k+1}C - D^*Y_{k+2}D)Z_{k+1} \\ &\leq (I - C^*Z_{k+1}C - D^*Y_{k+2}D)^{-1} \\ &\leq Z. \end{aligned} \quad (30)$$

Also, from the NMEs (24), we get that

$$\begin{aligned} Y_{k+2} - Y_{k+1} &= Y_{k+1} - Y_{k+1}(I - A^*Y_{k+1}A - B^*Z_{k+1}B)Y_{k+1} \\ &= Y_{k+1}[Y_{k+1}^{-1} - (I - A^*Y_{k+1}A - B^*Z_{k+1}B)]Y_{k+1}. \end{aligned} \quad (31)$$

Next, by using the fact $I - A^*Y_kA - B^*Z_kB \geq I - A^*Y_{k+1}A - B^*Z_{k+1}B > O$ together with the NMEs (24), Lemma 1, and the fact $Y_k > O$, it follows that

$$\begin{aligned} Y_{k+1} &= 2Y_k - Y_k(I - A^*Y_kA - B^*Z_kB)Y_k \\ &\leq (I - A^*Y_kA - B^*Z_kB)^{-1} \\ &\leq (I - A^*Y_{k+1}A - B^*Z_{k+1}B)^{-1}, \end{aligned} \quad (32)$$

that is, $Y_{k+1} \leq (I - A^*Y_{k+1}A - B^*Z_{k+1}B)^{-1}$ and $Y_{k+1}^{-1} \geq I - A^*Y_{k+1}A - B^*Z_{k+1}B$. Using this fact with $Y_{k+1} > O$, in equation (31), we have $Y_{k+2} \geq Y_{k+1}$. Using this fact with Lemma 2, we have

$$I - C^*Z_kC - D^*Y_{k+1}D \geq I - C^*Z_{k+1}C - D^*Y_{k+2}D > O. \quad (33)$$

This fact, together with the NMEs (24), Lemma 1, and the fact $Z_k > O$, yields

$$\begin{aligned} Z_{k+1} &= 2Z_k - Z_k(I - C^*Z_kC - D^*Y_{k+1}D)Z_k \\ &\leq (I - C^*Z_kC - D^*Y_{k+1}D)^{-1} \\ &\leq (I - C^*Z_{k+1}C - D^*Y_{k+2}D)^{-1}, \end{aligned} \quad (34)$$

that is, $Z_{k+1} \leq (I - C^*Z_{k+1}C - D^*Y_{k+2}D)^{-1}$ or $Z_{k+1}^{-1} \geq I - C^*Z_{k+1}C - D^*Y_{k+2}D$. Thereby,

$$\begin{aligned} Z_{k+2} - Z_{k+1} &= Z_{k+1} - Z_{k+1}(I - C^*Z_{k+1}C - D^*Y_{k+2}D)Z_{k+1} \\ &= Z_{k+1}[Z_{k+1}^{-1} - (I - C^*Z_{k+1}C - D^*Y_{k+2}D)]Z_{k+1} \geq O. \end{aligned} \quad (35)$$

From the above, $O < Y_n \leq Y_{n+1} \leq Y$ and $O < Z_n \leq Z_{n+1} \leq Z$ hold for $n = k + 1$.

Thus, by using the principle of induction, we have that the relations $O < Y_n \leq Y_{n+1} \leq Y$ and $O < Z_n \leq Z_{n+1} \leq Z$ hold for $n \in \mathbb{N}^*$. So, the sequence $\{(Y_n, Z_n)\}$ is well defined, increasing, and bounded above. Let

$$\begin{aligned} \lim_{n \rightarrow \infty} Y_n &= \hat{Y}, \\ \lim_{n \rightarrow \infty} Z_n &= \hat{Z}. \end{aligned} \quad (36)$$

Then, $(\hat{Y}, \hat{Z})$ is a PDS of the NMEs (4) by Algorithm 1. Since $(\hat{Y}, \hat{Z}) \leq (Y, Z)$ for any PDS (Y, Z) of NMEs (4), $(\hat{Y}, \hat{Z})$ is the minimal PSD of NMEs (4). $\square$

Remark 2. If $(\hat{X}_1, \hat{X}_2)$ is the maximal PDS of the system of NMEs (3), then by the relationship between the NMEs (3) and (4), we get that $(\hat{X}_1^{-1}, \hat{X}_2^{-1})$ is the minimal PDS of NMEs (4). Therefore, from Theorem 4, the sequence $\{(Y_n, Z_n)\}$ generated by Algorithm 1 with the initialization $Y_0 = I$ and $Z_0 = I$ converges to the inverse of the maximal PDS of NMEs (3), respectively.

5. Rate of Convergence

Lemma 4 (see [18]). If $0 < \theta \leq 1$ and $U, V \in P(n)$ with $U, V \geq cI > O$, then, for every unitarily invariant norm, $\|U^\theta - V^\theta\| \leq \theta c^{\theta-1} \|U - V\|$ and $\|U^{-\theta} - V^{-\theta}\| \leq \theta c^{-(\theta+1)} \|U - V\|$.

Theorem 5. If (Y, Z) is a PDS of the matrix equation (4) under Algorithm 1 and $\epsilon > 0$ is arbitrary, then we get the following.

(A)

$$\|Y_n - Y\| \leq (\|AY\| + \epsilon)^2 \|Y - Y_n\| + (\|BY\| + \epsilon)^2 \|Z - Z_n\|, \quad (37)$$

and

$$\|\hat{Y}_n - Y^{-1}\| \leq \|A\|^2 \|Y_n - Y\| + \|B\|^2 \|Z_n - Z\|. \quad (38)$$

(B)

$$\|Z_n - Z\| \leq (\|CZ\| + \epsilon)^2 \|Z_n - Z\| + (\|DZ\| + \epsilon)^2 \|Y_{n+1} - Y\|, \quad (39)$$

and

$$\|\hat{Z}_n - Z^{-1}\| \leq \|C\|^2 \|Z_n - Z\| + \|D\|^2 \|Y_{n+1} - Y\|, \quad (40)$$

for all n large enough.

Proof. To prove (A), let (Y, Z) be the PDS of equation (4). Then, from Algorithm 1,

$$\begin{aligned} Y_{n+1} &= 2Y_n - Y_n(I - A^*Y_nA - B^*Z_nB)Y_n \\ &= 2Y_n - Y_n[I - A^*(Y + Y_n - Y)A - B^*(Z + Z_n - Z)B]Y_n \\ &= 2Y_n - Y_n[I - A^*YA - B^*ZB + A^*(Y - Y_n)A + B^*(Z - Z_n)B]Y_n \\ &= 2Y_n - Y_nY^{-1}Y_n - Y_nA^*(Y - Y_n)AY_n - Y_nB^*(Z - Z_n)BY_n. \end{aligned} \quad (41)$$

Thus, since $Y - 2Y_n + Y_nY^{-1}Y_n = (Y - Y_n)Y^{-1}(Y - Y_n)$, we have

$$\begin{aligned} Y - Y_{n+1} &= (Y - Y_n)Y^{-1}(Y - Y_n) + Y_nA^*(Y - Y_n)AY_n \\ &\quad + Y_nB^*(Z - Z_n)BY_n. \end{aligned} \quad (42)$$

Now, since $\lim_{n \rightarrow \infty} Y_n = Y$, using Lemma 4, we have

$$\begin{aligned} \|Y - Y_{n+1}\| &\leq \|(Y - Y_n)Y^{-1}\| \|Y - Y_n\| + \|AY_n\|^2 \|Y - Y_n\| \\ &\quad + \|BY_n\| \|Z - Z_n\| \\ &\leq \left[\|(Y - Y_n)Y^{-1}\| + \|AY_n\|^2 \right] \|Y - Y_n\| + \|BY_n\|^2 \|Z - Z_n\| \\ &\leq (\|AY\| + \epsilon)^2 \|Y - Y_n\| + (\|BY\| + \epsilon)^2 \|Z - Z_n\|. \end{aligned} \quad (43)$$

which is (35).

Since

$$\hat{Y}_n - Y^{-1} = A^*(Y - Y_n)A + B^*(Z - Z_n)B, \quad (44)$$

using Lemma 4, we have relation (38).

In the same way, we can prove (B). $\square$

6. Numerical Examples

Two examples are presented in this section in support of Algorithm 1. Take Residue = $\|Y_{n+1} - Y_n\| + \|Z_{n+1} - Z_n\|$, tolerance = $1e-10$, and norm as Frobenius norm. For comparative analysis, a basic fixed point algorithm (BFP, for short) has been used:

$$\begin{aligned} X_{1(k+1)} &= I - A^*X_{1(k)}^{-1}A - B^*X_{2(k)}^{-1}B, \\ X_{2(k+1)} &= I - C^*X_{2(k)}^{-1}C - D^*X_{1(k)}^{-1}D. \end{aligned} \quad (45)$$

These examples, together with tables and graphs showing different input data, solutions, iteration number of different schemes, error, CPU time, average computational time, etc., are illustrated here. For a better understanding, we have used line graphs, bar graphs, pie graphs, and surface plotting.

Example 2. Take

$$\begin{aligned} A &= \begin{bmatrix} 0.0367 & 0.0893 & 0.0670 \\ 0.0574 & 0.1355 & 0.0717 \\ 0.1530 & 0.1339 & 0.1036 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.1052 & 0.0175 & 0.0861 \\ 0.0749 & 0.1339 & 0.1514 \\ 0.1036 & 0.0654 & 0.0367 \end{bmatrix}, \\ C &= \begin{bmatrix} 0.2367 & 0.0011 & 0.0011 \\ -0.0056 & 0.2221 & -0.0202 \\ 0.0112 & 0.0213 & 0.2670 \end{bmatrix}, \\ D &= \begin{bmatrix} -0.0101 & 0.0011 & 0.0011 \\ -0.0011 & -0.0168 & -0.0101 \\ 0.0045 & 0.0112 & -0.1055 \end{bmatrix}. \end{aligned} \quad (46)$$

After applying Algorithm 1 with the initial conditions $Y_0 = I$ and $Z_0 = I$, we obtain $X_1 = \begin{bmatrix} 0.9379 & -0.0581 & -0.0526 \\ -0.0581 & 0.9221 & -0.0617 \\ -0.0526 & -0.0617 & 0.9418 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 0.9401 & 0.0007 & -0.0035 \\ 0.0007 & 0.9470 & -0.0006 \\ -0.0035 & -0.0006 & 0.9090 \end{bmatrix}$.

Table 1 presents a comparison between our algorithm and the basic fixed point algorithm.

Figures 1 and 2 represent CPU time graph and iteration vs. error graph.

Figures 3 and 4 represent pie chart for average CPU time based on ten experiments and solution's surface plot.

Example 3. In the current example, a new, randomly chosen set of coefficients $A, B, C, D \in \mathbb{C}^{n \times n}$ is used. The construction is followed by

$$\begin{aligned}
 A_t(i, j) &= \frac{r}{i+j-1}, \\
 B_t(i, j) &= \frac{r}{1} \times A_t(i, j), \\
 C_t(i, j) &= \frac{r^2}{i+j-1}, \\
 D_t(i, j) &= \frac{1}{r-1} \times A_t(i, j), \\
 Q_1(i, j) &= I(i, j) + A_t^*(i, j) \times A_t(i, j), \\
 Q_2(i, j) &= I(i, j) + B_t^*(i, j) \times B_t(i, j), \\
 A_1(i, j) &= Q_1(i, j)^{-(1/2)} \times A_t(i, j) \times Q_1(i, j)^{-(1/2)}, \\
 B_1(i, j) &= Q_1(i, j)^{-(1/2)} \times B_t(i, j) \times Q_1(i, j)^{-(1/2)}, \\
 C_1(i, j) &= Q_2(i, j)^{-(1/2)} \times C_t(i, j) \times Q_2(i, j)^{-(1/2)}, \\
 D_1(i, j) &= Q_2(i, j)^{-(1/2)} \times D_t(i, j) \times Q_2(i, j)^{-(1/2)},
 \end{aligned} \tag{47}$$

$$N_1 = \text{norm}(A_1^* \times A_1 + B_1^* \times B_1, \text{fro}'),$$

$$N_2 = \text{norm}(C_1^* \times C_1 + D_1^* \times D_1, \text{fro}'),$$

$$A = \frac{1}{r} \times \left(\frac{A_1}{N_1} + i \frac{B_1}{N_1} \right),$$

$$B = \frac{1}{r} \times \left(\frac{B_1}{N_1} + i \frac{A_1}{N_1} \right),$$

$$C = \frac{1}{r} \times \left(\frac{C_1}{N_2} + i \frac{D_1}{N_2} \right),$$

$$D = \frac{1}{r} \times \left(\frac{D_1}{N_2} + i \frac{C_1}{N_2} \right).$$

Here, “ r ” and “ fro ” stand for dimension and Frobenius norm of the matrix, respectively. A different set of initial conditions has been chosen here as

$$\begin{aligned}
 Y_0 &= \left(\frac{1}{2} \right) \times A \times \text{conjugate-transpose}(A), \\
 Z_0 &= \left(\frac{1}{2} \right) \times B \times \text{conjugate-transpose}(B).
 \end{aligned} \tag{48}$$

The result of this experiment using Algorithm 1 is shown in Table 2.

Example 4. Here, we consider some randomly generated matrices:

$$\begin{aligned}
 A_1 &= I + \frac{1}{(2 \times r) \text{rand}(r)}, \\
 B_1 &= I + \frac{1}{(r)^2 \text{rand}(r)}, \\
 C_1 &= I + \frac{1}{(r)^3 \text{rand}(r)}, \\
 D_1 &= I + \frac{1}{(r \times 3) \text{rand}(r)}, \\
 A &= \frac{A_1}{\|A_1\|}, \\
 B &= \frac{B_1}{\|B_1\|}, \\
 C &= \frac{C_1}{\|C_1\|}, \\
 D &= \frac{D_1}{\|D_1\|},
 \end{aligned} \tag{49}$$

where r = dimension of the matrices, $\text{rand}(r)$ = random matrices of order r , and $\text{tol} = 1e-10$. After applying Algorithm 1 (ALgo1) and basic fixed-point method under the initial conditions $Y_0 = I$ and $Z_0 = I$, we get Table 3 that shows the various outputs for different r . The 5th column of Table 3 ensures positive definiteness of Example 4 w.r.t. different order matrices.

Associated graphs are (Figures 5–25)

- (i) $\dim = 3$ $\left\{ \begin{array}{l} \text{Fig 5: CPU time} \\ \text{Fig 6: it. no. vs. error} \\ \text{Fig 7: Sols surface plot'} \end{array} \right.$
- (ii) $\dim = 5$ $\left\{ \begin{array}{l} \text{Fig 8: CPU time} \\ \text{Fig 9: it. no. vs. error} \\ \text{Fig 10: Sols surface plot'} \end{array} \right.$
- (iii) $\dim = 8$ $\left\{ \begin{array}{l} \text{Fig 11: CPU time} \\ \text{Fig 12: it. no. vs. error} \\ \text{Fig 13: Sols surface plot'} \end{array} \right.$
- (iv) $\dim = 12$ $\left\{ \begin{array}{l} \text{Fig 14: CPU time} \\ \text{Fig 15: it. no. vs. error} \\ \text{Fig 16: Sols surface plot'} \end{array} \right.$

TABLE 1: Comparison of Algorithm 1 with basic fixed point iteration.

Method	r	No. of iteration	Error	Average CPU time
Algol	3	12	$0.5908e-10$	0.0070
BFP	3	12	$0.2180e-10$	0.0157

FIGURE 1: CPU time graph and iteration vs error graph.

FIGURE 2: CPU time graph and iteration vs error graph.

(v) $\dim = 20$ { Fig 17: CPU time
Fig 18: it. no. vs. error
Fig 19: Sols surface plot'

(vi) $\dim = 32$ { Fig 20: CPU time
Fig 21: it. no. vs. error
Fig 22: Sols surface plot'

(vii) $\dim = 64$ { Fig 23: CPU time
Fig 24: it. no. vs. error
Fig 25: Sols surface plot'

(viii) Figure 26 represents the average CPU time through bar graphs for different dimensions

Remark 3. We can infer from the above discussions that our algorithm is less expensive in terms of computation after evaluating these examples w.r.t. different sets of parameters.

Remark 4. We can infer from the above discussions that the solutions are positive definite.

FIGURE 3: Pie chart for average CPU time.

FIGURE 4: Solution surface plot.

TABLE 2: CPU analysis for different dimension.

Dim	No. of iteration	Error	CPU time (seconds)
3	8	$3.0319e-12$	0.007770
4	6	$6.4760e-12$	0.006651
5	5	$2.2293e-11$	0.008359
10	4	$2.0735e-12$	0.007447
200	3	$7.9969e-15$	0.030397
500	2	$1.1787e-11$	0.111352
1000	2	$7.3933e-13$	0.758877

TABLE 3: Comparison of Algorithm 1 with basic fixed point iteration for different dimension.

Dim		Algo 1	BFP	Min(eig) X_1, X_2
3	No. of iter.	17	19	0.7646, 0.7620
	Error ($\times 10^{-10}$)	0.6836	0.48240	
	Av. CPU time	0.0064	0.0098	
5	No. of iter.	11	13	0.8689, 0.8729
	Error ($\times 10^{-10}$)	0.1686	0.3111	
	Av. CPU time	0.0062	0.0096	
8	No. of iter.	9	10	0.9175, 0.9238
	Error ($\times 10^{-10}$)	0.8159	0.48613	
	Av. CPU time	0.0072	0.0101	
12	No. of iter.	8	9	0.9450, 0.9497
	Error ($\times 10^{-10}$)	0.3285	0.0948	
	Av. CPU time	0.0073	0.0104	
20	No. of iter.	7	7	0.9672, 0.9701
	Error ($\times 10^{-10}$)	0.2388	0.9457	
	Av. CPU time	0.0077	0.0109	
32	No. of iter.	6	7	0.9796, 0.9814
	Error ($\times 10^{-10}$)	0.7179	0.0356	
	Av. CPU time	0.0078	0.0116	
64	No. of iter.	6	6	0.9899, 0.9907
	Error ($\times 10^{-10}$)	0.1332	0.48082	
	Av. CPU time	0.0124	0.0168

FIGURE 5: CPU time vs no. of iteration for dimension 3.

FIGURE 6: Iteration no. vs error for dimension 3.

FIGURE 7: Solution surface plot for dimension 3.

FIGURE 8: CPU time vs no. of iteration for dimension 5.

FIGURE 9: Iteration no. vs error for dimension 5.

FIGURE 10: Solution surface plot for dimension 5.

FIGURE 11: CPU time vs no. of iteration for dimension 8.

FIGURE 12: Iteration no. vs error for dimension 8.

FIGURE 13: Solution surface plot for dimension 8.

FIGURE 14: CPU time vs no of iteration for dimension 12.

FIGURE 15: Iteration no. vs error for dimension 12.

FIGURE 16: Solution surface plot for dimension 12.

FIGURE 17: CPU time vs no. of iteration for dimension 20.

FIGURE 18: Iteration no. vs error for dimension 20.

FIGURE 19: Solution surface plot for dimension 20.

FIGURE 20: CPU time vs. no. of iteration for dimension 32.

FIGURE 21: Iteration no. vs error for dimension 32.

FIGURE 22: Solution surface plot for dimension 32.

FIGURE 23: CPU time vs no. of iteration for dimension 64.

FIGURE 24: Iteration no. vs error for dimension 64.

FIGURE 25: Solution surface plot for dimension 64.

FIGURE 26: Average CPU time for different dimensions.

7. Conclusion

In this study, a new iterative algorithm has been developed. All numerical tests are in agreement with the theoretical findings of this research done. Finally, based on the numerical results, we have concluded that the new iterative approach is extremely powerful and efficient in finding numerical solutions for a wide range of nonlinear matrix equations including complex ones. It also produces very accurate results with less iterations and lower computational costs, compared with the basic fixed-point approach.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors are grateful to the Science and Engineering Research Board (SERB), India, for providing project funding under the scheme-CRG/2018/000615.

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