Retraction

Retracted: A New Neutrosophic Negative Binomial Distribution: Properties and Applications

Journal of Mathematics

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] R. A. K. Sherwani, S. Iqbal, S. Abbas, M. Aslam, and A. H. AL-Marshadi, "A New Neutrosophic Negative Binomial Distribution: Properties and Applications," *Journal of Mathematics*, vol. 2021, Article ID 2788265, 12 pages, 2021.

Research Article

A New Neutrosophic Negative Binomial Distribution: Properties and Applications

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Many problems in real life exist that are full of confusion, vagueness, and ambiguity. The quantification of such issues in a scientific way is the need of time. The negative binomial distribution is an important discrete probability distribution from the account of classical probability distribution theory. The distribution was used to study the chance of kth success in n trials before $n - 1$ failures for crisp data. The literature lacks in dealing with the situations for interval-valued data under negative binomial distribution. In this research, the neutrosophic negative binomial distribution is proposed to generalize the classical negative binomial distribution. The generalized proposed distribution considers the indeterminacy and crisp form from interval-valued. Several properties of the proposed distribution, such as moment generating function, characteristic function, and probability generating function, are also derived. Furthermore, the derivation of reliability analysis properties such as survival, hazard rate, reversed hazard rate, cumulative hazard rate, mills ratio, and odds ratio are also presented. In addition, order statistics for the proposed distribution, including w^{th}, joint, median, minimum, and maximum order statistics are part of the paper. The proposed distribution is discussed from the real data applications perspective by considering the different case studies. This research opens the way to deal with the problems that follow conventional conveyances and include nonprecisely determined details simultaneously.

1. Introduction

The term neutrosophy was introduced by Smarandache [1], a modern philosophical branch inspired by famous fuzzy logic. It is a generalization of intuitionistic fuzzy logic [2]. The term is based on the logic of vague, unclear, blurred, fuzzy realms (problems, circumstances, and ideas) that are common in today's era. The fuzzy logic is a limiting case of precise reasoning used to quantify the imprecise modes of reasoning [3]. The theory helps articulate decisions for decision-making problems under an imprecise and uncertain environment [4, 5]. Zadeh recently presented the fuzzy logic theory and applications in a precise way [6].

Further extensions in fuzzy logic and its applications can be seen in [4,7–11]. The standard fuzzy sets deal with the exact values, and there are many situations in real-life data where it is hard to find single values data. In such situations, interval-valued fuzzy sets were introduced by [12]. Smarandache claimed both neutrosophic sets and neutrosophic statistics to generalize fuzzy logic [13, 14]. Neutrosophic statistics deals with the interval-valued data as the classical statistical methods are helpless in dealing with the situations that generate indefinite data in interval form. Some methodological and applied forms of neutrosophic statistics have been discussed in [15–18].

Probability is among the classical statistical methods that deal with the quantification of random phenomena. The techniques available in the literature paid less attention to uncertainty-related problems under the fuzzy environment. The classical probability ignored serious, aberrant, ambiguous values, so a new suitable instrument was used. Smarandache [19] introduced the basic definition of

neutrosophic sets in his well-defined book in 2014 purely in the statistical scenario, which provides a new basis containing indeterminate data to deal with many problems. The primary objective of neutrosophic logic is to define any logical argument of a statement under consideration in a 3-D neutrosophic space where each dimension represents truth (T), false (U), and indeterminacy (I), respectively. The symbols T, I, and U are the standard, a nonstandard real subset of $(-0, 1+)$ without any specific connection. Many researchers have extended the classical distributions under neutrosophic logic that includes neutrosophic binomial distribution and neutrosophic normal distribution [20, 21], neutrosophic multinomial distribution [19], neutrosophic Poisson, neutrosophic exponential, and neutrosophic uniform distribution [22], neutrosophic gamma distribution [23], neutrosophic Weibull distribution and its several families [24], and neutrosophic beta distribution [25]. In this paper, we extended the concept of two parameters negative binomial distribution to neutrosophic negative binomial distribution using neutrosophic logic.

2. Neutrosophic Negative Binomial Distribution (NNBD)

The classical negative binomial distribution is generalized neutrosophically, which ensures some indeterminacy related to the probabilistic experiment. Suppose each trial of an experiment results in an outcome, labeled as success (S) and

failure (U) and also with some indeterminacy (I). For example, tossing a coin on an unbalanced surface may have cracks, a coin may fall on its edge inside the crack, and one may get neither head nor tail but some indeterminacy.

The neutrosophic negative binomial random variable is defined as a variable number of trials to obtain the fixed number of successes. It is known as a neutrosophic negative binomial distribution. First, obtaining indeterminacy for every trial means there will be indeterminacy for all trials. Secondly, obtaining indeterminacy for no trial means no indeterminacy for all trials. There may exist a situation when we get indeterminacy for a few trials and determinacy for other trials. In that case, we introduce an indeterminacy threshold. Let th^* be the number of trials that result in indeterminacy and $\text{th}^* = \{0, 1, 2, \ldots, \infty\}$. Cases where threshold $> \text{th}^*$ will belong to indeterminate part, and when threshold $< \text{th}^*$, those cases will belong to a determinate part.

For $x = \{\acute{s}, \acute{s} + 1, \acute{s} + 2, \ldots\}$, Np_r (occurrences of a fixed number of successes for a variable number of trials) = (T_x, U_x, I_x), probability mass function and cumulative distribution function of neutrosophic negative binomial distribution are, respectively, given as

$$T_x = \left(p_r(S)\right)^{\acute{s}} \sum_{t=0}^{\text{th}^*} \binom{x - \acute{s}}{t} \left(p_r(I)\right)^t \left(p_r(U)\right)^{x - \acute{s} - t}. \quad (1)$$

Similarly,

$$U_x = \sum_{\substack{y = \acute{s} \\ y \neq x}}^{\infty} T_y = \left(p_r(S)\right)^{\acute{s}} \sum_{\substack{y = \acute{s} \\ y \neq x}}^{\infty} \sum_{t=0}^{\text{th}^*} \binom{y - \acute{s}}{t} \left(p_r(I)\right)^t \left(p_r(U)\right)^{y - \acute{s} - t}, \quad (2)$$

$$I_x = \left(p_r(S)\right)^{\acute{s}} \sum_{z^* = \text{th}^* + 1}^{x - \acute{s}} \binom{x - \acute{s}}{z^*} \left(p_r(I)\right)^{z^*} \sum_{x = \acute{s}}^{\infty} \sum_{t=0}^{x - \acute{s} - z^*} \binom{x - \acute{s} - z^*}{t} \left(p_r(U)\right)^{x - \acute{s} - z^* - t}. \quad (3)$$

The CDF corresponding to (1) is given by

$$F(x) = 1 - \left(p_r(S)\right)^{\acute{s}} \left(p_r(U)\right)^{x - \acute{s} + 1} \sum_{m' = 1}^{\infty} \binom{x - \acute{s} + m'}{m' - 1} \left(p_r(I)\right)^{m' - 1}. \quad (4)$$

The CDF corresponding to (3) is given by

$$F(x) = 1 - \left(p_r(S)\right)^{\acute{s}} \sum_{m' = 1}^{\infty} \binom{x - \acute{s} + m'}{m'} \left(p_r(I)\right)^{m'} \sum_{m' = 0}^{\infty} \binom{x - \acute{s}}{m'} \left(p_r(U)\right)^{x - \acute{s} - m'}. \quad (5)$$

2.1. A Special Case of Neutrosophic Negative Binomial Distribution (NNBD). The neutrosophic geometric distribution

is a special case of NNBD when the number of successes $\acute{s} = 1$.

2.2. Physical Conditions

(i) Each trial results in three mutually exclusive and exhaustive outcomes such as success, failure, and indeterminacy

(ii) All the trials must be independent

(iii) The probability of success remains fixed or constant for each trial

(iv) An experiment is repeated a variable number of times to produce a fixed number of successes

2.3. Case Studies

2.3.1. Case Study 1. Assume that you are surveying individuals existing at the polling booth and asking them if they voted independently. It was observed that 20% of people voted independently, 60% did not vote independently, and 10% were unsure whether they voted independently or not. What is the probability of eight people, you must ask before you find three people who voted independently?

Using the information in the problem mentioned above, we compute the probability for all three parts of pmf.

$X \longrightarrow$ number of people who voted

$\acute{s} \longrightarrow$ number of people who voted independently will be considered a success

So, $X = 3, 4, \ldots, 8$ and $\acute{s} = 3$

Let the threshold $th^* = 2$

$$p_r \text{ (people voted independently)} = p_r(S) = 0.2$$

(i) p_r (people not voted independently) $= p_r(U) = 0.6$

(ii) p_r (people not sure about their voting) $= p_r(I) = 0.1$

$$T_x = T_3 = (0.2)^3 \sum_{t=0}^{2} \binom{8-3}{t}(0.1)^t(0.6)^{8-3-t}$$

$$= (0.2)^3 \left\{ \binom{5}{0}(0.1)^0(0.6)^5 + \binom{5}{1}(0.1)^1(0.6)^4 + \binom{5}{2}(0.1)^2(0.6)^3 \right\}$$

$$= (0.2)^3 \{0.7776 + 0.0648 + 0.0216\}$$

$$= 0.0013132,$$

(6)

$$I_3 = (0.2)^3 \sum_{z^*=3}^{8-3=5} \frac{(8-3)!}{z^*!}(0.1)^{z^*} \sum_{t=0}^{5-z^*} \frac{(0.6)^{5-z^*-t}}{t!(5-z^*-t)!}$$

$$= (0.2)^3 \left[\left\{ \frac{5!}{3!}(0.1)^3 \sum_{t=0}^{2} \frac{(0.6)^{2-t}}{t!(2-t)!} \right\} + \left\{ \frac{5!}{4!}(0.1)^4 \sum_{t=0}^{1} \frac{(0.6)^{1-t}}{t!(1-t)!} \right\} + \left\{ \frac{5!}{5!}(0.1)^5 \sum_{t=0}^{0} \frac{(0.6)^{0-t}}{t!(0-t)!} \right\} \right]$$

$$= (0.2)^3 \{1.28 + 8 \times 10^{-4} + 0.1^5\}$$

$$= 0.010246.$$

U_3 can be easily calculated in the following way rather than using combinational formula.

As we know that

$$T_x + I_x + U_x = (p_r(S) + p_r(I) + p_r(U))^n, \quad (7)$$

here instead of n, we will use x, as x is the number of trials.

$$T_x + I_x + U_x = (p_r(S) + p_r(I) + p_r(U))^x. \quad (8)$$

So, we may compute U_3 as given below:

$$U_3 = (p_r(S) + p_r(I) + p_r(U))^x - T_3 - I_3$$
$$= (0.2 + 0.1 + 0.6)^8 - 0.0013132 - 0.010246 \quad (9)$$
$$= 0.41891.$$

If the computed vector is normalized,

$$(T_3, I_3, U_3) = (0.0013132, 0.41891, 0.010246), \quad (10)$$

by dividing each component of a vector with their total sum

$$0.0013132 + 0.41891 + 0.010246 = 0.4304692. \quad (11)$$

Hence, we get

$$(T_3, I_3, U_3) = (0.0030506, 0.023802, 0.973147). \quad (12)$$

2.3.2. Case Study 2. A specified location has 35% rain on any specific day, 70% chances that day will be sunny, and 15% indeterminacy that neither the day will be sunny nor rainy.

What is the probability that there will be rain on three specific days in a week?

Using the information in the problem mentioned above, we compute the probability for all three parts of pmf.

$X \longrightarrow$ number of days in a week

$ś \longrightarrow$ number of rainy days

So, $X = 3, 4, 5, 6, 7$ and $ś = 3$. Let the threshold be 3, i.e., $th^* = 3$

p_r (the day will be rainy) $= p_r(S) = 0.35$

(i) p_r (the day will be sunny) $= p_r(U) = 0.7$

(ii) p_r (the day will be neither rainy nor sunny) $= p_r(I) = 0.15$

$$T_x = T_3 = (0.35)^3 \sum_{t=0}^{3} \binom{7-3}{l} (0.15)^t (0.7)^{7-3-t}$$

$$= (0.35)^3 \left\{ \binom{4}{0}(0.15)^0(0.7)^4 + \binom{4}{1}(0.15)^1(0.7)^3 + \binom{4}{2}(0.15)^2(0.7)^2 + \binom{4}{3}(0.15)^3(0.7)^1 \right\}$$

$$= (0.35)^3 \{0.2401 + 0.2058 + 0.06615 + 0.00945\}$$

$$= 0.022359,$$

$$I_3 = (0.35)^3 \sum_{z^*=4}^{7-4=3} \frac{(7-4)!}{z^*!}(0.15)^z \sum_{t=0}^{4-z^*} \frac{(0.7)^{4-z^*-t}}{t!(4-z^*-t)!}$$

$$= (0.35)^3 \cdot \sum_{z^*=4}^{4} \frac{3!}{4!}(0.15)^4 \sum_{t=0}^{0} \frac{(0.7)^{4-4-t}}{t!(4-4-t)!}$$

$$= (0.35)^3 \left\{ \frac{3!}{4!}(0.15)^4(1) \right\}$$

$$= 5.42636 \times 10^{-6},$$

$$U_3 = (p_r(S) + p_r(I) + p_r(U))^x - T_3 - I_3$$

$$= (0.35 + 0.15 + 0.7)^7 - 0.022359 - 5.42636 \times 10^{-6}$$

$$= 3.560816.$$

$$\text{(13)}$$

If the computed vector is normalized,

$$(T_3, I_3, U_3) = (0.022359, 5.42636 \times 10^{-6}, 3.560816), \tag{14}$$

by dividing each component of a vector with their total sum

$$0.022359 + 5.42636 + 3.560816 = 3.583180. \tag{15}$$

Hence, we get

$$(T_3, I_3, U_3) = (0.0062399, 1.51440 \times 10^{-6}, 0.99376). \tag{16}$$

2.3.3. Case Study 3. Jackson is a football player. His success rate of hitting the goal is 70%, the failure rate is 40%, and 15% chance that the situation may provide no clear evidence about the goal whether hitting or not. What is the probability that Jackson hits 2nd goal on his fifth attempt? Using the information in the problem mentioned above, we compute the probability for all three parts of pmf.

$$\begin{aligned} ś &= 2, \\ X &= 2, 3, 4, 5. \end{aligned} \tag{17}$$

Let the threshold be 1; $th^* = 1$.

p_r (goal hit) $= p_r(S) = 0.7$

$p_r \text{ (goal not hit)} = p_r(U) = 0.4$

$p_r \text{ (no evidence about hitting or not hitting of goal)} = p_r(I) = 0.15$

$$T_x = T_2 = (0.7)^2 \sum_{t=0}^{1} \binom{5-2}{t} (0.15)^t \cdot (0.4)^{5-2-t}$$

$$= (0.7)^2 \left\{ \binom{3}{0} (0.15)^0 (0.4)^3 + \binom{3}{1} (0.15)^1 (0.4)^2 \right\}$$

$$= (0.7)^2 \{0.064 + 0.072\}$$

$$= 0.06664,$$

$$I_2 = (0.7)^2 \sum_{z^*=2}^{5-2=3} \frac{3!}{z^*!} (0.15)^{z^*} \sum_{t=0}^{3-z^*} \frac{(0.4)^{3-z^*-t}}{t!(3-z^*-t)!}$$

$$= (0.7)^2 \left[\left\{ \frac{3!}{2!} (0.15)^2 \cdot \sum_{t=0}^{1} \frac{(0.4)^{1-t}}{t!(1-t)!} \right\} + \left\{ \frac{3!}{3!} (0.15)^3 \cdot \sum_{t=0}^{0} \frac{(0.4)^{0-t}}{t!(0-t)!} \right\} \right]$$

$$= (0.7)^2 \{0.0945 + (0.15)^3\}$$

$$I_2 = 0.047959,$$

$$U_2 = \left(p_r(S) + p_r(I) + p_r(U) \right)^x - T_2 - I_2$$

$$= (0.7 + 0.15 + 0.4)^5 - 0.06664 - 0.047959$$

$$= 2.937159.$$

$$(18)$$

If the computed vector is normalized,

$$\left(T_2, I_2, U_2 \right) = (0.06664, 0.047959, 2.937159), \tag{19}$$

by dividing each component of a vector with their total sum

$$0.06664 + 0.047959 + 2.937159 = 3.051758. \tag{20}$$

Hence, we get

$$\left(T_2, I_2, U_2 \right) = (0.021836, 0.962448, 0.015715). \tag{21}$$

3. Main Properties of NNBD

3.1. Moment Generating Function.
Moment generating function (m.g.f) of $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given by

$$M_0(t) = \left(\frac{p_r(S)}{p_r(U)} \right)^{\acute{s}} \left(1 - \frac{p_r(U)}{p_r(U)} \right)^{-1} \sum_{m'=0}^{\infty} \left(p_r(U) e^t \right)^{\acute{s}+m'}, \tag{22}$$

for indeterminate part of pmf is given as

$$M_0(t) = A^* \left(p_r(S) \right)^{\acute{s}} \left(\sum_{n=1}^{\infty} \frac{\left(p_r(I) \right)^n}{n} \right) \left(\sum_{m'=0}^{\infty} \left(p_r(U) \cdot e^t \right)^{\acute{s}+m'} \right), \tag{23}$$

where

$$A^* = -\frac{\left(p_r(U) \right)^{-(\acute{s}+1)}}{1!} + \frac{\left(p_r(U) \right)^{-(\acute{s}+3)}}{2!} - \frac{\left(p_r(U) \right)^{-(\acute{s}+5)}}{2!3!} + \cdots. \tag{24}$$

3.2. Characteristic Function.
The characteristic function of $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given by

$$\phi_i(t) = \left(\frac{p_r(S)}{p_r(U)} \right)^{\acute{s}} \left(1 - \frac{p_r(I)}{p_r(U)} \right)^{-1} \sum_{m'=0}^{\infty} \left(p_r(U) e^{it} \right)^{\acute{s}+m'}, \tag{25}$$

for indeterminate part of pmf is given as

$$\phi_i(t) = A^* \left(p_r(S) \right)^{\acute{s}} \cdot \left\{ \sum_{n=1}^{\infty} \frac{\left(P(I) \right)^n}{n} \right\} \left\{ \sum_{m'=0}^{\infty} \left(p_r(U) e^{it} \right)^{\acute{s}+m'} \right\}. \tag{26}$$

3.3. Probability Generating Function.
Probability generating function of $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given by

$$G(\theta) = \left(\frac{p_r(S)}{p_r(U)}\right)^{\acute{s}} \left(1 - \frac{p_r(I)}{p_r(U)}\right)^{-1} \sum_{m'=0}^{\infty} \left(p_r(U) \cdot \theta\right)^{\acute{s}+m'},$$

(27)

for indeterminate part of pmf is given as

$$G(\theta) = A^* \left(p_r(S)\right)^{\acute{s}} \left\{\sum_{n=1}^{\infty} \frac{\left(p_r(I)\right)^n}{n}\right\} \left\{\sum_{m'=0}^{\infty} \left(p_r(U) \cdot \theta\right)^{\acute{s}+m'}\right\}.$$

(28)

4. Reliability Analysis

This section finds various reliability properties like survival function, hazard rate function, reversed hazard rate function, and cumulative hazard rate function. In addition, the mills ratio and odds ratio for the new proposed distribution are derived.

4.1. Survival Function.

Survival function of r.v $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given as

$$S(x) = \left(p_r(S)\right)^{\acute{s}} \left(p_r(U)\right)^{x-\acute{s}+1} \sum_{m'=1}^{\infty} \binom{x - \acute{s} + m'}{m' - 1} \left(p_r(I)\right)^{m'-1}.$$

(29)

The indeterminate part of pmf is given as

$$S(x) = \left(p_r(S)\right)^{\acute{s}} \sum_{m'=1}^{\infty} \binom{x - \acute{s} + m'}{m'} \left(p_r(I)\right)^{m'} \sum_{m'=0}^{\infty} \binom{x - \acute{s}}{m'} \left(p_r(U)\right)^{x-\acute{s}-m'}.$$

(30)

4.2. Hazard Rate or Failure Rate Function.

Hazard rate function of r.v $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given as

The indeterminate part of pmf is given as

$$h(x) = \frac{\sum_{t=0}^{th^*} \binom{x - \acute{s}}{t} \left(p_r(I)\right)^t \left(p_r(U)\right)^{x-\acute{s}-t}}{\left(p_r(U)\right)^{x-\acute{s}+1} \sum_{m'=1}^{\infty} \binom{x - \acute{s} + m'}{m' - 1} \left(p_r(I)\right)^{m'-1}}.$$

(31)

$$h(x) = \frac{\sum_{z^*=th^*+1}^{x-\acute{s}} \binom{x - \acute{s}}{z^*} \left(p_r(I)\right)^{z^*} \sum_{x=\acute{s}}^{\infty} \sum_{t=0}^{x-\acute{s}-z^*} \binom{x - \acute{s} - z^*}{t} \left(p_r(U)\right)^{x-\acute{s}-z^*-t}}{\sum_{m'=1}^{\infty} \binom{x - \acute{s} + m'}{m'} \left(p_r(I)\right)^{m'} \sum_{m'=0}^{\infty} \binom{x - \acute{s}}{m'} \left(p_r(U)\right)^{x-\acute{s}-m'}}.$$

(32)

4.3. Reversed Hazard Rate Function.

Reversed hazard rate function (RHRF) of r.v $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given as

$$\partial(x) = \frac{\left(p_r(S)\right)^{\acute{s}} \sum_{t=0}^{th^*} \binom{x - \acute{s}}{t} \left(p_r(I)\right)^t \left(p_r(U)\right)^{x-\acute{s}-t}}{1 - \left(p_r(S)\right)^{\acute{s}} \left(p_r(U)\right)^{x-\acute{s}+1} \sum_{m'=1}^{\infty} \binom{x - \acute{s} + m'}{m' - 1} \left(p_r(I)\right)^{m'-1}}.$$

(33)

For the indeterminate part of pmf, RHRF is given as

$$\partial(x) = \frac{(p_r(S))^{\acute{s}} \sum_{z^*=\text{th}^*+1}^{x-\acute{s}} \binom{x-\acute{s}}{z^*}(p_r(I))^{z^*} \sum_{x=\acute{s}}^{\infty} \sum_{t=0}^{x-\acute{s}-z^*} \binom{x-\acute{s}-z^*}{t}(p_r(U))^{x-\acute{s}-z^*-t}}{1-(p_r(S))^{\acute{s}} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'}(p_r(I))^{m'} \sum_{m'=0}^{\infty} \binom{x-\acute{s}}{m'}(p_r(U))^{x-\acute{s}-m'}}. \tag{34}$$

4.4. Cumulative Hazard Rate Function.
Cumulative hazard rate function (CHRF) of r.v $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given as

$$H(x) = -\ln\left\{ (p_r(S))^{\acute{s}}(p_r(U))^{x-\acute{s}+1} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1} \right\}. \tag{35}$$

For the indeterminate part of pmf, CHRF is given as

$$H(x) = -\ln\left\{ (p_r(S))^{\acute{s}} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'}(p_r(I))^{m'} \sum_{m'=0}^{\infty} \binom{x-\acute{s}}{m'}(p_r(U))^{x-\acute{s}-m'} \right\}. \tag{36}$$

4.5. Mills Ratio.
Mills ratio of r.v $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given as

$$m(x) = \frac{(p_r(U))^{x-\acute{s}+1} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}}{\sum_{t=0}^{\text{th}^*} \binom{x-\acute{s}}{t}(p_r(I))^t (p_r(U))^{x-\acute{s}-t}}. \tag{37}$$

For indeterminate part of pmf, mills ratio is given as

$$m(x) = \frac{\sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'}(p_r(I))^{m'} \sum_{m'=0}^{\infty} \binom{x-\acute{s}}{m'}(p_r(U))^{x-\acute{s}-m'}}{\sum_{z^*=\text{th}^*+1}^{x-\acute{s}} \binom{x-\acute{s}}{z^*}(p_r(I))^{z^*} \sum_{x=\acute{s}}^{\infty} \sum_{t=0}^{x-\acute{s}-z^*} \binom{x-\acute{s}-z^*}{t}(p_r(U))^{x-\acute{s}-z^*-t}}. \tag{38}$$

4.6. Odds Ratio.
The odds ratio of r.v $X \sim \text{NNBD}(x; \acute{s}, p_r(S))$ for true part of pmf is given as

$$\varphi(x) = \frac{1 - (p_r(S))^{\acute{s}}(p_r(U))^{x-\acute{s}+1}\sum_{m'=1}^{\infty}\begin{pmatrix} x-\acute{s}+m' \\ m'-1 \end{pmatrix}(p_r(I))^{m'-1}}{(p_r(S))^{\acute{s}}(p_r(U))^{x-\acute{s}+1}\sum_{m'=1}^{\infty}\begin{pmatrix} x-\acute{s}+m' \\ m'-1 \end{pmatrix}(p_r(I))^{m'-1}}. \tag{39}$$

For the indeterminate part of pmf, the odds ratio is given as

$$\varphi(x) = \frac{1 - (p_r(S))^{\acute{s}}\sum_{m'=1}^{\infty}\begin{pmatrix} x-\acute{s}+m' \\ m' \end{pmatrix}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\begin{pmatrix} x-\acute{s} \\ m' \end{pmatrix}(p_r(U))^{x-\acute{s}-m'}}{(p_r(S))^{\acute{s}}\sum_{m'=1}^{\infty}\begin{pmatrix} x-\acute{s}+m' \\ m' \end{pmatrix}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\begin{pmatrix} x-\acute{s} \\ m' \end{pmatrix}(p_r(U))^{x-\acute{s}-m'}}. \tag{40}$$

5. Order Statistics

In this section, we derived the order statistics for the new proposed distribution NNBD, such as the wth order statistics, joint, largest, and smallest order statistics, maximum and minimum, and median order statistics, as well as smallest and largest order statistics.

5.1. wth Order Statistics. Let $X_1, X_2, \ldots, X_w$ be the random sample from NNBD and let $X_{(1)}, X_{(2)}, \ldots, X_{(w)}$ be the corresponding order statistics. wth order statistics for the true part of NNBD can be given as

$$f_{w:n}(x) = \frac{n!}{(w-1)!(n-w)!}(p_r(S))^{\acute{s}+n-w}\sum_{t=0}^{th^*}\begin{pmatrix} x-\acute{s} \\ t \end{pmatrix}(p_r(I))^t(p_r(U))^{x-\acute{s}-t}\left[1 - (p_r(S))^{\acute{s}}(p_r(U))^{x-\acute{s}+1}\sum_{m'=1}^{\infty}\begin{pmatrix} x-\acute{s}+m' \\ m'-1 \end{pmatrix}(P(I))^{m'-1}\right]^{w-1}$$

$$\left[(p_r(U))^{x-\acute{s}+1}\sum_{m'=1}^{\infty}\begin{pmatrix} x-\acute{s}+m' \\ m'-1 \end{pmatrix}(p_r(I))^{m'-1}\right]^{n-w}. \tag{41}$$

wth order statistics for the indeterminate part of NNBD can be given as

$$f_{w:n}(x) = \frac{n!}{(w-1)!(n-w)!}(p_r(S))^{n+\acute{s}-w}$$

$$\sum_{z^*=th^*+1}^{x-\acute{s}}\begin{pmatrix} x-\acute{s} \\ z^* \end{pmatrix}(p_r(I))^{z^*}\sum_{x=k}^{\infty}\sum_{t=0}^{x-\acute{s}-z^*}\begin{pmatrix} x-\acute{s}-z^* \\ t \end{pmatrix}(p_r(U))^{x-\acute{s}-z^*-t}$$

$$\left[1 - (p_r(S))^{\acute{s}}\sum_{m'=1}^{\infty}\begin{pmatrix} x-\acute{s}+m' \\ m' \end{pmatrix}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\begin{pmatrix} x-\acute{s} \\ m' \end{pmatrix}(p_r(U))^{x-\acute{s}-m'}\right]^{w-1} \tag{42}$$

$$\left[\sum_{m'=1}^{\infty}\begin{pmatrix} x-\acute{s}+m' \\ m' \end{pmatrix}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\begin{pmatrix} x-\acute{s} \\ m' \end{pmatrix}(p_r(U))^{x-\acute{s}-m'}\right]^{n-w}.$$

5.2. Joint Order Statistics.
Joint order statistics of $y_{1:\,b}$ and $y_{u:\,b}$ for true part of NNBD is derived as follows:

$$
f_{a:\,u:\,b}(x) = \frac{b!}{(a-1)!(u-a-1)!(b-u)!}(p_r(S))^{\acute{s}(b-a+1)}\left(\sum_{t=0}^{\mathrm{th}^*}\binom{y-\acute{s}}{t}(p_r(I))^t(p_r(U))^{y-\acute{s}-t}\right)
$$

$$
\left(\sum_{t=0}^{\mathrm{th}^*}\binom{z-\acute{s}}{t}(p_r(I))^t(p_r(U))^{z-\acute{s}-t}\right)\left[1-(p_r(S))^{\acute{s}}(p_r(U))^{y-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{y-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}\right]^{a-1}
$$

$$
\left[(p_r(U))^{y-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{y-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}-(p_r(U))^{z-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{z-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}\right]^{u-a-1}
$$

$$
\left[(p_r(U))^{y-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{y-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}\right]^{b-u}. \tag{43}
$$

Joint order statistics for the indeterminate part of NNBD is given as

$$
f_{a:\,u:\,b}(x) = \frac{b!}{(a-1)!(u-a-1)!(b-u)!}(p_r(S))^{\acute{s}(b-a+1)}
$$

$$
\left\{\sum_{z^*=\mathrm{th}+1}^{y-\acute{s}}\binom{y-\acute{s}}{z^*}(p_r(I))^{z^*}\sum_{x=\acute{s}}^{\infty}\sum_{t=0}^{y-\acute{s}-z^*}\binom{y-\acute{s}-z^*}{t}(p_r(U))^{y-\acute{s}-z^*-t}\right\}
$$

$$
\left\{\sum_{z^*=\mathrm{th}^*+1}^{z-\acute{s}}\binom{z-\acute{s}}{z^*}(p_r(I))^{z^*}\sum_{z=\acute{s}}^{\infty}\sum_{t=0}^{z-\acute{s}-z^*}\binom{z-\acute{s}-z^*}{t}(p_r(U))^{z-\acute{s}-z^*-t}\right\}
$$

$$
\left[\sum_{m'=1}^{\infty}\binom{y-\acute{s}+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{y-\acute{s}}{m'}(p_r(U))^{y-\acute{s}-m'}-\sum_{m'=1}^{\infty}\binom{z-\acute{s}+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{z-\acute{s}}{m'}(p_r(U))^{x-\acute{s}-m'}\right]^{u-a-1}
$$

$$
\left[\sum_{m'=1}^{\infty}\binom{y-\acute{s}+m'}{m'}(p_r(I))^{m'}\sum_{m'=0}^{\infty}\binom{y-\acute{s}}{m'}(p_r(U))^{y-\acute{s}-m'}\right]^{b-u}. \tag{44}
$$

5.3. Largest Order Statistics.
For $=\acute{\eta}$ the largest order statistics for the true part of NNBD is given as

$$
f_{:\,\acute{\eta}}(x) = \acute{\eta}(p_r(S))^{\acute{s}}\sum_{t=0}^{\mathrm{th}^*}\binom{x-\acute{s}}{t}(p_r(I))^t(p_r(U))^{x-\acute{s}-t}\left[1-(p_r(S))^{\acute{s}}(p_r(U))^{x-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{x-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}\right]^{\acute{\eta}}-1. \tag{45}
$$

Largest order statistics for an indeterminate part of NNBD is given as

$$f_{\overset{\cdot}{\eta}}: \overset{\cdot}{\eta}(x) = \overset{\cdot}{\eta}(p_r(S))^{\acute{s}} \sum_{z^*=\mathrm{th}^*+1}^{x-\acute{s}} \binom{x-\acute{s}}{z^*}(p_r(I))^{z^*} \sum_{x=\acute{s}}^{\infty} \sum_{t=0}^{x-\acute{s}-z^*} \binom{x-\acute{s}-z^*}{t}(p_r(U))^{x-\acute{s}-z^*-t}$$

$$\left[1-(p_r(S))^{\acute{s}} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'}(p_r(I))^{m'} \sum_{m'=0}^{\infty} \binom{x-\acute{s}}{m'}(p_r(U))^{x-\acute{s}-m'}\right]^{\overset{\cdot}{\eta}} - 1. \tag{46}$$

5.4. Smallest Order Statistics. For $u = 1$, smallest order statistics for the true part of NNBD is given as

The smallest order statistics for the indeterminate part of NNBD is given as

$$f_{1:n}(x) = n(p_r(S))^{n\acute{s}} \sum_{t=0}^{\mathrm{th}^*} \binom{x-\acute{s}}{t}(p_r(I))^t (p_r(U))^{x-\acute{s}-t}$$

$$\left[(p_r(U))^{x-\acute{s}+1} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}\right]^{n-1}. \tag{47}$$

$$f_{1:n}(x) = n(p_r(S))^{n\acute{s}} \sum_{z^*=\mathrm{th}^*+1}^{x-\acute{s}} \binom{x-\acute{s}}{z^*}(p_r(I))^{z^*} \sum_{x=\acute{s}}^{\infty} \sum_{t=0}^{x-\acute{s}-z^*} \binom{x-\acute{s}-z^*}{t}(p_r(U))^{x-\acute{s}-z^*-t}$$

$$\left[\sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'}(p_r(I))^{m'} \sum_{m'=0}^{\infty} \binom{x-\acute{s}}{m'}(p_r(U))^{x-\acute{s}-m'}\right]^{n-1}. \tag{48}$$

5.5. Median Order Statistics. For $u = m + 1$, median order statistics for true part of NNBD is given as

$$f_{m+1:n}(x) = \frac{(2m+1)!}{m!n!}(p_r(S))^{\acute{s}(m+1)} \left[1-(p_r(S))^{\acute{s}}(p_r(U))^{x-\acute{s}+1} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}\right]^m$$

$$\left[(p_r(U))^{x-\acute{s}+1} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'-1}(p_r(I))^{m'-1}\right]^m \sum_{t=0}^{\mathrm{th}^*} \binom{x-\acute{s}}{t}(p_r(I))^t (p_r(U))^{x-\acute{s}-t}. \tag{49}$$

Median order statistics for the indeterminate part of NNBD is given as

$$f_{m+1:n}(x) = \frac{(2m+1)!}{m!n!}(p_r(S))^{\acute{s}(m+1)} \left[1-(p_r(S))^{\acute{s}} \sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'}(p_r(I))^{m'} \sum_{m'=0}^{\infty} \binom{x-\acute{s}}{m'}(p_r(U))^{x-\acute{s}-m'}\right]^m$$

$$\left[\sum_{m'=1}^{\infty} \binom{x-\acute{s}+m'}{m'}(p_r(I))^{m'} \sum_{m'=0}^{\infty} \binom{x-\acute{s}}{m'}(p_r(U))^{x-\acute{s}-m'}\right]^m$$

$$\sum_{z^*=\mathrm{th}^*+1}^{x-\acute{s}} \binom{x-\acute{s}}{z^*}(p_r(I))^{z^*} \sum_{x=\acute{s}}^{\infty} \sum_{t=0}^{x-\acute{s}-z^*} \binom{x-\acute{s}-z^*}{t}(p_r(U))^{x-\acute{s}-z^*-t}. \tag{50}$$

5.6. *Minimum and Maximum Joint Order Statistics.*

Minimum and maximum joint order statistics for the true part of NNBD is given as

$$f_{1:\,a:a}(y) = n(n-1)\left(p_r(S)\right)^{a\acute{s}}\left\{\sum_{t=0}^{th^*}\binom{y-\acute{s}}{t}\left(p_r(I)\right)^t\left(p_r(U)\right)^{y-\acute{s}-t}\right\}$$

$$\left\{\sum_{t=0}^{th^*}\binom{z-\acute{s}}{t}\left(p_r(I)\right)^t\left(p_r(U)\right)^{z-\acute{s}-t}\right\}$$

$$\left[\left(p_r(U)\right)^{y-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{y-\acute{s}+m'}{m'-1}\left(p_r(I)\right)^{m'-1} - \left(p_r(U)\right)^{z-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{z-\acute{s}+m'}{m'-1}\left(p_r(I)\right)^{m'-1}\right]^{a-2}.$$

(51)

Minimum and maximum joint order statistics for the indeterminate part of NNBD is given as

$$f_{1:\,a:a}(y) = a(a-1) = n(n-1)\left(p_r(S)\right)^{a\acute{s}}$$

$$\left\{\sum_{z^*=th^*+1}^{y-\acute{s}}\binom{y-\acute{s}}{z^*}\left(p_r(I)\right)^{z^*}\sum_{x=\acute{s}}^{\infty}\sum_{t=0}^{y-\acute{s}-z^*}\binom{y-\acute{s}-z^*}{t}\left(p_r(U)\right)^{y-\acute{s}-z^*-t}\right\}$$

$$\left\{\sum_{z^*=th^*+1}^{z-\acute{s}}\binom{x-\acute{s}}{z^*}\left(p_r(I)\right)^{z^*}\sum_{x=\acute{s}}^{\infty}\sum_{t=0}^{z-\acute{s}-z^*}\binom{z-\acute{s}-z^*}{t}\left(p_r(U)\right)^{z-\acute{s}-z^*-t}\right\}$$

$$\left[\left(p_r(U)\right)^{y-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{y-\acute{s}+m'}{m'-1}\left(p_r(I)\right)^{m'-1} - \left(p_r(U)\right)^{z-\acute{s}+1}\sum_{m'=1}^{\infty}\binom{z-\acute{s}+m'}{m'-1}\left(p_r(I)\right)^{m'-1}\right]^{a-2}.$$

(52)

6. Conclusion

This paper proposes a discrete neutrosophic negative binomial probability distribution using the neutrosophic logic. We have discussed various case studies under the proposed distribution. Several mathematical properties, including mgf, characteristics function, and probability generating function of the proposed distribution, have been derived and presented. On reliability review, we have presented characteristics such as survival function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, mills ratio, and odds ratio. Furthermore, we have obtained the order statistics for the proposed distribution. The proposed NNBD was useful in modeling the k th successes in a sequence of n independent trials before a specified number of failures took place.

Data Availability

The data is given in the paper.

Disclosure

This research is part of the thesis with Turnitin similarity report ID: 1481816761 dated December 29, 2020, submitted to Punjab University Library, Lahore

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

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