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Research Article

An Improved Asymptotic on the Representations of Integers as Sums of Products

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In this paper, we improve the error terms of Chace's results in the study by Chace (1994) on the number of ways of writing an integer N as a sum of k products of l factors, valid for $k \geq 3$ and $l = 2, 3$. More precisely, for $l = 2, 3$, we improve the upper bound $N^{k-1-(2(k-2)/(k-1)(l+1))+\varepsilon}$, $k \geq 3$ for the error term, to $N^{2-(2/2l+1)+\varepsilon}$ when $k = 3$ and $N^{k-1-(4(k-2)/(l+1)(k+l-2))+\varepsilon}$ when $k \geq 4$.

1. Introduction

In this paper, we study the number of representations of a natural number N as a sum of k terms, each being a product of l factors. Let $\nu(N; k, l)$ denote this number. This problem was studied by Estermann [1, 2] in the case $k = 2$ or 3 and $l = 2$ by using some properties of Dirichlet L-function. His method is not easy to be generalized. Later, Chace [3] generalized Estermann's result to $k \geq 3$ and $l \geq 2$. For such k and l, he got

$$\nu(N; k, l) = \mu(N; k, l) + E(N; k, l), \tag{1}$$

where

$$\mu(N; k, l) \asymp N^{k-1} \log^{k(l-1)} N, \tag{2}$$

and

$$E(N; k, l) \ll_{\varepsilon} \begin{cases} N^{k-1-(2(k-2)/(k-1)(l+1))+\varepsilon}, & \text{if } k = 3 \text{ or } l = 2,\ l = 3, \\ N^{k-1-(3(k-2)/(k-1)(l+2))+\varepsilon}, & \text{if } k \text{ and } l \geq 4. \end{cases} \tag{3}$$

Here, $\mu(N; k, l)$ is defined in (72). The way he studied the problem is different from Estermann. The main tools he used are the Hardy–Littlewood method and some results from the divisor problem in arithmetic progressions. He got the main term $\mu(N; k, l)$ which is a sum of terms of the form $\mathcal{S}\mathcal{J}$, where $\mathcal{S}$ are the "singular series" and $\mathcal{J}$ are the "singular integrals." They occur in the applications of the Hardy–Littlewood method. In this paper, we improve Chace's result in the cases $k \geq 3$ and $l = 2, 3$. We get the following result.

Theorem 1. *Suppose $k \geq 3$ and $l = 2, 3$. Then,*

$$\nu(N; k, l) = \mu(N; k, l) + E(N; k, l), \tag{4}$$

where $\mu(N; k, l)$ defined by (72) satisfies (2) and the error term satisfies

$$E(N; k, l) \ll_{\varepsilon} \begin{cases} N^{2-2/(2l+1)+\varepsilon}, & \text{if } k = 3, \\ N^{k-1-(4(k-2)/(l+1)(k+l-2))+\varepsilon}, & \text{if } k \geq 4. \end{cases} \tag{5}$$

We can compare the exponents in (3) with our results (5). For $l = 2, 3$, we see that $2 - 2/(2l + 1) < k - 1 - (2(k - 2)/(k - 1)(l + 1)) = 2 - 1/(l + 1)$ when $k = 3$ and $k - 1 - (4(k - 2)/(l + 1)(k + l - 2)) < k - 1 - (3(k - 2)/(k - 1)(l + 2))$ when $k \geq 4$. Our error terms are better than Chace's.

The proof of the theorem is an application of the Hardy–Littlewood method (cf, Chapter 3 of [4]), the Voronoi summation formula, and some results from the Kloosterman sums. The estimates on the minor arc were studied by Chace, and his result is sufficient for us. Hence, we will not focus on the minor arc in this paper. The main difficulty arises in treating the error term of the major arcs. In Section 2, we make some preparations for our proof. We state the Voronoi summation formula and give some lemmas related to the Kloosterman sums in this section. In Sections 3.1 and 3.2, we obtain a bound for the contribution from the minor arcs. In I3.3, we prove our theorem.

2. Applications of the Voronoi Summation Formula

Let $X > 0$. Suppose that $f(x)$ is a smooth function compactly supported in $(X - X^{\delta}, 2X + X^{\delta})$ with $0 < \delta < 1$, satisfying $f(x) = 1$ in $[X, 2X]$ and

$$f^{(j)}(x) \ll X^{-\delta j}, \quad j \geq 0, \tag{6}$$

$$\int \left| f^{(j)}(x) \right| \mathrm{d}x \ll X^{-\delta (j-1)}, \quad j \geq 1. \tag{7}$$

Let

$$F(s) = \int_0^{\infty} f(x) x^{s-1} \mathrm{d}x, \tag{8}$$

be the Mellin transformation of $f(x)$. To state Voronoi's summation formula, we introduce the notations below. For $\Re s > 1$, let

$$E_l\left(s, \frac{h}{q}\right) = \sum_{n=1}^{\infty} \frac{d_l(n) e(nh/q)}{n^s}, \tag{9}$$

where

$$d_l(n) = \sum_{n_1 \cdots n_l = n} 1. \tag{10}$$

We define

$$A_l^{\pm}\left(n, \frac{a}{q}\right) = \frac{1}{2} \sum_{n_1 \cdots n_l = n} \left(I(n_1, \ldots, n_l, a; q) \pm I(n_1, \ldots, n_l, a; q) \right), \tag{11}$$

$$U_l^{\pm}(x) = \frac{1}{2\pi i} \int_{\Re s = \sigma} \gamma_l^{\pm}(s) \frac{\mathrm{d}s}{x^s}, \tag{12}$$

where

$$I(n_1, \ldots, n_l, a; q) = \sum_{x_1, \ldots, x_l (\mathrm{mod} q)} e\left(\frac{n_1 x_1 + \cdots + n_l x_l + a x_1 \cdots x_l}{q} \right), \tag{13}$$

$$\gamma_l^{\pm}(s) = \frac{\Gamma^l (s/2) + (1 \pm (-1))/4)}{\Gamma^l (1-s)/2 + (1 \pm (-1))/4)}, \tag{14}$$

for $x > 0$ and $0 < \Re s < (1/2) - (1/l)$. From Theorem 2 in [5], we know the following lemma.

Lemma 1 (Voronoi's summation formula). *With the above notations, we have, for $l \geq 2$,*

$$\sum_{n=1}^{\infty} f(n) d_l(n) e\left(\frac{nh}{q}\right) = \mathrm{Res}_{s=1} F(s) E_l\left(s, \frac{h}{q}\right) + \frac{\pi^{(l/2)}}{q^l} \sum_{n=1}^{\infty} A_l^{+}\left(n, \frac{h}{q}\right) \int_0^{\infty} f(x) U_l^{+}\left(\frac{\pi^l n x}{q^l}\right) \mathrm{d}x + \frac{i^{3l} \pi^{(l/2)}}{q^l} \sum_{n=1}^{\infty} A_l^{-}\left(n, \frac{h}{q}\right) \int_0^{\infty} f(x) U_l^{-}\left(\frac{\pi^l n x}{q^l}\right) \mathrm{d}x. \tag{15}$$

To simplify the integrals in (15), we define

$$G_l^{\pm}(x) = \frac{1}{2\pi i} \int_{\Re s = 1-\sigma} \frac{F(s)}{\gamma_l^{\pm}(s)} x^s \mathrm{d}s, \tag{16}$$

where $\sigma > 0$. By (12), it is clear that

$$\begin{aligned} \int_0^{\infty} f(x) U_l^{\pm}(xy) \mathrm{d}x &= \frac{1}{2\pi i} \int_0^{\infty} \int_{\Re s = \sigma} f(x) \gamma_l^{\pm}(s) \frac{\mathrm{d}s}{(xy)^s}, \\ &= -\frac{1}{2\pi i} \int_{\Re s = 1-\sigma} \int_0^{\infty} f(x) x^{s-1} \mathrm{d}x \, \gamma_l^{\pm}(1-s) y^{s-1} \mathrm{d}s, \\ &= -\frac{1}{2\pi i} \int_{\Re s = 1-\sigma} \frac{F(s)}{\gamma_l^{\pm}(s)} y^{s-1} \mathrm{d}s, \\ &= -\frac{1}{y} G_l^{\pm}(y). \end{aligned} \tag{17}$$

To use Lemma 1 in practice, we need to estimate the residue in (15), $G_l^{\pm}(x)$ and $A_l^{\pm}(n,(h/q))$. We first compute the residue. For $0 \le j \le l-1$, we define

$$A_j(q) = \sum_{b=1}^{q} e\left(\frac{ab}{q}\right) c_{j+1}(b,q), \tag{18}$$

where $a > 0$ is an integer, and the coefficients $c_j(b,q)$ are sums of terms of the form

$$\sum_{b_1 b_2 \equiv b (\bmod q)} f(b_1), \tag{19}$$

for some function f. The coefficients $c_j(b,q)$ are given explicitly in equation (2.13) in [6]. It has been shown in [6] that for $(a,q)=1$, $A_j(q)$ is independent of a.

Lemma 2. *Suppose $1 \le a \le q$, $(a,q)=1$. We have*

$$\mathrm{Res}_{s=1} F(s) E_l\left(s,\frac{a}{q}\right) = \sum_{n=0}^{l-1} A_n(q) \int_0^{\infty} f(x) \frac{\log^n x}{n!} \mathrm{d}x. \tag{20}$$

Proof. By (9), we can rewrite $E_l(s,(a/q))$ as

$$E_l\left(s,\frac{a}{q}\right) = \sum_{h \bmod q} e\left(\frac{ah}{q}\right) Z(s;h,q), \tag{21}$$

where

$$Z(s;h,q) = \sum_{\substack{n \ge 1 \\ n = h (\bmod q)}} \frac{d_l(n)}{n^s}, \tag{22}$$

for $\mathfrak{R}s > 1$. Therefore,

$$\mathrm{Res}_{s=1} F(s) E_l\left(s,\frac{a}{q}\right) = \sum_{h=1}^{q} e\left(\frac{ah}{q}\right) \int_0^{\infty} f(x) d\, \mathrm{Res}_{s=1}\left\{Z(s;h,q)\frac{x^s}{s}\right\}. \tag{23}$$

Chace (Theorem 1 and (2.1) in [6]) gave that

$$\mathrm{Res}_{s=1}\left\{Z(s;h,q)\frac{x^s}{s}\right\} = \sum_{n=0}^{l-1} c_{n+1}(h,q) L_n(x), \tag{24}$$

where

$$L_n(x) = x \sum_{j=0}^{n} (-1)^{n-j} \frac{\log^j x}{j!}. \tag{25}$$

Substituting this into (23), we have

$$\mathrm{Res}_{s=1} F(s) E_l\left(s,\frac{a}{q}\right) = \sum_{n=0}^{l-1} A_n(q) \int_0^{\infty} f(s) \frac{\log^n x}{n!} \mathrm{d}x. \tag{26}$$

The proof of the lemma is complete.

Our next lemma, proved by Jiang and Lü (Lemma 2.7 in [7]), is to evaluate the integrals in (15). □

Lemma 3. *Let $G_l^{\pm}(x)$ be defined as in (16). Then, we have, for $l \ge 2$,*

$$G_l^{\pm}(x) \ll \begin{cases} X^{-A}, & \text{if } x > X^{l(1-\delta)-1+\varepsilon}, \\ (xX)^{(l-1)/(2l)}, & \text{if } X^{-1} \ll x \le X^{l(1-\delta)-1+\varepsilon}, \\ (xX)^{(1/2)} X^{(1-\delta)\varepsilon}, & \text{if } x \ll X^{-1}. \end{cases} \tag{27}$$

Proof. This can be proved by taking $J = X^{1-\delta}$ with $0<\delta<1$ in Lemma 2.7 of [7].

The following two lemmas give estimates for $A_l^{\pm}(n,(a/q))$. This two lemmas play an important role in our proof. We use some results for the Kloosterman sum to obtain the power saving in the q aspect. □

Lemma 4. *Suppose $1 \le a \le q$, $(a,q)=1$, and let $A_l^{\pm}(n,(a/q))$ be defined as in (11). We have*

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} A_l^{\pm}\left(n,\frac{a}{q}\right) e\left(-\frac{aN}{q}\right) \ll_{\varepsilon} q^{(l+1)/2+\varepsilon} n^{\varepsilon}. \tag{28}$$

Proof. By (13), we obtain

$$\sum_{\substack{a \bmod q\\(a,q)=1}} I(m_1,\ldots,m_l,a;q)e\left(-\frac{aN}{q}\right) = \sum_{x_1,\ldots,x_l \bmod q} e\left(\frac{m_1x_1+\cdots m_lx_l}{q}\right)\sum_{\substack{a \bmod q\\(a,q)=1}} e\left(\frac{ax_1\cdots x_l - aN}{q}\right),$$

$$= \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{x_1,\ldots,x_l \bmod q\\ x_1\cdots x_l = N \bmod d}} e\left(\frac{m_1x_1+\cdots m_lx_l}{q}\right), \tag{29}$$

$$= \sum_{\substack{d|q\\(q/d)|\mathbf{m}}} d\mu\left(\frac{q}{d}\right)\left(\frac{q}{d}\right)^l S_l\left(\frac{d}{q}\mathbf{m};N,d\right),$$

where

$$S_l(\mathbf{m};N,d) := \sum_{\substack{x_1,\ldots,x_l \bmod d\\ x_1\cdots x_l = N \bmod d}} e\left(\frac{m_1x_1+\cdots+m_lx_l}{d}\right), \tag{30}$$

for $\mathbf{m} = (m_1,\ldots,m_l) \in \mathbb{Z}^l$. Here, the notation $d|\mathbf{m}$ means that d divides each component of $\mathbf{m}$. Then, we obtain that the left hand side of the desired equation in our lemma equals to

$$\frac{1}{2}\sum_{m_1\cdots m_l=n}\sum_{\substack{d|q\\(q/d)|\mathbf{m}}} d\left(\frac{q}{d}\right)^l \mu\left(\frac{q}{d}\right)\left(S_l\left(\frac{\mathbf{m}}{(q/d)};N,d\right)+S_l\left(\frac{\mathbf{m}}{(q/d)};-N,d\right)\right)$$

$$= \frac{1}{2}\sum_{d|q} d\left(\frac{q}{d}\right)^l \mu\left(\frac{q}{d}\right)\sum_{m_1\cdots m_l = n(d/q)^l}\left(S_l(\mathbf{m};N,d)+S_l(\mathbf{m};-N,d)\right). \tag{31}$$

Now, we define

$$K_{l-1}(\mathbf{m};w) = \sum_{\substack{x_1,\ldots,x_{l-1} \bmod w\\ (x_j,w)=1, 1\le j\le l-1}} e\left(\frac{m_1x_1+\cdots+m_{l-1}x_{l-1}+m_l\overline{x_1\cdots x_{l-1}}}{w}\right), \tag{32}$$

for $\mathbf{m} = (m_1, \ldots, m_l) \in \mathbb{Z}^l$ and $w \in \mathbb{Z}$. Here, $\overline{x}$ denotes the multiplicative inverse of x modulo w. In Section 6 in [8], Smith gave that

$$\sum_{m_1 \cdots m_l = n} S_l(\mathbf{m}; N, d) = \sum_{\substack{b \mid d \\ b^l \mid n}} b^{l-1} d_l\left(n/b^l\right) K_{l-1}\left(e\left[Nn/b^l\right]; (d/b).\right), \tag{33}$$

where $e[t] = (1, \ldots, 1, t) \in \mathbb{Z}^l$ for all $t \in \mathbb{Z}$. Therefore, (31) can be written as

$$\frac{1}{2} \sum_{\substack{q = q_1 d_1 b \\ b^l \mid \left(n/q_1^l\right)}} d_1 q_1^l b^l \mu(q_1) d_l\left(\frac{n}{q_1^l b^l}\right)\left\{K_{l-1}\left(e\left[\frac{Nn}{q_1^l b^l}\right]; d_1\right) + K_{l-1}\left(e\left[-\frac{Nn}{q_1^l b^l}\right]; d_1\right)\right\}. \tag{34}$$

Taking the change of variables that $m = q_1 b$, we get

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} A_l^{\pm}\left(n, \frac{a}{q}\right) e\left(-\frac{aN}{q}\right) = \frac{q}{2} d_l(n) \left(K_{l-1}(e[Nn]; q) \pm K_{l-1}(e[-Nn]; q)\right). \tag{35}$$

From Theorem 6 in [9], we know that

$$\left|K_{l-1}(e[t]; q)\right| \leq q^{(l-1)/2} d_l(q). \tag{36}$$

The lemma follows immediately. □

Lemma 5. *Suppose* $1 \leq a \leq q$, $(a, q) = 1$, *and* $l = 2, 3$. *Let* $A_l^{\pm}(n, (a/q))$ *be defined as before. Then, we have*

$$A_l^{\pm}\left(n, \frac{a}{q}\right) \ll_{\varepsilon} q^{(l/2)+\varepsilon} n^{\varepsilon} (n, q)^{(1/2)}. \tag{37}$$

Proof. By (13), for $l = 2$, we have

$$I(n_1, n_2, a; q) = \sum_{x_1, x_2 \bmod q} e\left(\frac{n_1 x_1 + n_2 x_2 + a x_1 x_2}{q}\right) = \sum_{x_1 \bmod q} e\left(\frac{n_1 x_1}{q}\right) \sum_{x_2 \bmod q} e\left(\frac{(n_2 + a x_1) x_2}{q}\right) = q \sum_{\substack{x_1 \bmod q \\ a x_1 = -n_2 \bmod q}} e\left(\frac{n_1 x_1}{q}\right) \ll q. \tag{38}$$

For $l = 3$, one has

$$I(n_1, n_2, n_3, a; q) = q \sum_{d \mid (n_2, n_3, q)} d S\left(n_1, -\frac{n_2 n_3 \overline{a}}{d}; \frac{q}{d}\right). \tag{39}$$

Now, if we use Weil's classical bound

$$|S(m, n; c)| \leq (m, n, c)^{(1/2)} d_2(c) c^{(1/2)}, \tag{40}$$

then it follows from (39) that

$$I(n_1, n_2, n_3, a; q) \ll_{\varepsilon} q^{(3/2)+\varepsilon} \sum_{d \mid (n_2, n_3, q)} (n_1 d, n_2 n_3 \overline{a}, q)^{(1/2)}. \tag{41}$$

For $d \mid (n_2, n_3, q)$ and $(a, q) = 1$, noting that $n = n_1 n_2 n_3$ by (11), we have $(n_1 d, n_2 n_3 \overline{a}, q) \leq (n_1 n_2 n_3 \overline{a}, q) = (n, q)$. Therefore,

$$I(n_1, n_2, n_3, a; q) \ll_{\varepsilon} q^{(3/2)+\varepsilon} (n, q)^{(1/2)}. \tag{42}$$

Hence, we have

$$A_l^{\pm}\left(n, \frac{a}{q}\right) \ll_{\varepsilon} q^{(l/2)+\varepsilon} (n, q)^{(1/2)} \sum_{n_1 n_2 n_3 = n} 1 \ll_{\varepsilon} q^{(l/2)+\varepsilon} n^{\varepsilon} (n, q)^{(1/2)}, \tag{43}$$

for $l = 2, 3$. The proof of the lemma is complete. □

3. Proof of Theorem 1

In this section, we prove Theorem 1. The main difficulty arises in treating the major arcs. The results from Kloosterman's sums will play a role in our proof. Throughout this section, $1 \leq X \leq (N/2)$. We choose a smooth function f compactly supported in $[X - X^{\delta}, 2X + X^{\delta}]$ with $0 < \delta < 1$, satisfying $f(x) = 1$ in $[X, 2X]$ and (6) and (7). To apply the circle method, we choose the parameters P and Q such that

$$\begin{aligned} PQ &= N,\\ P &\le N^{(1/l)}. \end{aligned} \tag{44}$$

By Dirichlet's lemma on rational approximations, each $\alpha \in I := [(1/Q), 1 + (1/Q)]$ may be written in the form as follows:

$$\begin{aligned} \alpha &= \frac{a}{q} + \beta,\\ |\beta| &< \frac{1}{qQ}, \end{aligned} \tag{45}$$

for some integers a and q with $1 \le a \le q \le Q$ and $(a, q) = 1$. We denote by $\mathfrak{M}(q, a)$ the set of α satisfying (45) and define the major arc by

$$\mathfrak{M} = \bigcup_{1\le q\le P} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} \mathfrak{M}(a, q). \tag{46}$$

It follows from $P \le N^{(1/l)} < (Q/2)$ that the major arcs $\mathfrak{M}(q, a)$ are disjoint. The union $\mathfrak{m}$ of minor arcs is just the complement of $\mathfrak{M}$ in I. We denote

$$S_l(\alpha) = \sum_{n\le N} d_l(n) e(n\alpha). \tag{47}$$

Then, we have

$$\begin{aligned} \nu(N; k, l) &= \int_I S_l^k(\alpha) e(-N\alpha)\mathrm{d}\alpha,\\ &= \int_{\mathfrak{M}} S_l^k(\alpha) e(-N\alpha)\mathrm{d}\alpha + \int_{\mathfrak{m}} S_l^k(\alpha) e(-N\alpha)\mathrm{d}\alpha. \end{aligned} \tag{48}$$

The upper bound of the integral of $S_l^k(\alpha)$ on $\mathfrak{m}$ is given by Chace [3].

Lemma 6. *Suppose $k \ge 3$ and $l \ge 2$ are integers. Then,*

$$\int_{\mathfrak{m}} S_l^k(\alpha) e(-N\alpha)\mathrm{d}\alpha \ll_{\varepsilon} N^{k-1+\varepsilon} P^{-k+2}. \tag{49}$$

This estimate on the minor arc is sufficient for us. It remains to consider the integral of $S_l(\alpha)$ on the major arc. Let

$$S_l(\alpha, X) = \sum_{X<n\le 2X} d_l(n) e(n\alpha). \tag{50}$$

Now, for $\alpha = a/q + \beta \in \mathfrak{M}$, we have

$$S_l(\alpha, X) = \sum_{n=1}^{\infty} f_\beta(n) d_l(n) e\left(\frac{an}{q}\right) + R_{1,l}(\alpha, X), \tag{51}$$

where $f_\beta(x) := f(x) e(x\beta)$ and

$$R_{1,l}(\alpha, X) = -\left(\sum_{X-X^\delta<n\le X} + \sum_{2X<n<2X+X^\delta} f(n) d_l(n) e(n\alpha) \ll_\varepsilon X^{\delta+\varepsilon}\right). \tag{52}$$

It is clear that $f_\beta(x)$ is a smooth function compactly supported in $[X - X^\delta, 2X + X^\delta]$. Moreover, it satisfies

$$f_\beta^{(\nu)}(x) \ll \left(\frac{|\beta|X + X^{1-\delta}}{X}\right)^{\nu}, \tag{53}$$

for any $\nu \ge 0$, and

$$\begin{aligned} \int \left|f_\beta^{(\nu)}(x)\right|\mathrm{d}x &\ll \sum_{0\le j\le \nu} |\beta|^{\nu-j} \int \left|f^{(j)}(x)\right|\mathrm{d}x,\\ &\ll |\beta|^{\nu} X + |\beta|^{\nu-1} \sum_{1\le j\le \nu} \left(\frac{X^{1-\delta}}{|\beta|X}\right)^{j-1},\\ &\ll |\beta|^{\nu} X + |\beta|^{\nu-1} + X^{-\delta(\nu-1)},\\ &\ll (1 + |\beta|X)\left(\frac{|\beta|X + X^{1-\delta}}{X}\right)^{\nu-1}, \end{aligned} \tag{54}$$

for any $\nu \ge 1$. Now, by Lemma 1, we get

$$\sum_{n=1}^{\infty} f_\beta(n) d_l(n) e\left(\frac{an}{q}\right) = \mathrm{Res}_{s=1} F_\beta(s) E_l\left(\frac{a}{q}\right) + R_{2,l}\left(\frac{a}{q} + \beta, X\right), \tag{55}$$

where

$$F_\beta(s) = \int_0^\infty f_\beta(x) x^{s-1}\mathrm{d}x, \tag{56}$$

is the Mellin transformation of $f_\beta(x)$ and

$$R_{2,l}\left(\frac{a}{q} + \beta, X\right) := \frac{\pi^{(l/2)}}{q^l} \sum_{n=1}^{\infty} A_l^+\left(n, \frac{a}{q}\right) \int_0^\infty f(x) U_l^+\left(\frac{\pi^l nx}{q^l}\right)\mathrm{d}x + \frac{i^{3l}\pi^{(l/2)}}{q^l} \sum_{n=1}^{\infty} A_l^-\left(n, \frac{a}{q}\right) \int_0^\infty f(x) U_l^-\left(\frac{\pi^l nx}{q^l}\right)\mathrm{d}x. \tag{57}$$

It follows from Lemma 2 that

$$\mathrm{Res}_{s=1} F_\beta(s) E_l\left(\frac{a}{q}\right) = \sum_{n=0}^{l-1} A_n(q) \int_0^\infty f_\beta(s) \frac{\log^n x}{n!} d\beta\ . \tag{58}$$

From Equation (4.8) in [3], we know that

$$A_j(q) \ll_\varepsilon q^{-1+\varepsilon}. \tag{59}$$

Then, we have

$$\mathrm{Res}_{s=1} F_\beta(s) E_l\left(s, \frac{a}{q}\right) = \sum_{n=0}^{l-1} A_n(q) \int_X^{2X} e(\beta x) \frac{\log^n x}{n!} dx + R_{3,l}(\alpha, X), \tag{60}$$

where

$$R_{3,l}(\alpha, X) \ll_{\varepsilon} q^{-1+\varepsilon} X^{\delta+\varepsilon}. \tag{61}$$

Now, for $\alpha = a/q + \beta \in \mathfrak{M}$, by (51), (55), and (60), we write

$$S_l(\alpha, X) = T_l(\alpha, X) + R_l(\alpha, X), \tag{62}$$

where

$$T_l\left(\frac{a}{q} + \beta, X\right) = \sum_{n=0}^{l-1} A_n(q) \int_X^{2X} e(\beta x) \frac{\log^n x}{n!} dx, \tag{63}$$

$$R_l(\alpha, X) = R_{1,l}(\alpha, X) + R_{2,l}(\alpha, X) + R_{3,l}(\alpha, X). \tag{64}$$

Hence, by dyadic analysis, for $\alpha = a/q + \beta \in \mathfrak{M}$, formally, we write

$$S_l(\alpha) = T_l(\alpha) + R_l(\alpha), \tag{65}$$

where

$$R_l(\alpha) = \sum_{j=1}^{[\log_2 N]-1} \sum_{t=1}^{3} R_{t,l}(\alpha, (N/2)) + O(1), \tag{66}$$

$$T_l\left(\frac{a}{q} + \beta\right) = \sum_{n=0}^{l-1} A_n(q) I_n(\beta), \tag{67}$$

with

$$I_n(\beta) = \int_1^N e(\beta x) \frac{\log^n x}{n!} \mathrm{d}x, \tag{68}$$

for $n = 0, 1, \ldots, l-1$. Therefore, we have

$$\int_{\mathfrak{M}} S_l^k(\alpha) e(-N\alpha) d\alpha = \sum_{j=0}^{k} \binom{k}{j} M_{j,l}(N), \tag{69}$$

where

$$M_{j,l}(N) = \int_{\mathfrak{M}} T_l^{k-j}(\alpha) R_l^j(\alpha) e(-N\alpha) \mathrm{d}\alpha. \tag{70}$$

It has been proved by Chace (Section 5 in [3]) that

$$M_{0,l}(N) = \mu(N; k, l) + O_{\varepsilon}\left(N^{k-1+\varepsilon} P^{-k+2}\right), \tag{71}$$

where

$$\mu(N; k, l) = \sum_{q=1}^{\infty} C_q(N) \int_{-\infty}^{\infty} \left[\sum_{n=0}^{l-1} A_n(q) I_n(\beta)\right]^k e(-N\beta) d\beta, \tag{72}$$

satisfying

$$\mu(N; k, l) \asymp N^{k-1} \log^{k(l-1)} N. \tag{73}$$

In the following subsections, we will estimate $M_{j,l}(N)$ for $1 \le j \le k$.

3.1. *The Estimate of* $M_{1,l}(N)$.

In this section, we give the upper bound of $M_{1,l}(N)$.

Lemma 7. *Let $l = 2$ or 3. We have*

$$M_{1,l}(N) \ll_{\varepsilon} \begin{cases} N^{1+(1-\delta)(l-1)/2+\varepsilon} P^{(l-1)/2} + N^{1+\delta+\varepsilon}, & \text{if } k = 3, \\ N^{k-2+(1-\delta)(l-1)/2+\varepsilon} + N^{k-2+\delta+\varepsilon}, & \text{if } k \ge 4. \end{cases} \tag{74}$$

Proof. By our definition of major arc (46), we have

$$M_{1,l}(N) = \sum_{q \le P} \sum_{\substack{a \bmod q \\ (a,q)=1}} \int_{|\beta| < \frac{1}{qQ}} T_l^{k-1}\left(\frac{a}{q} + \beta\right) R_l\left(\frac{a}{q} + \beta\right) e(-N\alpha) d\alpha. \tag{75}$$

Because for $\alpha = a/q + \beta \in \mathfrak{M}$, $T_l(\alpha, N)$ is independent of a, and we can interchange the order of summation over a and the integral and then take the summation of $R_l(\alpha, N) e(-Na/q)$ over a first. Therefore,

$$M_{1,l}(N) \ll \sum_{q \le P} \int_{|\beta| < \frac{1}{qQ}} \left|T_l\left(\frac{a}{q} + \beta\right)\right|^{k-1} \left|\sum_{\substack{a \bmod q \\ (a,q)=1}} R_l\left(\frac{a}{q} + \beta\right) e\left(-\frac{aN}{q}\right)\right| d\alpha. \tag{76}$$

By (59) and (68), it is clear that

$$T_l\left(\frac{a}{q}+\beta\right) \ll_{\varepsilon} q^{-1+\varepsilon} N^{\varepsilon} \min\left\{N, |\beta|^{-1}\right\}. \tag{77}$$

By the definition of $R_l(\alpha)$ in (66), we have

$$\sum_{\substack{a \bmod q\\(a,q)=1}} R_{t,l}\left(\frac{a}{q}+\beta\right) e\left(-\frac{aN}{q}\right) \ll \sum_{t=1}^{3} \max_{1\leq X\leq (N/2)} \left| \sum_{\substack{a \bmod q\\(a,q)=1}} R_l\left(\frac{a}{q}+\beta, X\right) e\left(-\frac{aN}{q}\right) \right| + q. \tag{78}$$

We deduce that

$$\begin{aligned}\sum_{\substack{a \bmod q\\(a,q)=1}} R_{1,l}\left(\frac{a}{q}+\beta, X\right) e\left(-\frac{aN}{q}\right) &= \sum_{\substack{X-X^{\delta}\leq n\leq X\\ \text{or}\\ 2X\leq n\leq 2X+X^{\delta}}} d_l(n)(n) \sum_{\substack{a \bmod q\\(a,q)=1}} e\left(\frac{a(n-N)}{q}\right) e(n\beta)\\ &= -\sum_{d\mid q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{X-X^{\delta}\leq n\leq X\\ \text{or}\\ 2X\leq n\leq 2X+X^{\delta},\\ n\equiv N \bmod d}} d_l(n) f(n) e(n\beta) \ll_{\varepsilon} X^{\delta+\varepsilon}.\end{aligned} \tag{79}$$

It is obvious from (61) that

$$\sum_{\substack{a \bmod q\\(a,q)=1}} R_{3,l}\left(\frac{a}{q}+\beta, X\right) e\left(-\frac{aN}{q}\right) = \ll_{\varepsilon} X^{\delta+\varepsilon}. \tag{80}$$

It remains to consider the summation of $R_{2,l}$. We write $G_{l,\beta}^{\pm}$ be $G_l^{\pm}$ with F replaced by F_β in (16). Similar to the assertions of Lemma 3, we have

$$G_{l,\beta}^{\pm}(x) \ll \begin{cases} X^{-A} & \text{if } x > \left(|\beta| X + X^{1-\delta}\right)^{l+\varepsilon} X^{-1},\\ (1+|\beta|X)(xX)^{(l-1)/(2l)} & \text{if } X^{-1} \ll x \leq \left(|\beta| X + X^{1-\delta}\right)^{l+\varepsilon} X^{-1},\\ (1+|\beta|X)(xX)^{(1/2)}\left(|\beta| X + X^{1-\delta}\right)^{\varepsilon} & \text{if } x \ll X^{-1}.\end{cases} \tag{81}$$

Notice that

$$\int_0^{\infty} f_\beta(x)U_l^{\pm}(xy)\mathrm{d}x = -\frac{1}{y}G_{\beta,l}^{\pm}(y). \tag{82}$$

Then, by Lemma 4, we have

$$\begin{aligned} q^{-l}\sum_{n=1}^{\infty}\sum_{\substack{a\bmod q\\(a,q)=1}} A_l^{\pm}\left(n,\frac{a}{q}\right)e\left(-\frac{aN}{q}\right)\int_0^{\infty} f_\beta(x)U_l^{\pm}\left(\frac{\pi^l nx}{q^l}\right)\mathrm{d}x &\ll_\varepsilon q^{-(l-1)/2+\varepsilon}\sum_{n=1}^{\infty} n^{\varepsilon}\frac{q^l}{n}\left|G_{l,\beta}^{\pm}\left(\frac{\pi^l n}{q^l}\right)\right| \\ &\ll_\varepsilon q^{-(l-1)/2+\varepsilon}\left\{\sum_{n\le\left(|\beta|X+X^{1-\delta}\right)^{l+\varepsilon}q^lX^{-1}} n^{\varepsilon}(1+|\beta|X)\frac{q^l}{n}\left(\frac{nX}{q^l}\right)^{(l-1)/2l} + \sum_{n\ll q^lX^{-1}} n^{\varepsilon}\frac{q^l}{n}(1+|\beta|X)\left(\frac{nX}{q^l}\right)^{(1/2)}\left(|\beta|X+X^{1-\delta}\right)^{\varepsilon}\right\} \\ &\ll_\varepsilon q^{(l+1)/2+\varepsilon}\left(|\beta|X+X^{1-\delta}\right)^{(l-1)/2+\varepsilon}(1+|\beta|X). \end{aligned} \tag{83}$$

Therefore, we get

$$\sum_{\substack{a\bmod q\\(a,q)=1}} R_{2,l}\left(\frac{a}{q}+\beta, X\right)e\left(-\frac{aN}{q}\right) \ll_\varepsilon q^{(l+1)/2+\varepsilon}\left(|\beta|X+X^{1-\delta}\right)^{(l-1)/2+\varepsilon}(1+|\beta|X). \tag{84}$$

Hence, by (79), (80), and (84), for $l=2,3$, one has

$$\sum_{\substack{a\bmod q\\(a,q)=1}} R_l\left(\frac{a}{q}+\beta\right)e\left(-\frac{aN}{q}\right) \ll \sum_{t=1}^{3}\max_{1\le X\le N/2}\left|\sum_{\substack{a\bmod q\\(a,q)=1}} R_{t,l}\left(\frac{a}{q}+\beta, X\right)e\left(-\frac{aN}{q}\right)\right| \ll_\varepsilon q^{(l+1)/2+\varepsilon}\left(|\beta|N+N^{1-\delta}\right)^{(l-1)/2+\varepsilon}(1+|\beta|N)+N^{\delta+\varepsilon}. \tag{85}$$

Substituting this and (77) into (76), we obtain

$$\begin{aligned} M_{1,l}(N) &\ll \sum_{q\le P}\int_{|\beta|<\frac{1}{qQ}}\left|T_l\left(\frac{a}{q}+\beta\right)\right|^{k-1}\left|\sum_{\substack{a\bmod q\\(a,q)=1}} R_l\left(\frac{a}{q}+\beta\right)e\left(-\frac{aN}{q}\right)\right|d\alpha \\ &\ll \sum_{q\le P} q^{-k+1+(l+1)/2+\varepsilon}N^{\varepsilon}\int_{|\beta|<\frac{1}{qQ}}\min\left\{N,|\beta|^{-1}\right\}^{k-1}\left(|\beta|N+N^{1-\delta}\right)^{(l-1)/2+\varepsilon}(1+|\beta|X)d\beta \\ &\quad + \sum_{q\le P} q^{-k+1+\varepsilon}\int_{|\beta|<\frac{1}{qQ}}\min\left\{N,|\beta|^{-1}\right\}^{k-1}N^{\delta+\varepsilon}d\beta . \end{aligned} \tag{86}$$

By some elementary calculations, we get the desired result. □

3.2. Integral of $M_{j,l}(N)$, $2 \le j \le k$, on the Major Arcs. The way we treat for $M_{j,l}(N)$ for $j \ge 2$ is different from that of $M_{1,l}(N)$. We get the following lemma.

Lemma 8. *Let $M_{j,l}(N)$ and $2 \le j \le k$ be defined as before. We have, for $l = 2, 3$,*

$$\sum_{j=2}^{k} \binom{k}{j} M_{j,l}(N) \ll_{\varepsilon} \begin{cases} P^{l+1} N^{(1-\delta)(l-1)+\varepsilon} + P^{l+2} N^{\varepsilon} + P N^{2\delta+\varepsilon}, & \text{if } k = 3, \\ N^{k-3+\varepsilon}\left(P^{l} N^{(1-\delta)(l-1)} + P^{l+1}\right) + N^{k-3+2\delta+\varepsilon}, & \text{if } k \ge 4. \end{cases} \tag{87}$$

Proof. The strategy is similar to that of Section 5 in [3]. Note that

$$\sum_{j=2}^{k} T_l^{k-j}(\alpha) R_l^{j}(\alpha) \ll \left|R_l(\alpha)\right|^2 \left(\left|S_l(\alpha)\right|^{k-2} + \left|T_l(\alpha)\right|^{k-2}\right). \tag{88}$$

We obtain

$$\sum_{j=2}^{k} \binom{k}{j} M_{j,l}(N) \ll \int_{\mathfrak{M}} \left| \sum_{j=2}^{k} T_l^{k-j}(\alpha) R_l^{j}(\alpha) \right| \mathrm{d}\alpha \ll \max_{\alpha \in \mathfrak{M}} \left|R_l(\alpha)\right|^2 \int_{\mathfrak{M}} \left(\left|S_l(\alpha)\right|^{k-2} + \left|T_l(\alpha)\right|^{k-2}\right) \mathrm{d}\alpha. \tag{89}$$

Similar to (83), by (81) and Lemma 5, we have, for $l = 2, 3$ and $\alpha = a/q + \beta \in \mathfrak{M}$,

$$R_{2,l}(\alpha, X) \ll_{\varepsilon} q^{(l/2)+\varepsilon} \left(|\beta| X + J\right)^{(l-1)/2+\varepsilon} \left(1 + |\beta| X\right). \tag{90}$$

Hence, by (52), (61), (64), (90), and (66), we have

$$\begin{aligned} \max_{\alpha \in \mathfrak{M}} \left|R_l(\alpha)\right| &\ll \max_{\alpha \in \mathfrak{M}} \max_{1 \le X \le N/2} \left(\sum_{t=1}^{3} \left|R_{t,l}(\alpha, X)\right| + 1 \right) \\ &\ll_{\varepsilon} \max_{(a/q)+\beta \in \mathfrak{M}} \left(q^{(l/2)+\varepsilon} \left(1 + |\beta| N\right) \left(|\beta| N + N^{1-\delta}\right)^{(l-1)/2+\varepsilon} + N^{\delta+\varepsilon} \right) \\ &\ll_{\varepsilon} P^{(l/2)} N^{(1-\delta)(l-1)/2+\varepsilon} + P^{(l+1)/2+\varepsilon} + N^{\delta+\varepsilon}. \end{aligned} \tag{91}$$

Now, for $k = 3$, by Cauchy's inequality, the definition of major arc (46), and Parseval's identity, we obtain

$$\int_{\mathfrak{M}} \left|S_l(\alpha)\right| d\alpha \ll |\mathfrak{M}|^{(1/2)} \left(\int_0^1 \left|S_l(\alpha)\right|^2 d\alpha \right)^{(1/2)} \ll_{\varepsilon} P N^{\varepsilon}. \tag{92}$$

For $k \ge 4$, we have

$$\int_{\mathfrak{M}} \left|S_l(\alpha)\right|^{k-2} d\alpha \ll \max_{\alpha \in \mathfrak{M}} \left|S_l(\alpha)\right|^{k-4} \int_0^1 \left|S_l(\alpha)\right|^2 d\alpha \ll_{\varepsilon} N^{k-3+\varepsilon}, \tag{93}$$

by the fact that $S_l(\alpha) \ll_{\varepsilon} N^{1+\varepsilon}$. One then shows that

$$\int_{\mathfrak{M}} T_l(\alpha)^{k-2} d\alpha \ll_{\varepsilon} \begin{cases} P N^{\varepsilon}, & \text{if } k = 3, \\ N^{k-3+\varepsilon}, & \text{if } k \ge 4, \end{cases} \tag{94}$$

by using the definition of the major arc and (77). Combining these results, we complete the proof of the lemma. □

3.3. Proof of Theorem 1. By Lemma 6 and (69)–(73), we have, for $k \ge 3$, $l = 2, 3$,

$$\nu(N; k, l) = \mu(N; k, l) + k M_{1,l}(N) + \sum_{j=2}^{k} \binom{k}{j} M_{j,l}(N) + O_{\varepsilon}\left(N^{k-1+\varepsilon} P^{-k+2}\right), \tag{95}$$

where

$$\mu(N;k,l) = \sum_{q=1}^{\infty} C_q(N) \int_{-\infty}^{\infty} \left[\sum_{n=0}^{l-1} A_n(q) I_n(\beta)\right]^k e(-N\beta) d\beta , \tag{96}$$

satisfying

$$\mu(N;k,l) \asymp N^{k-1} \log^{k(l-1)} N. \tag{97}$$

Now, by Lemmas 7 and 8, taking

$$\delta = \begin{cases} \dfrac{2l-1}{2l+1}, & \text{if } k = 3, \\ \dfrac{l-1}{l+1}, & \text{if } k \geq 4, \end{cases} \quad \text{and } P = \begin{cases} N^{2/(2l+1)}, & \text{if } k = 3, \\ N^{(4/(l+1)(k+l-2))}, & \text{if } k \geq 4, \end{cases} \tag{98}$$

we have, for $l = 2, 3$,

$$\nu(N;k,l) = \begin{cases} \mu(N;3,l) + O_\varepsilon\left(N^{2-2/(2l+1)+\varepsilon}\right), & \text{if } k = 3, \\ \mu(N;k,l) + O_\varepsilon\left(N^{k-1-(4(k-2)/(l+1)(k+l-2))+\varepsilon}\right), & \text{if } k \geq 4. \end{cases} \tag{99}$$

We complete the proof of Theorem 1.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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