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Research Article

Multiple Positive Solutions for a Class of Boundary Value Problem of Fractional (p,q)-Difference Equations under (p,q)-Integral Boundary Conditions

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This paper is mainly concerned with a class of fractional (p,q)-difference equations under (p,q)-integral boundary conditions. Multiple positive solutions are established by using the topological degree theory and Krein–Rutman theorem. Finally, two examples are worked out to illustrate the main results.

1. Introduction

The q-difference operator was first systematically studied by Jackson [1]. Then, q-calculus has been studied extensively. See [2–4] and references therein. q-calculus and q-difference equations have been used by many researchers to solve physical problems such as molecular problems and chemical physics [1, 5–7]. For example, in 1967, Floreanini and Vinet [3] studied the behaviors of hydrogen atoms by using Schrödinger equation and q-calculus. Diaz and Osler [8] investigated the q-field theory.

In the last decades, the theory of quantum calculus based on two-parameter (p,q)-integer has been studied since it can be used efficiently in many fields such as difference equations, Lie group, hypergeometric series, and physical sciences. The (p,q)-calculus was first studied by Chakrabarti and Jagannathan [2] in the field of quantum algebra in 1991. Njionou Sadjang [9] systematically established the basic theory of (p,q)-calculus and some (p,q)-Taylor formula. Milovanovic and Gupta [10] developed the concept of (p,q)-beta and (p,q)-gamma functions. These basic concepts and theories promote the development of (p,q)-calculus. For detailed results on (p,q)-calculus, please see [9–13] and references therein.

On the contrary, the research of fractional calculus in discrete settings was initiated in [8, 11, 14]. In 2020, Soontharanonl and Sitthiwirattham [15] introduced the fractional (p,q)-calculus, which has been found in a wide range of applications in many fields such as concrete mathematical models of quantum mechanics and fluid mechanics [7, 13, 15].

As we all know, in recent decades, more and more researchers pay much attention to the fractional differential equations and have obtained substantial achievements, we refer the readers to see [16–34] and references therein. Although the results of discrete fractional calculus are similar to those of continuous fractional calculus, the theory of discrete fractional calculus remains much less developed than that of continuous fractional calculus [35, 36]. Therefore, it is very important to develop discrete calculus. In particular, the fractional (p,q)-difference equations involving (p,q)-integral boundary conditions have rarely been studied. In order to make up for this gap, the paper mainly studies the following boundary value problem of fractional

(p,q)-difference equations under (p,q)-integral boundary conditions:

$$\begin{cases} D_{p,q}^{\alpha} x(t) + f(t,x) = 0, t \in (0,1), \\ x(0) = D_{p,q} x(0) = 0, D_{p,q} x(1) = \int_0^1 h(t) D_{p,q} x(t) d_{p,q} t, \end{cases} \tag{1}$$

where $2 < \alpha < 3$, $0 < q < p \le 1$, $f \in C([0,1] \times R^+, R^+)$, and $D_{p,q}^{\alpha}$ is fractional (p,q)-difference operator.

It should be pointed out that the boundary conditions of BVP equation (1) are more extensive. Furthermore, two parameters in the discrete environment makes the boundary value problem more complex. In order to overcome these difficulties, we constructed a special cone. The existence and multiplicity of the positive solution for the BVP equation (1) are obtained by using the topological degree theory, Krein–Rutman theorem.

This paper is structured as follows. In Section 2, we introduce some definitions of (p,q)-fractional integral and differential operator together with some basic properties and lemmas. The main results are given and proved in Section 3. Finally, in Section 4, two examples are given to show the applicability of our main results.

2. Preliminaries

In this section, we list some basic definitions and lemmas that will be used in this paper. For $0 < q < p \le 1$, we let

$$[k]_{p,q} := \begin{cases} \dfrac{p^k - q^k}{p - q} = p^{k-1} [k] m q_{q/p}, & k \in N, \\ 1, & k = 0. \end{cases} \tag{2}$$

The (p,q)-analogue of the power function $(a-b)_{p,q}^n$ with $n \in N_0 \backslash$coloneq 0, 1, 2, ..., is given by

$$(a-b)_{p,q}^0 =: 1, (a-b)_{p,q}^n := \prod_{k=0}^{n-1} \left(a p^k - b q^k\right), \quad a, b \in R. \tag{3}$$

For $\alpha \in R$,

$$(a-b)_{p,q}^{\alpha} := a^{\alpha} \prod_{i=0}^{\infty} \left[\frac{1 - (b/a)(q/p)^i}{1 - (b/a)(q/p)^{\alpha+i}} \right], \quad a \ne 0. \tag{4}$$

By [15], we obtain

$$(a-b)_{p,q}^{\alpha} = p^{(\alpha/2)} (a-b)_{q/p}^{\alpha} = a^{\alpha} \prod_{i=0}^{\infty} \frac{1}{p^{\alpha}} \left[\frac{1 - b/a (q/p)^i}{1 - b/a (q/p)^{\alpha+i}} \right], \quad a \ne 0. \tag{5}$$

Note that $a_q^{\alpha} = a_{p,q}^{\alpha} = a^{\alpha}$ and $(0)_q^{\alpha} = (0)_{p,q}^{\alpha} = 0$ for $\alpha > 0$. The (p,q)-gamma and (p,q)-beta functions are defined by

$$\Gamma_{p,q}(x) := \begin{cases} \dfrac{(p-q)_{p,q}^{x-1}}{(p-q)^{x-1}} = \dfrac{(1-q/p)_{p,q}^{x-1}}{(1-q/p)^{x-1}}, & x \in R \setminus \{0, -1, -2, \ldots,\}, \\ [x-1]_{p,q}!, & x \in N, \end{cases} \tag{6}$$

$$B_{p,q}(x,y) := \int_0^1 t^{x-1} (1-qt)_{p,q}^{y-1} d_{p,q} t = p^{\frac{1}{2}(y-1)(2x+y-2)} \frac{\Gamma_{p,q}(x) \Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)},$$

respectively.

Definition 1 (see [15]). For $0 < q < p \le 1$ and $f\colon [0,T] \longrightarrow R$, we define the (p,q)-difference of f as

$$D_{p,q} f(t) := \frac{f(pt) - f(qt)}{(p-q)(t)}, \quad t \ne 0, \tag{7}$$

where $D_{p,q} f(0) = f'(0)$, provided that f is differentiable at 0.

Definition 2 (see [15]). For $N-1 < \alpha < N$, $0 < q < p \le 1$, and $f\colon I_{p,q}^T \longrightarrow R$, the fractional (p,q)-difference is defined by

$$D_{p,q}^{\alpha} f(t) = \frac{(p-q) t^{N-\alpha}}{p^{\binom{N-\alpha}{2}} \Gamma_{p,q}(N-\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1 - \left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{N-\alpha-1} D_{p,q}^N f\left(\frac{q^k}{p^{k+N-\alpha}} t\right), \tag{8}$$

where $I_{p,q}^{T} = \{(q^k/p^{k+1})T\colon k \in \mathbb{N}_0\} \bigcup \{0\}$.

Definition 3 (see [15]). Let I be any closed interval of R containing a, b, and 0. Assuming that $f\colon I \longrightarrow R$ is a given function, we define (p,q)-integral of f from a to b by

$$\int_a^b f(t) d_{p,q}t = \int_0^b f(t) d_{p,q}t - \int_0^a f(t) d_{p,q}t, \tag{9}$$

where

$$\begin{aligned} I_{p,q} f(x) &= \int_a^x f(t) d_{p,q}t \\ &= (p-q)x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} x\right), \quad x \in I, \end{aligned} \tag{10}$$

provided that the series converges at $x = a$ and $x = b$. f is called (p,q)-integrable on $[a,b]$.

Definition 4 (see [15]). For $\alpha > 0$, $0 < q < p \le 1$, and $f\colon [0,T] \longrightarrow R$, the fractional (p,q)-integral is defined by

$$I_{p,q}^{\alpha} f(t) := \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{\alpha-1} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s, \tag{11}$$

and $(I_{p,q}^{0} f)(t) = f(t)$.

Lemma 1 (see [15]). *Let f, g be (p,q)-differentiable. The properties of (p,q)-difference operator are as follows:*

(i) $D_{p,q}[f(t) + g(t)] = D_{p,q} f(t) + D_{p,q} g(t)$

(ii) $D_{p,q}[\alpha f(t)] = \alpha D_{p,q} f(t)$, *for* $\alpha \in R$

Lemma 2 (see [15]). *For $0 < q < p \le 1$, $\alpha \ge 1$, and $a \in R$, we have*

(i) $D_{p,q}(t-a)_{p,q}^{\alpha} = [\alpha]_{p,q}(pt-a)_{p,q}^{\alpha-1}$

(ii) $D_{p,q}(a-t)_{p,q}^{\alpha} = -[\alpha]_{p,q}(a-qt)_{p,q}^{\alpha-1}$

Lemma 3 (see [15]). *For $\alpha, \beta \ge 0$ and $0 < q < p \le 1$, (p,q)-integral and (p,q)-difference operators have the following properties:*

(i) $I_{p,q}^{\alpha}[I_{p,q}^{\beta} f(x)] = I_{p,q}^{\beta}[I_{p,q}^{\alpha} f(x)] = I_{p,q}^{\alpha+\beta} f(x)$

(ii) $D_{p,q} I_{p,q} f(x) = f(x)$ *and* $I_{p,q} D_{p,q} f(x) = f(x) - f(0)$

Lemma 4. *Assume $h \ge 0$, $A = 1 - \int_0^1 h(t) t^{\alpha-2} d_{p,q}t > 0$, and $\alpha \in (2,3)$. If $g \in C[0,1]$, then the following boundary value problem,*

$$\begin{cases} D_{p,q}^{\alpha} x(t) + g(t) = 0, & t \in (0,1), \\ x(0) = D_{p,q} x(0) = 0, & D_{p,q} x(1) = \int_0^1 h(t) D_{p,q} x(t) d_{p,q}t, \end{cases} \tag{12}$$

has a unique solution

$$x(t) = \int_0^1 G(t,qs) g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s, \tag{13}$$

where

$$\begin{aligned} G(t,qs) &= G_0(t,qs) + \frac{t^{\alpha-1}}{A} \int_0^1 h(t) G_1(t,qs) d_{p,q}t, \\ G_0(t,qs) &= \frac{1}{p^{(\alpha-1/2)} \Gamma_{p,q}(\alpha)} \begin{cases} \dfrac{1}{p^{2\alpha-2/\alpha-2}} t^{\alpha-1} (1-qs)^{(\alpha-2)} - (t-qs)^{(\alpha-1)}, & 0 \le s \le t \le 1, \\ \dfrac{1}{p^{2\alpha-2/\alpha-2}} t^{\alpha-1} (1-qs)^{(\alpha-2)}, & 0 \le t \le s \le 1, \end{cases} \\ G_1(t,qs) &= \frac{1}{p^{(\alpha-1/2)} \Gamma_{p,q}(\alpha)} \begin{cases} \dfrac{1}{p^{2\alpha-2/\alpha-2}} t^{\alpha-2} (1-qs)^{(\alpha-2)} - (t-qs)^{(\alpha-2)}, & 0 \le s \le t \le 1; \\ \dfrac{1}{p^{2\alpha-2/\alpha-2}} t^{\alpha-2} (1-qs)^{(\alpha-2)}, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{14}$$

Proof. According to $D_{p,q}^{\alpha} x(t) = -g(t)$, we have

$$x(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} - I_{p,q}^{\alpha} g(t). \tag{15}$$

From $x(0) = D_{p,q}x(0) = 0$, one can easily obtain $C_2 = C_3 = 0$. Hence,

$$\begin{aligned} x(t) &= C_1 t^{\alpha-1} - I_{p,q}^{\alpha} g(t), \\ D_{p,q}x(1) &= C_1 \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha-1)} - \frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &= \int_0^1 h(t) D_{p,q}x(t) d_{p,q}t \\ &= \int_0^1 h(t)\left[C_1 \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha-1)} t^{\alpha-2}\right. \\ &\quad \left. - \frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha-1)} \int_0^t (t-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s\right] d_{p,q}t. \end{aligned} \tag{16}$$

Based on the hypothesis in Lemma 4, we can deduce that

$$C_1 = \frac{1}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s - \frac{1}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 h(t) d_{p,q}t \int_0^t (t-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s. \tag{17}$$

Thus,

$$\begin{aligned} x(t) &= C_1 t^{\alpha-1} - I_{p,q}^{\alpha} g(t) \\ &= \frac{1}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 t(1-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &\quad - \frac{1}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 h(t) d_{p,q}t \int_0^t t^{\alpha-1}(t-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &\quad - \frac{1}{p^{(\alpha/2)}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\ &= \frac{1}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 t^{\alpha-2}(1-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &\quad - \frac{1}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 h(t) d_{p,q}t. \\ &\quad - \frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^t \frac{1}{p^{2\alpha-2/\alpha-2}}(t-qs)^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s + \frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 \frac{t^{\alpha-1}}{p^{2\alpha-2/\alpha-2}}(1-qs)^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &\quad - \frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 \frac{t^{\alpha-1}}{p^{2\alpha-2/\alpha-2}}(1-qs)^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &= \int_0^1 G_0\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s + \frac{t^{\alpha-1}}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 h(t) t^{\alpha-2} d_{p,q}t \int_0^1 \frac{1}{p^{2\alpha-2/\alpha-2}}(1-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\ &\quad - \frac{t^{\alpha-1}}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 h(t) d_{p,q}t \int_0^t \frac{1}{p^{2\alpha-2/\alpha-2}}(t-qs)^{(\alpha-2)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &= \frac{t^{\alpha-1}}{Ap^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)} \int_0^1 \left[\int_0^1 \frac{1}{p^{2\alpha-2/\alpha-2}} h(t) t^{\alpha-2}(1-qs)^{(\alpha-2)} d_{p,q}t - \int_s^1 \frac{1}{p^{2\alpha-2/\alpha-2}} h(t)(t-qs)^{(\alpha-2)} d_{p,q}t\right] g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &= \int_0^1 G_0(t,qs) g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s + \frac{t^{\alpha-1}}{A} \int_0^1 \int_0^1 G_1(t,qs) g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s \\ &= \int_0^1 G(t,qs) g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s. \end{aligned} \tag{18}$$

This completes the proof. □

Lemma 5. *The functions $G_i\,(i=0,1)$ have the following properties:*

(1) $G_i\,(t,qs)\geq 0$, for $t,s\in[0,1]$

(2) $t^{\alpha-1}G_0\,(1,qs)\leq G_0\,(t,qs)\leq G_0\,(1,qs)$, for $t,s\in[0,1]$

Proof

(1) On the one hand, for $0\leq s\leq t\leq 1$, we know

$$G_0\,(t,qs)=\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[\frac{1}{p^{2\alpha-2/\alpha-2}}t^{\alpha-1}(1-qs)^{(\alpha-2)}-(t-qs)^{(\alpha-1)}\right]. \tag{19}$$

Thus, for $t\neq 0$, it is easy to see that

$$\begin{aligned}G_0\,(t,qs)&=\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[\frac{1}{p^{2\alpha-2/\alpha-2}}t^{\alpha-1}(1-qs)^{(\alpha-2)}-(t-qs)^{(\alpha-1)}\right]\\&=\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[\frac{1}{p^{2\alpha-2/\alpha-2}}t^{\alpha-1}(1-qs)^{(\alpha-2)}-t^{\alpha-1}\left(1-q\frac{s}{t}\right)^{(\alpha-1)}\right]\\&\geq\frac{t^{\alpha-1}}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}\right]\\&\geq 0.\end{aligned} \tag{20}$$

Similarly, for $0\leq s\leq t\leq 1$, we know

$$G_1\,(t,qs)=\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[\frac{1}{p^{2\alpha-2/\alpha-2}}t^{\alpha-2}(1-qs)^{(\alpha-2)}-(t-qs)^{(\alpha-2)}\right]. \tag{21}$$

Thus, for $t\neq 0$, it is also easy to see that

$$\begin{aligned}G_1\,(t,qs)&=\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[\frac{1}{p^{2\alpha-2/\alpha-2}}t^{\alpha-2}(1-qs)^{(\alpha-2)}-(t-qs)^{(\alpha-2)}\right]\\&=\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[\frac{1}{p^{2\alpha-2/\alpha-2}}t^{\alpha-2}(1-qs)^{(\alpha-2)}-t^{\alpha-2}\left(1-q\frac{s}{t}\right)^{(\alpha-2)}\right]\\&\geq\frac{t^{\alpha-2}}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-2)}\right]\\&=0.\end{aligned} \tag{22}$$

On the other hand, for $0 \le t \le s \le 1$, it is easy to see that, from Lemma 4, the conclusion is obviously established. Therefore, $G_i(t, qs) \ge 0$, for $t, s \in [0, 1]$.

(2) Firstly, for $0 \le s \le t \le 1$, one can easily obtain that

$$\begin{aligned} {}^{t}D_{p,q}G_0(t,qs) &= {}_{t}D_{p,q}\left\{\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[\frac{1}{p^{2\alpha-2/\alpha-2}}t^{\alpha-1}(1-qs)^{(\alpha-2)}-(t-qs)^{(\alpha-1)}\right]\right\} \\ &= \left\{\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[\frac{1}{p^{2\alpha-2/\alpha-2}}[\alpha-1]t^{\alpha-2}(1-qs)^{(\alpha-2)}-[\alpha-1](t-qs)^{(\alpha-2)}\right]\right\} \\ &= \left\{\frac{1}{p^{(\alpha-1/2)}\Gamma_{p,q}(\alpha)}\left[[\alpha-1]t^{\alpha-2}(1-qs)^{(\alpha-2)}-[\alpha-1]t^{\alpha-2}\left(1-q\frac{s}{t}\right)^{(\alpha-2)}\right]\right\} \\ &\ge 0. \end{aligned} \tag{23}$$

For $0 \le t \le s \le 1$ and $2 < \alpha < 3$, we know that $G(t, qs) = t^{\alpha-1}(1-qs)^{(\alpha-2)}$ is obviously an increasing function with respect to t.

Therefore, for $t, s \in [0, 1]$, $G_0(t, qs)$ is an increasing function with respect to t. Then, $G_0(t, qs) \le G_0(1, qs)$.

Secondly, for $0 \le s \le t \le 1$, we have

$$\begin{aligned} \frac{G_0(t,qs)}{G_0(1,qs)} &= \frac{1/p^{2\alpha-2/\alpha-2}t^{\alpha-1}(1-qs)^{(\alpha-2)}-(t-qs)^{(\alpha-1)}}{1/p^{2\alpha-2/\alpha-2}(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}} \\ &= \frac{t^{\alpha-1}\left[1/p^{2\alpha-2/\alpha-2}(1-qs)^{(\alpha-2)}-(1-q(s/t))^{(\alpha-1)}\right]}{1/p^{2\alpha-2/\alpha-2}(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}} \\ &\ge \frac{t^{\alpha-1}\left[1/p^{2\alpha-2/\alpha-2}(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}\right]}{1/p^{2\alpha-2/\alpha-2}(1-qs)^{(\alpha-2)}-(1-qs)^{(\alpha-1)}} \\ &= t^{\alpha-1}. \end{aligned} \tag{24}$$

For $0 \le t \le s \le 1$, we have

$$\frac{G_0(t,qs)}{G_0(1,qs)} = t^{\alpha-1}. \tag{25}$$

Therefore, $t^{\alpha-1}G_0(1, qs) \le G_0(t, qs)$, for $t, s \in [0, 1]$. □

Lemma 6. *From Lemma 5, the following conclusions are established:*

$$\begin{aligned} &t^{\alpha-1}\theta_1(qs) \le G(t,qs) \le \theta_1(qs) \text{ for } t,s \in [0,1], \\ &G(t,qs) \le t^{\alpha-1}\theta_2(qs) \text{ for } t,s \in [0,1], \end{aligned} \tag{26}$$

where

$$\begin{aligned} \theta_1(s) &= G_0(1,s) + \frac{1}{A}\int_0^1 h(t)G_1(t,s)d_{p,q}s, \\ \theta_2(s) &= \frac{1}{\Gamma_{p,q}(\alpha)}\left(1-s^{(\alpha-2)}\right) + \frac{1}{A}\int_0^1 h(t)G_1(t,s)d_{p,q}s. \end{aligned} \tag{27}$$

Lemma 7 (see [37]). *Let Ω be a bounded open set in a Banach space E, and $T: \Omega \longrightarrow E$ is a continuous compact operator. If there exists $x_0 \in E\setminus\{0\}$ such that*

$$x - Tx \ne \mu x_0, \quad \forall x \in \partial\Omega, \mu \ge 0, \tag{28}$$

then the topological degree $\deg(I - T, \Omega, 0) = 0$.

Lemma 8 (see [37]). *Let Ω be a bounded open set in a Banach space E with $0 \in \Omega$, and $T: \Omega \longrightarrow E$ is a continuous compact operator. If*

$$Tx \ne \mu x, \quad \forall x \in \partial\Omega, \mu \ge 1, \tag{29}$$

then the topological degree $\deg(I - T, \Omega, 0) = 1$.

Let $E := C[0, 1]$, $\|x\| := \max_{t\in[0,1]}|x(t)|$, *and* $P := \{x \in E: x(t) \ge t^{\alpha-1}\|x\|, \forall t \in [0, 1]\}$. *Then,* $(E, \|\cdot\|)$ *is a real Banach space and P is a cone on E. From Lemma 4, we can define operator $T: E \longrightarrow E$ as follows:*

$$(Tx)(t) := \int_0^1 G(t,qs)f\left(\frac{s}{p^{\alpha-2}}, x\left(\frac{s}{p^{\alpha-2}}\right)\right)d_{p,q}s, \quad x \in E, \tag{30}$$

where G is determined in Lemma 4. Obviously, T is a completely continuous operator.

In addition, from Lemma 4, we can obtain that the solution of BVP equation (12) is equivalent to

$$x(t) = \lambda \int_0^1 G(t, qs) g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s, \quad t \in [0,1]. \tag{31}$$

For our purposes, we need to define the operator L by

$$(Lx)(t) = \int_0^1 G(t, qs) x\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s, \quad t \in [0,1], x \in E. \tag{32}$$

It is easy to prove that $L\colon E \longrightarrow E$ is a linear completely continuous operator and $L(P) \subset P$. Obviously, we know that L has a spectral radius, denoted by $r(L)$, that is not equal to 0. From Krein–Rutman theorem, we know that L has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(L))^{-1}$, i.e., $\varphi_1 = \lambda_1 L\varphi_1$.

3. Main Results

In this section, we shall establish the existence and multiplicity results of BVP equation (1), which is based on the topological degree theory. For convenience, let λ_1 be the first eigenvalue of the following eigenvalue problem:

$$\begin{cases} D_{p,q}^{\alpha} x(t) + \lambda x(t) = 0, & t \in (0,1), \\ x(0) = D_{p,q}x(0) = 0, & D_{p,q}x(1) = \int_0^1 h(t) D_{p,q}x(t) d_{p,q}t. \end{cases} \tag{33}$$

Now, let us list the following assumptions satisfied throughout the paper:

($H1$) $\liminf_{x\longrightarrow 0} f(t,x)/x > \lambda_1$ uniformly with respect to $t \in [0,1]$.

($H2$) $\limsup_{x\longrightarrow \infty} f(t,x)/x < \lambda_1$ uniformly with respect to $t \in [0,1]$.

($H3$) $\limsup_{x\longrightarrow 0} f(t,x)/x < \lambda_1$ uniformly with respect to $t \in [0,1]$.

($H4$) $\liminf_{x\longrightarrow \infty} f(t,x)/x > \lambda_1$ uniformly with respect to $t \in [0,1]$.

($H5$) There exist $r^* > 0$ and a continuous function ϕ_{r^*} such that

$$\begin{gathered} f(t,x) \geq \phi_{r^*}(t), \forall t \in [0,1], x \in \left[t^{\alpha-1} r^*, r^*\right], \\ \max_{t\in[0,1]} \int_0^1 t^{\alpha-1} G(1, qs) \phi_{r^*}\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s > r^*. \end{gathered} \tag{34}$$

($H6$) There exist $r_* > 0$ and a continuous function ψ_{r_*} such that

$$\begin{gathered} f(t,x) \leq \psi_{r_*}(t), \quad \forall t \in [0,1], \\ x \in [0, r_*], \\ \int_0^1 G(1, qs) \psi_{r_*}\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s < r_*. \end{gathered} \tag{35}$$

Now, we are in a position to give our main results.

Theorem 1. *Under assumptions (H1) and (H2), BVP equation (1) admits at least one positive solution.*

Proof. First, assumption ($H1$) implies that there exists $r > 0$ such that

$$f(t,x) > \lambda_1 x, \quad \forall x \in [0,r], t \in [0,1]. \tag{36}$$

We claim that, for $\mu \geq 0$,

$$x(t) - Tx(t) \neq \mu\varphi_1(t), \forall x \in \partial B_r \cap P, t \in [0,1]. \tag{37}$$

Suppose, on the contrary, that there exist $x_1 \in \partial B_r \cap P, \mu_1 > 0$ such that

$$x_1(t) - Tx_1(t) = \mu_1\varphi_1(t), \quad t \in [0,1]. \tag{38}$$

Without loss of generality, suppose $\mu_1 > 0$. Then, $x_1(t) \geq \mu_1\varphi_1(t)$, for $t \in [0,1]$.

Let

$$\mu^* = \sup\{\mu\colon x_1(t) \geq \mu_1\varphi_1(t), t \in [0,1]\}. \tag{39}$$

Obviously, $0 < \mu_1 \leq \mu^* < +\infty$ and $x_1(t) \geq \mu^*\varphi_1(t)$, for $t \in [0,1]$.

Thus,

$$\begin{aligned} x_1(t) &= Tx_1(t) + \mu_1\varphi_1(t) \\ &= \int_0^1 G(t, qs) f\left(\frac{s}{p^{\alpha-2}}, x_1\left(\frac{s}{p^{\alpha-2}}\right)\right) d_{p,q}s + \mu_1\varphi_1(t) \\ &\geq \lambda_1 \int_0^1 G(t, qs) x_1\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s + \mu_1\varphi_1(t) \\ &\geq \lambda_1\mu^* \int_0^1 G(t, qs) \varphi_1\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s + \mu_1\varphi_1(t) \\ &= (\mu^* + \mu_1)\varphi_1(t). \end{aligned} \tag{40}$$

It is a contradiction with the definition of μ^*. According to Lemma 7, one obtains

$$\deg(T, B_r \cap P, P) = 0. \tag{41}$$

On the contrary, we can choose $\varepsilon_0 > 0$ such that $0 < (\lambda_1 - \varepsilon_0)\|L\| < 1$. Then, from ($H2$), there exists $R > 0$ such that

$$f(t,x) \leq (\lambda_1 - \varepsilon_0)x, \quad x \geq R, t \in [0,1]. \tag{42}$$

Let $m = \max_{(t,x)\in[0,1]\times[0,R]} f(t,x)$. Thus, one can easily find that

$$f(t,x) \leq (\lambda_1 - \varepsilon_0)x + m, \quad \forall x \geq 0, t \in [0,1]. \tag{43}$$

Choose $R_0 > \max\{R, r, m\int_0^1 \theta_1(qs) d_{p,q}s / 1 - (\lambda_1 - \varepsilon_0)\|L\|\}$. We claim that, for $\mu \geq 1$,

$$Tx(t) \neq \mu x(t), \quad \forall x \in \partial B_{R_0} \cap P, t \in [0,1]. \tag{44}$$

Suppose, on the contrary, that there exist $x_2 \in \partial B_{R_0} \cap P$ and $\mu_2 \geq 1$ such that

$$Tx_2(t) = \mu_2 x_2(t), \quad t \in [0,1]. \tag{45}$$

Hence,

$$\begin{aligned} x_2(t) \le \mu_2 x(t) = Tx_2(t) &= \int_0^1 G(t,qs) f\left(\frac{s}{p^{\alpha-2}}, x_2\left(\frac{s}{p^{\alpha-2}}\right)\right) d_{p,q}s \\ &\le \int_0^1 G(t,qs)\left[(\lambda_1-\varepsilon_0)x_2\left(\frac{s}{p^{\alpha-2}}\right)+m\right] d_{p,q}, \end{aligned} \tag{46}$$

noticing that $0 < (\lambda_1 - \varepsilon_0)\|L\| < 1$. We know that the inverse operator of $I - (\lambda_1 - \varepsilon_0)L$ exists, and

$$[I-(\lambda_1-\varepsilon_0)L]^{-1} = \sum_{n=0}^{\infty} (\lambda_1-\varepsilon_0)^n L^n, \tag{47}$$

which shows that $[I - (\lambda_1 - \varepsilon_0)L]^{-1}(P) \subseteq P$.

Thus,

$$x_2(t) \le [I-(\lambda_1-\varepsilon_0)L]^{-1} m \int_0^1 G(t,qs) d_{p,q}s. \tag{48}$$

In addition, by $\|[I - (\lambda_1 - \varepsilon_0)L]^{-1}\| \le 1/1 - (\lambda_1 - \varepsilon_0)\|L\|$ and Lemma 6, one can obtain

$$\begin{aligned} R_0 = \|x_2(t)\| &\le \left\|[I-(\lambda_1-\varepsilon_0)L]^{-1}\right\| m \int_0^1 \theta_1(qs) d_{p,q}s \\ &\le \frac{m\int_0^1 \theta_1(qs) d_{p,q}s}{1-(\lambda_1-\varepsilon_0)\|L\|} \\ &< R_0, \end{aligned} \tag{49}$$

which is a contradiction. By Lemma 8, we obtain

$$\deg\left(T, B_{R_0} \cap P, P\right) = 1. \tag{50}$$

Therefore,

$$\deg\left(T, B_{R_0} \backslash \overline{B}_r \cap P, P\right) = \deg\left(T, B_{R_0} \cap P, P\right) - \deg\left(T, B_r \cap P, P\right) = 1 - 0 = 1. \tag{51}$$

which means that BVP equation (1) has at least one positive solution. □

Theorem 2. *Under assumptions (H3) and (H4), BVP equation (1) admits at least one positive solution.*

Proof. On the one hand, assumption $(H3)$ implies that there exist $\varepsilon \in (0, \lambda_1)$ and $r_1 > 0$ such that

$$f(t,x) < (\lambda_1 - \varepsilon), |x| < r_1. \tag{52}$$

We claim that, for $\mu \in [0,1]$,

$$x(t) \ne \mu T x(t), \quad \forall x \in \partial B_{r_1} \cap P, t \in [0,1]. \tag{53}$$

Suppose, on the contrary, that there exist $x_1 \in \partial B_{r_1} \cap P$ and $\mu \in [0,1]$ such that

$$x_1(t) = \mu_1 T x_1(t), \quad t \in [0,1]. \tag{54}$$

Consequently, we have

$$x_1(t) = \mu_1 T x_1(t) < (\lambda_1 - \varepsilon)\int_0^1 G(t,qs) x_1\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s = (\lambda_1-\varepsilon)(Lx_1)(t). \tag{55}$$

The nth iteration of this inequality shows that

$$x_1(t) < (\lambda_1-\varepsilon)^n (L^n x_1)(t) (n = 1, 2, \ldots,). \tag{56}$$

Then,

$$\|x_1\| < (\lambda_1-\varepsilon)^n \|L^n\| \|x_1\|, \text{i.e.}, 1 < (\lambda_1-\varepsilon)^n \|L^n\|. \tag{57}$$

It means that

$$1 \le (\lambda_1-\varepsilon) \lim_{n\longrightarrow\infty} \sqrt[n]{\|L^n\|} = (\lambda_1-\varepsilon) r(L) = \frac{(\lambda_1-\varepsilon)}{\lambda_1} < 1, \tag{58}$$

which is a contradiction. It follows from Lemma 8 that

$$\deg\left(T, B_{r_1} \cap P, P\right) = 1. \tag{59}$$

On the other hand, let

$$L_n x(t) = \int_{1/n}^1 G(t,qs) x\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s, \quad t \in [0,1], \tag{60}$$

where $n > 1$. It is easy to see that L_n: $P \longrightarrow P$ is completely continuous operator and spectral radius $r(L_n) > 0$, denoted by $\lambda_n = r^{-1}(L_n)$. We know $\lim_{n\longrightarrow+\infty}\lambda_n = \lambda_1$.

It follows that there exist N_0, ε_0 such that $\lambda_{N_0} < \lambda_1 + \varepsilon_0$, namely, $r(L_{N_0}) > 1/r^{-1}(L) + \varepsilon_0$. From the Krein–Rutman theorem, there exists a $\varphi_{N_0}(t) \in E\backslash\{\theta\}$ such that

$$\varphi_{N_0}(t) = r^{-1}\left(L_{N_0}\right)\int_{1/N_0}^{1} G(t,qs)\varphi_{N_0}\left(\frac{s}{p^{\alpha-2}}\right)d_{p,q}s, \quad t \in [0,1]. \tag{61}$$

By the proof of Lemma 5, we know

$$G(t,qs) \geq t^{\alpha-1}G(\tau,qs). \tag{62}$$

Hence,

$$\varphi_{N_0}(t) \geq r^{-1}\left(L_{N_0}\right)\int_{1/N_0}^{1} t^{\alpha-1}G(\tau,qs)\varphi_{N_0}\left(\frac{s}{p^{\alpha-2}}\right)d_{p,q}s = t^{\alpha-1}\varphi_{N_0}(\tau), \quad \forall, t,\tau \in [0,1], \tag{63}$$

which means that $\varphi_{N_0}(t) \geq t^{\alpha-1}\|\varphi_{N_0}\|$, namely, $\varphi_{N_0}(t) \in P\backslash\{\theta\}$.

Thus, from $(H4)$, there exists $R_1 > 0$ such that

$$f(t,x) > (\lambda_1 + \varepsilon)x, \quad t \in [0,1], \forall, x \geq R_1. \tag{64}$$

Choose $\overline{R}_0 > \max\{r_1, N_0^{\alpha-1}R_1\}$. Thus,

$$x(t) \geq t^{\alpha-1}\|x\| = t^{\alpha-1}\overline{R}_0 > R_1, \quad t \in \left[\frac{1}{N_0}, 1\right], x \in \partial B_{\overline{R}_0} \cap P. \tag{65}$$

Now, we prove that, for $\mu \geq 0$,

$$x(t) - Tx(t) \neq \mu\varphi_{N_0}(t), \quad \forall x \in \partial B_{\overline{R}_0} \cap P, t \in [0,1]. \tag{66}$$

Similar to the proof of Theorem 1, this conclusion is clearly established. So, according to Lemma 7,

$$\deg\left(T, B_{\overline{R}_0} \cap P, P\right) = 0. \tag{67}$$

Therefore,

$$\deg\left(T, B_{\overline{R}_0}\backslash\overline{B}_{r_1} \cap P, P\right) = \deg\left(T, B_{\overline{R}_0} \cap P, P\right) - \deg\left(T, B_{r_1} \cap P, P\right) = 0 - 1 = -1, \tag{68}$$

which means that BVP equation (1) has at least one positive solution.

Up to now, some existence results of BVP equation (1) have been obtained by using the topological degree theory and Krein–Rutman theorem. In the following, the multiple solutions will be considered for BVP equation (1). □

Theorem 3. *Suppose that (H2), (H3), and (H5) are satisfied. Then, BVP equation (1) has at least two positive solutions.*

Proof. By $(H5)$, we know

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,qs)f\left(\frac{s}{p^{\alpha-2}}, x\left(\frac{s}{p^{\alpha-2}}\right)\right)d_{p,q}s \\ &\geq \int_0^1 t^{\alpha-1}G(1,qs)\phi_{r^*}\left(\frac{s}{p^{\alpha-2}}\right)d_{p,q}s, \quad x \in \partial B_{r^*} \cap P.\end{aligned} \tag{69}$$

Consequently,

$$\|Tx\| \geq \max_{t\in[0,1]}\int_0^1 t^{\alpha-2}G(1,qs)\phi_{r^*}\left(\frac{s}{p^{\alpha-2}}\right)d_{p,q}s > r^* = \|x\|. \tag{70}$$

So, similar to the previous proof of Theorem 1, it is easy to know, for $\mu > 0$,

$$x(t) - Tx(t) \neq \mu\varphi_1(t), \quad \forall t \in [0,1], x \in \partial B_{r^*} \cap P. \tag{71}$$

By Lemma 7, one can immediately obtain that

$$\deg(T, B_r \cap P, P) = 0. \tag{72}$$

By the proof of Theorems 1 and 2, we know that there exist $r_1 \in (0, r^*)$ and $R_1 \geq \max\{r^*, R_0\}$ such that

$$\begin{aligned}\deg\left(T, B_{r_1} \cap P, P\right) &= 1, \\ \deg\left(T, B_{R_1} \cap P, P\right) &= 1.\end{aligned} \tag{73}$$

Consequently,

$$\begin{aligned}\deg\left(T, \left(B_{r^*}\backslash\overline{B}_{r_1}\right) \cap P, P\right) &= \deg(T, B_{r^*} \cap P, P) - \deg\left(T, B_{r_1} \cap P, P\right) = 0 - 1 = -1, \\ \deg\left(T, \left(B_{R_1}\backslash\overline{B}_{r^*}\right) \cap P, P\right) &= \deg\left(T, B_{R_1} \cap P, P\right) - \deg(T, B_{r^*} \cap P, P) = 1 - 0 = 1,\end{aligned} \tag{74}$$

which means that BVP equation (1) has at least two positive solutions. □

Theorem 4. *Suppose that (H1), (H4), and (H6) are satisfied. Then, BVP equation (1) has at least two positive solutions.*

Proof. By $(H6)$, we know

$$(Tx)(t) = \int_0^1 G(t,qs) f\left(\frac{s}{p^{\alpha-2}}, x\left(\frac{s}{p^{\alpha-2}}\right)\right) d_{p,q}s$$
$$\leq \int_0^1 G(1,qs)\psi_{r_*}\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s, \quad x \in \partial B_{r_*} \cap P. \tag{75}$$

Therefore,

$$\|Tx\| \leq \int_0^1 G(1,qs)\psi_{r_*}\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s < r_* = \|x\|. \tag{76}$$

So, similar to the previous proof of Theorem 1, it is easy to know, for $\mu \geq 1$,

$$Tx(t) \neq \mu x(t), \quad \forall x \in \partial B_{r_*} \cap P, t \in [0,1]. \tag{77}$$

By Lemma 8, one can immediately obtain that

$$\deg\left(T, B_{r_*} \cap P, P\right) = 1. \tag{78}$$

By the proof of Theorems 1 and 2, we know that there exist $\overline{r}_1 \in (0, r_*)$ and $\overline{R}_1 \geq \max\{r_*, \overline{R}_0\}$ such that

$$\begin{aligned} \deg\left(T, B_{\overline{r}_1} \cap P, P\right) &= 0, \\ \deg\left(T, B_{\overline{R}_1} \cap P, P\right) &= 0. \end{aligned} \tag{79}$$

Consequently,

$$\begin{aligned} \deg\left(T, \left(B_{r_*} \backslash \overline{B}_{\overline{r}_1}\right) \cap P, P\right) &= \deg\left(T, B_{r_*} \cap P, P\right) - \deg\left(T, B_{\overline{r}_1} \cap P, P\right) = 1 - 0 = 1, \\ \deg\left(T, \left(B_{\overline{R}_1} \backslash \overline{B}_{r_*}\right) \cap P, P\right) &= \deg\left(T, B_{\overline{R}_1} \cap P, P\right) - \deg\left(T, B_{r_*} \cap P, P\right) = 0 - 1 = -1, \end{aligned} \tag{80}$$

which means that BVP equation (1) has at least two positive solutions. □

4. Examples

Example 1. Consider the following boundary value problem:

$$\begin{cases} \left(D^{5/2}_{(1,1/2)} x\right)(t) + \dfrac{28x^2 t}{x^2+1} = 0, \quad 0 < t < 1, \\ x(0) = D_{(1,1/2)} x(0) = 0, D_{(1,1/2)} x(1) = \displaystyle\int_0^1 t^{(1/2)} D_{(1,1/2)} x(t) d_{1,1/2} t. \end{cases} \tag{81}$$

Conclusion: BVP equation (81) has at least two positive solutions.

Proof. BVP equation (81) can be regarded as a BVP of the form of equation (1), where

$$f(t,x) = \frac{28x^2 t}{x^2+1}, \tag{82}$$

$p = 1, q = 1/2, \alpha = 5/2$, and $h(t) = t^{1/2}$. Choose $\phi_{r^*}(t) = t^2 + 2$ and $r^* = 1/5$. Obviously, $f\colon [0,1] \times [1/5t^{3/2}, 1/5] \longrightarrow [0,+\infty)$ is continuous and $f(t,x) \geq \phi_{r^*}(t)$ for $(t,x) \in [0,1] \times [1/5t^{3/2}, 1/5]$.

Consequently,

$$\begin{aligned} \lim_{x \longrightarrow 0} \sup_{t \in [0,1]} \frac{f(t,x)}{x} &= \frac{28x^2 t}{x\left(x^2+1\right)} = 0 < \lambda_1, \\ \lim_{x \longrightarrow +\infty} \sup_{t \in [0,1]} \frac{f(t,x)}{x} &= \frac{28x^2 t}{x\left(x^2+1\right)} = 0 < \lambda_1. \end{aligned} \tag{83}$$

From the definition of function G, one can obtain that

$$G\left(t, \frac{s}{2}\right) = G_0\left(t, \frac{s}{2}\right) + \frac{t^{\alpha-1}}{A} \int_0^1 h(t) G_1\left(t, \frac{s}{2}\right) d_{1,1/2} t, \tag{84}$$

where

$$G_0\left(t,\frac{s}{2}\right)=\frac{1}{1^{(\alpha/2)}\Gamma_{(1,1/2)}(5/2)}\begin{cases} t^{3/2}\left(1-\frac{s}{2}\right)^{(1/2)}-\left(t-\frac{s}{2}\right)^{(3/2)}, & 0\le s\le t\le 1,\\ t^{3/2}\left(1-\frac{s}{2}\right)^{(1/2)}, & 0\le t\le s\le 1,\end{cases}$$
$$G_1\left(t,\frac{s}{2}\right)=\frac{1}{1^{(\alpha/2)}\Gamma_{(1,1/2)}(5/2)}\begin{cases} t^{1/2}\left(1-\frac{s}{2}\right)^{(1/2)}-\left(t-\frac{s}{2}\right)^{(1/2)}, & 0\le s\le t\le 1,\\ t^{1/2}\left(1-\frac{s}{2}\right)^{(1/2)}, & 0\le t\le s\le 1.\end{cases} \tag{85}$$

Consequently,

$$\max_{t\in[0,1]}\int_0^1 t^{\alpha-1}G\left(1,\frac{s}{2}\right)\phi_{r^*}(s)d_{p,q}s\approx 0.235>\frac{1}{5}=r^*. \tag{86}$$

Therefore, by Theorem 3, BVP equation (81) has at least two positive solutions. □

Example 2. Consider the following boundary value problem:

$$\begin{cases}\left(D^{5/2}_{(1,(1/2))}x\right)(t)+\left(3x^{3/2}+2t\right)=0, & 0<t<1,\\ x(0)=D_{(1,(1/2))}x(0)=0, \quad D_{(1,(1/2))}x(1)=\int_0^1 t^{(1/2)}D_{(1,(1/2))}x(t)d_{1,1/2}t.\end{cases} \tag{87}$$

Conclusion: BVP equation (87) has at least two positive solutions.

Proof. BVP equation (87) can be regarded as a BVP of the form of equation (1), where

$$f(t,x)=3x^{3/2}+2t, \tag{88}$$

$p=1$, $q=(1/2)$, $\alpha=(5/2)$, and $h(t)=t^{(1/2)}$. Choose $\psi_{r_*}(t)=2t+9$, $r_*=3\sqrt{3}$. Obviously, $f\colon[0,1]\times[0,3\sqrt{3}]\longrightarrow[0,+\infty)$ is continuous and $f(t,x)\le\psi_{r_*}(t)$ for $(t,x)\in[0,1]\times[0,3\sqrt{3}]$.

Consequently,

$$\lim_{x\longrightarrow 0}\inf_{t\in[0,1]}\frac{f(t,x)}{x}=\frac{3x^{3/2}+2t}{x}=+\infty>\lambda_1,$$
$$\lim_{x\longrightarrow +\infty}\inf_{t\in[0,1]}\frac{f(t,x)}{x}=\frac{3x^{3/2}+2t}{x}=+\infty>\lambda_1. \tag{89}$$

From the definition of function G, one can obtain that

$$G\left(t,\frac{s}{2}\right)=G_0\left(t,\frac{s}{2}\right)+\frac{t^{\alpha-1}}{A}\int_0^1 h(t)G_1\left(t,\frac{s}{2}\right)d_{1,(1/2)}t, \tag{90}$$

where

$$G_0\left(t,\frac{s}{2}\right)=\frac{1}{1^{(\alpha/2)}\Gamma_{(1,(1/2))}(5/2)}\begin{cases} t^{1/2}\left(1-\frac{s}{2}\right)^{(1/2)}-\left(t-\frac{s}{2}\right)^{(1/2)}, & 0\le s\le t\le 1;\\ t^{3/2}\left(1-\frac{s}{2}\right)^{(1/2)}, & 0\le t\le s\le 1,\end{cases}$$
$$G_1\left(t,\frac{s}{2}\right)=\frac{1}{1^{(\alpha/2)}\Gamma_{(1,(1/2))}(5/2)}\begin{cases} t^{1/2}\left(1-\frac{s}{2}\right)^{(1/2)}-\left(t-\frac{s}{2}\right)^{(1/2)}, & 0\le s\le t\le 1;\\ t^{1/2}\left(1-\frac{s}{2}\right)^{(1/2)}, & 0\le t\le s\le 1.\end{cases} \tag{91}$$

Thus,

$$\int_0^1 G\left(1,\frac{s}{2}\right)\psi_{r_*}(s)d_{p,q}s \approx 1.324 < 3\sqrt{3} = r_*. \tag{92}$$

Therefore, by Theorem 4, BVP equation (87) has at least two positive solutions. □

5. Conclusions

This paper is mainly concerned with a class of fractional (p,q)-difference equations with (p,q)-integral boundary conditions. We first give the definition of fractional (p,q)-difference operator and fractional (p,q)-integral operator. Then, the existence and multiplicity of positive solutions for boundary value problems are obtained by using topological degree theory and Krein–Rutman theorem. Finally, two illustrative examples are given to show the practical usefulness of the analytical results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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