Research Article

Characterization of the Congestion Lemma on Layout Computation

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An embedding of a guest network GN into a host network HN is to find a suitable bijective function between the vertices of the guest and the host such that each link of GN is stretched to a path in HN . The layout measure is attained by counting the length of paths in HN corresponding to the links in GN and with a complexity of finding the best possible function overall graph embedding. This measure can be computed by summing the minimum congestions on each link of HN , called the congestion lemma. In the current study, we discuss and characterize the congestion lemma by considering the regularity and optimality of the guest network. The exact values of the layout are generally hard to find and were known for very restricted combinations of guest and host networks. In this series, we derive the correct layout measures of circulant networks by embedding them into the path- and cycle-of-complete graphs.

1. Introduction

Nowadays, there is an emerging demand for high-performance concurrent functional in different fields which can be successfully achieved through parallel processing techniques. The core of a parallel processing system is the interconnected network by which the system processors are connected. One of the important challenges in parallel processing techniques is how to allocate the subprocesses to the processors within the system in such a way that the total communication cost is minimized. This issue in parallel processing can be reduced to a graph embedding problem [1, 2]. For this purpose, the network topology is formulated as a simple graph, in which the vertex set denotes the system processors and the edge set denotes the links connecting them.

In this paper, the collection of vertices and edges of a simple graph network GN are, respectively, represented by $V(GN)$ and $E(GN)$. A graph embedding of a guest network GN into a host network HN is a kind of vertex and edge labeling denoted by a 1 – 1 and onto mapping $\mu: V(GN) \rightarrow V(HN)$ together with 1 – 1 mapping $R: E(GN) \rightarrow \mathcal{R}(HN)$ such that $R(e)$ is

a $\mu(x)$ to $\mu(y)$ path in HN , where $e = (x, y)$ and $\mathcal{R}(HN)$ contains the collection of routes or paths in HN [2, 3]. The congestion of an edge s of HN is measured by counting the routes in $\{R(e)\}_{e \in E(GN)}$ such that s is in the route $R(e)$ and denoted by $EC_\mu(s)$. In other words, $EC_\mu(s) = |\{e \in E(GN): s \in E(R(e))\}|$. The layout/wire length [4, 5] of GN by embedding μ in HN is defined as

$$L_\mu(GN, HN) = \sum_{e \in E(GN)} |E(R(e))| = \sum_{s \in E(HN)} EC_\mu(s). \quad (1)$$

Let D be any subset of $E(HN)$. If we represent $EC_\mu(D) = \sum_{e \in D} EC_\mu(e)$, then $L_\mu(GN, HN) = \sum_{i=1}^p EC_\mu(F_i)$, where $E(HN) = \{F_1, F_2, \dots, F_p\}$ is a partition. For $\lambda \geq 1$, construct a set based on the edges of HN such that each edge in HN is duplicated λ -times. Such a set is denoted by $E^\lambda(HN)$. Then,

$$L_\mu(GN, HN) = \frac{1}{\lambda} \sum_{s \in E(HN)} EC_\mu(s). \quad (2)$$

Furthermore, if $E^\lambda(HN) = \{D_1, D_2, \dots, D_m\}$, then $L_\mu(GN, HN) = (1/\lambda) \sum_{i=1}^m EC_\mu(D_i)$. The correct layout of GN by embedding in HN is measured by

$$L(GN, HN) = \min_{\mu} L_{\mu}(GN, HN). \quad (3)$$

The main objective of parallel computing is to execute embeddings with the correct layout, and we certainly fix the accompanying function R such that each edge e of GN is to a shortest $R(e)$ path under μ , see Figure 1. Apart from that, the important topological descriptor, Wiener index [6], which is used in the characterization of chemical compounds can be obtained through $L(K_q, MS)$, where K_q is the complete graph and MS is the considered molecular structure.

The minimum layout problem plays an important role in finding an optimal solution for very large-scale integration (VLSI) chips physical layout [2], minimizing time delay of simulations in parallel computer systems, computer aided designs, structural engineering, cloning and visual stimuli models, and parallel architecture [7, 8]. The computation of layout measure has been already studied in a variety of papers, see [9, 10] and the references cited therein, for more details. The present study continues the layout computation of circulant graphs into path- and cycle-of-complete graphs.

2. Congestion Lemma

The combinatorial isoperimetric problems have emerged with important applications in the fields of communication systems and computer and physical sciences related disciplines. Harper [11, 12] has discovered the primary significance of edge isoperimetric problem (EIP), and it has been categorized into two types as follows [13].

Definition 1 (EIP(1)). For a graph network GN , if $F \subseteq V(GN)$, then $\Theta_{GN}(F) = \{(v, w) \in E(GN) : v \in F \text{ and } w \notin F\}$. Given a positive integer k , $\Theta_{GN}(k) = \min_{F \subseteq V(GN), |F|=k} |\Theta_{GN}(F)|$. Then, EIP(1) finds $F \subseteq V(GN)$ and $|F| = k$ such that $\Theta_{GN}(k) = |\Theta_{GN}(F)|$.

Definition 2 (EIP(2)). For a graph network GN , if $F \subseteq V(GN)$, then $I_{GN}(F) = \{(v, w) \in E(GN) : v, w \in F\}$. Given a positive integer k , $I_{GN}(k) = \max_{F \subseteq V(GN), |F|=k} |I_{GN}(F)|$. Then, EIP(2) finds $F \subseteq V(GN)$ and $|F| = k$ such that $I_{GN}(k) = |I_{GN}(F)|$.

In such a case, F is identified as optimal set corresponding to the EIP.

Lemma 1 (see [11]). (i) For a graph network GN , $\Theta_{GN}(V(GN) - F) = \Theta_{GN}(F)$ for all $F \subseteq V(GN)$. (ii) If GN is an r -regular graph, $|\Theta_{GN}(F)| + 2|I_{GN}(F)| = r|F|$ and, for any positive integer k , $\Theta_{GN}(k) = rk - 2I_{GN}(k)$.

The minimum layout of the hypercube network by embedding in a grid structure is derived using congestion lemma [14]. The generalized version of the congestion lemma appeared in [15] and the modified version in [16]. Here, we present a more general result that exemplifies the regularity and the optimality on the guest network.

In what follows, let GN and HN be two given networks and $F \subseteq E(HN)$. Suppose the removal of F from HN splits the network into m components, namely, $HN_1, HN_2, \dots, HN_m$. A graph embedding μ of GN into HN is F -repulsive if, when we let $GN_j = GN[\mu^{-1}(HN_j)]$, $1 \leq j \leq m$, the following conditions hold:

- (i) If $e \in E(GN_j)$, $1 \leq j \leq m$, then $|E(R(e)) \cap F| = 0$
- (ii) If $e = (v, w)$, such that $v \in V(GN_j)$ and $w \in V(GN_k)$, for $j < k$, then $|E(R(e)) \cap F| = 1$

Lemma 2. Let $F \subseteq E(HN)$ and μ be an F -repulsive graph embedding of GN into HN . We have $EC_{\mu}(F) = |E(GN)| - \sum_{j=1}^m |E(GN_j)|$. Moreover, among all the graph embeddings of GN into HN , $EC_{\mu}(F)$ is minimum if and only if the value $\sum_{j=1}^m |E(GN_j)|$ is maximum among all partitions of $V(GN) = \cup_{j=1}^m V_j$ with $|V_j| = |V(GN_j)|$, $j = 1, 2, \dots, m$.

Proof. Let $X = \{(x, y) \in E(GN) : x \in V(GN_j), y \in V(GN_k) \text{ for } j < k\}$. Since any edge (x, y) in GN either belongs to one of GN_j or x in GN_j and y in GN_k for some j and k , we obtain

$$|E(GN)| = \sum_{j=1}^m |E(GN_j)| + |X|. \quad (4)$$

We now easily compute $EC_{\mu}(F)$ bearing the conditions of the F -repulsive embedding. By assumption (i), the contribution to $EC_{\mu}(F)$ from the edges of GN_j , $1 \leq j \leq m$, is zero. By assumption (ii), every edge (x, y) of X increases $EC_{\mu}(F)$ by 1. Thus, $EC_{\mu}(F) = |X| = |E(GN)| - \sum_{j=1}^m |E(GN_j)|$.

Assume that $EC_{\mu}(F)$ is minimum. Suppose we had a partition giving a larger value than $\sum_{j=1}^m |E(GN_j)|$, and we could define an embedding β using this partition such that $EC_{\beta}(F) < EC_{\mu}(F)$, a contradiction. Conversely, let $EC_{\mu}(F)$ be not minimum. Suppose there exists a graph embedding γ such that $EC_{\gamma}(F) < EC_{\mu}(F)$, and consequently, we can find a partition with a larger value than $\sum_{j=1}^m |E(GN_j)|$, which is not possible because of the F -repulsive embedding under μ . $\square$

Lemma 3. Let $F \subseteq E(HN)$ and μ be an F -repulsive graph embedding of GN into HN . Among all the graph embeddings of GN into HN , (a) $EC_{\mu}(F)$ is minimum if GN_j s are optimal with respect to EIP(1) and $EC_{\mu}(F) = (1/2) \sum_{j=1}^m \Theta_{GN}(m_j)$, $m_j = |V(GN_j)|$, and (b) when GN is an r -regular network, $EC_{\mu}(F)$ is minimum if GN_j s are optimal with respect to EIP(2) and $EC_{\mu}(F) = (r/2)|V(GN)| - \sum_{j=1}^m |E(GN_j)|$.

Proof. We assume that GN_j s are optimal sets with respect to EIP(1). Such a case results in $\sum_{j=1}^m \Theta_{GN}(m_j)$ is minimum. Hence, $EC_{\mu}(F) = (1/2) \sum_{j=1}^m \Theta_{GN}(m_j)$ is minimum [15]. By extending the idea to EIP(2), we can easily derive the case of r -regular network by applying the simple fact $2|E(GN)| = r|V(GN)|$. $\square$

It is interesting as well as crucial to note that all GN_j s are not optimal, yet imply that $EC_{\mu}(F)$ is minimum. Furthermore, when $m = 2$, the above lemma is reduced to the

FIGURE 1: The correct layout of the embedding circulant network into path by $\mu(x) = x$.

modified congestion lemma [16], as in Case (a), and the congestion lemma [14], as in Case (b).

3. Layout Computation

The purpose of the section is to derive the layout of circulant networks into a few graph structures generated from the path and the cycle. We begin with the basic results on circulant networks [2, 17, 18].

Definition 3 (see [2]). A circulant network, denoted by $CN(q; \pm D)$, $D \subseteq \{1, 2, \dots, \lfloor q/2 \rfloor\}$, $q \geq 3$, is constructed from $V(CN) = \{0, 1, \dots, q-1\}$ such that $E(CN) = \{(i, j): |j - i| \equiv d \pmod{q}, d \in \pm D\}$.

With the optimum fault tolerance and best routing functionality, the circulant network is considered an excellent network over the years on account of its applications in the areas of computer binary code designs and

telecommunication network systems. Particularly, circulant network is a natural generalized form of the double loop network, and in addition, a matrix representation generates the circulant if all its rows are periodic rotations of the first one. From the construction of circulant networks, one can easily see that $CN(q; \pm 1)$ and $CN(q; \pm \{1, 2, \dots, \lfloor q/2 \rfloor\})$ are, respectively, the cycle C_q and the complete graph K_q . In our study, we denote the cycle $CN(q; \pm 1)$ as a peripheral cycle. From the symmetry of circulant network, we have that $CN(q; \pm \{1, 2, \dots, l\})$, $1 \leq l < \lfloor q/2 \rfloor$, is a regular network of degree $2l$.

Lemma 4 (see [19]). A set of m consecutive vertices of $CN(q; \pm 1)$, $1 \leq m \leq q$, is an optimal set with respect to $EIP(2)$ in $G(q; \pm D)$, $D = \{1, 2, \dots, l\}$, $1 \leq l < \lfloor q/2 \rfloor$.

Lemma 5 (see [19]). For a circulant network $CN(q; \pm D)$, $D = \{1, 2, \dots, l\}$ and $1 \leq l < \lfloor q/2 \rfloor$, $1 \leq m \leq q$, we have

$$I_{CN}(m) = \begin{cases} m \frac{(m-1)}{2}; & m \leq l+1 \\ ml - \frac{l(l+1)}{2}; & l+1 < m \leq q-l \\ \frac{1}{2} \{ (q-m)^2 + (4l+1)m - (2l+1)q \}; & q-l < m \leq q \end{cases} \quad (5)$$

Definition 4. A path-of-complete graph is obtained by unifying a bone path $v_1, v_2, \dots, v_m$ and $m-1$ complete graphs $K_{q_1}, K_{q_2}, \dots, K_{q_{m-1}}$ such that the edge (v_i, v_{i+1}) , $1 \leq i \leq m-1$, of the bone path shares an edge of the complete graph K_{q_i} . We denote it by $PC(m; q_1, q_2, \dots, q_{m-1})$. In an analogous way, we can define a cycle-of-complete graph by combining a bone cycle of length m and $K_{q_1}, K_{q_2}, \dots, K_{q_m}$ complete graphs. This graph is denoted by $CC(m; q_1, q_2, \dots, q_m)$.

Clearly, the number of vertices in $PC(m; q_1, q_2, \dots, q_{m-1})$ and $CC(m; q_1, q_2, \dots, q_m)$ are $q_1 + q_2 + \dots + q_{m-1} - (m-2)$

and $q_1 + q_2 + \dots + q_m - m$, respectively. The different cases of path- and cycle-of-complete graphs are shown in Figure 2. In the literature, these structures are sometimes called necklace graphs and also sharing between graphs by vertices, see [20, 21].

In what follows, $q_0 = 0$ and $q_\alpha = q_{\alpha-m}$, $\alpha > m$.

Theorem 1. The minimum layout of circulant network $GN = CN(q; \pm \{1, 2, \dots, l\})$, $1 \leq l < \lfloor q/2 \rfloor$, into path-of-complete graph $HN = PC(m; q_1, q_2, \dots, q_{m-1})$ such that $q = q_1 + q_2 + \dots + q_{m-1} - (m-2)$ is given by $L(GN, HN) = m|E(G)| - \sum_{i=1}^m [I_G(q_1 + q_2 + \dots + q_{i-1} - i + 2) + I_G(q - (q_1 + q_2 + \dots + q_i - i))]$.

FIGURE 2: Path-of-complete graphs. (a) $PC(7; 3(6 - \text{times}))$. (b) $PC(7; 4(6 - \text{times}))$. (c) Cycle-of-complete graph $CC(6; 4(6 - \text{times}))$.

Proof. We begin with the embedding method of GN and HN . Label the peripheral cycle vertices of GN as $0, 1, \dots, q-1$ and the bone path vertices v_i , $1 \leq i \leq m$, of HN as $q_1 + q_2 + \dots + q_{i-1} - i + 1$ in such a way that label the vertices (except on the path) of i^{th} complete graph K_{q_i} , $1 \leq i \leq m-1$, from $q_1 + q_2 + \dots + q_{i-1} - i + 2$ to $q_1 + q_2 + \dots + q_i - i - 1$. We prove that the graph embedding μ of GN into HN defined by $\mu(w) = w$ yields the minimum layout. For $1 \leq i \leq m-1$, let S_i be the set of edges in the complete graph K_{q_i} . Then, $HN - S_i$ reduced to q_i components $HN_1, HN_2, \dots, HN_{q_i}$ with $V(HN_1) = \{0, 1, \dots, q_1 + q_2 + \dots + q_{i-1} - i + 1\}$, $2 \leq j \leq q_i - 1$, $V(HN_j) = \{q_1 + q_2 + \dots + q_{i-1} - i + j\}$, and $V(HN_{q_i}) = \{q_1 + q_2 + \dots + q_i - i, q_1 +$

$q_2 + \dots + q_i - i + 1, \dots, q - 1\}$. By Lemma 4, the induced subgraph by $\{\mu^{-1}(v) : v \in V(HN_i)\}$ on GN is an optimal set. We now conclude that μ is S_i -repulsive embedding of GN into HN . By Lemma 3, $EC_\mu(S_i)$ is minimum and $EC_\mu(S_i) = |E(GN)| - [I_{GN}(q_1 + q_2 + \dots + q_{i-1} - i + 2) + I_{GN}(q - (q_1 + q_2 + \dots + q_i - i))]$. Therefore, $L(GN, HN) = \sum_{i=1}^m EC_\mu(S_i) = m|E(GN)| - \sum_{i=1}^m [I_{GN}(q_1 + q_2 + \dots + q_{i-1} - i + 2) + I_{GN}(q - (q_1 + q_2 + \dots + q_i - i))]$. $\square$

Theorem 2. The minimum layout of circulant network $GN = CN(q; \pm \{1, 2, \dots, l\})$, $1 \leq l < [q/2]$, into cycle-of-complete graph $HN = CC(m; q_1, q_2, \dots, q_m)$ such that $q = q_1 + q_2 + \dots + q_{m-1} - (m-1)$ is given by

$$L(GN, HN) = \frac{m}{2} |E(GN)| - \begin{cases} \sum_{i=1}^{m/2} [I_{GN}(q_{i+1} + \dots + q_{m/2+i-1} - m/2 + 1) \\ + I_{GN}(q_{m/2+i+1} + \dots + q_{i-1} - m/2 + 1)] & : m \text{ even} \\ \frac{1}{2} \sum_{i=1}^m [I_{GN}(q_{i+1} + \dots + q_{(m-1)/2+i-1} - (m-3)/2 + 1) \\ + I_{GN}(q_{(m-1)/2+i+1} + \dots + q_{i-1} - (m-1)/2 + 1)] & : m \text{ odd} \end{cases} \quad (6)$$

Proof. We first give the labeling of GN into HN and followed by embedding algorithm. Label the peripheral cycle vertices of GN as $0, 1, \dots, q-1$ and the bone cycle vertices v_i , $1 \leq i \leq m$, of HN as $q_1 + q_2 + \dots + q_{i-1} - i + 1$ along the vertices (except on the cycle) of i^{th} complete graph K_{q_i} , $1 \leq i \leq m$, from $q_1 + q_2 + \dots + q_{i-1} - i + 2$ to $q_1 + q_2 + \dots + q_i - i - 1$. Let μ be an embedding from GN into HN defined by $\mu(w) = w$.

Case 1 (m even): for $1 \leq i \leq m/2$, let S_i be the set of edges in the complete graphs K_{q_i} and $K_{q_{m/2+i}}$. Then, $HN - S_i$ reduced to $q_i + q_{m/2+i} - 2$ number of components HN_i s in which $q_i + q_{m/2+i} - 4$ components have cardinalities one each and the remaining components with $q_{i+1} + \dots + q_{m/2+i-1} - m/2 + 1$ and $q_{m/2+i+1} + \dots + q_{i-1} - m/2 + 1$ vertices. By Lemma 4, the induced subgraph by $\{\mu^{-1}(v) : v \in V(HN_j)\}$ on GN is an optimal set. Therefore, μ is S_i -repulsive embedding of GN into HN .

By Lemma 3, $EC_\mu(S_i)$ is minimum and $EC_\mu(S_i) = |E(GN)| - [I_{GN}(q_{i+1} + \dots + q_{m/2+i} - i - 1 - m/2 + 1) + I_{GN}(q_{m/2+i+1} + \dots + q_{i-1} - m/2 + 1)]$. By the construction of the edge cuts, we infer that $\{S_i : 1 \leq i \leq m/2\}$ is a partition of $E(HN)$, and hence, $L(GN, HN) = \sum_{i=1}^{m/2} EC_\mu(S_i) = (m/2)|E(GN)| - \sum_{i=1}^{m/2} [I_{GN}(q_{i+1} + \dots + q_{m/2+i-1} - m/2 + 1) + I_{GN}(q_{m/2+i+1} + \dots + q_{i-1} - m/2 + 1)]$.

Case 2 (m odd): for $1 \leq i \leq m$, let S_i be the set of edges in the complete graphs K_{q_i} and $K_{q_{(m-1)/2+i}}$. Then, $HN - S_i$ reduced to $q_i + q_{(m-1)/2+i} - 2$ number of components HN_j s in which $q_i + q_{(m-1)/2+i} - 4$ components have cardinalities one each and the remaining components with $q_{i+1} + \dots + q_{(m-1)/2+i-1} - (m-3)/2 + 1$ and $q_{(m-1)/2+i+1} + \dots + q_{i-1} - (m-1)/2 + 1$ vertices. By Lemma 4, the induced subgraph by $\{\mu^{-1}(v) : v \in V(HN_j)\}$ is an optimal set. Therefore, μ is

S_i -repulsive embedding of GN into HN . By Lemma 3, $EC_\mu(S_i)$ is minimum and $EC_\mu(S_i) = |E(GN)| - [I_{GN}(q_{i+1} + \dots + q_{(m-1)/2+i-1} - (m-3)/2 + 1) + I_{GN}(q_{(m-1)/2+i+1} + \dots + q_{i-1} - (m-1)/2 + 1)]$. We note that $\{S_i; 1 \leq i \leq m\}$ is a partition of $E^2(HN)$, and hence, $L(GN, HN) = (1/2) \sum_{i=1}^m EC_\mu(S_i) = (m/2)|E(GN)| - (1/2) \sum_{i=1}^m [I_{GN}(q_{i+1} + \dots + q_{(m-1)/2+i-1} - (m-3)/2 + 1) + I_{GN}(q_{(m-1)/2+i+1} + \dots + q_{i-1} - (m-1)/2 + 1)]$. $\square$

4. Conclusion

In analyzing the structural properties of a connected network, the measure such as graph embedding is of greater interest. As we know that the path, cycle, complete, and circulant graphs are important interconnection networks due to their simplicity, in this article, we successfully embedded circulant graphs into path- and cycle-of-complete graphs with minimum layout by the way of characterizing the congestion lemma with respect to regularity and optimality on the guest network.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

M. A. and A. J. S. conceptualized the study; J. B. L. and J. N. D. investigated the study; M. A. wrote and prepared the original draft; A. J. S. and J. N. D. reviewed and edited the manuscript; J. B. L. supervised the study

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