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Research Article

On Soft Quantum B-Algebras and Fuzzy Soft Quantum B-Algebras

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This paper aims to make a combination between the quantum B-algebras (briefly, $\mathscr{X}$-$\mathbb{A}$s) and two interesting theories (e.g., soft set theory and fuzzy soft set theory). Firstly, we propose the novel notions of soft quantum B-algebras (briefly, $\mathbb{SQB}$-$\mathbb{A}$s), a soft deductive system of $\mathbb{QB}$-$\mathbb{A}$s, and deducible soft quantum B-algebras (briefly, $\mathbb{DSQB}$-$\mathbb{A}$s). Then, we discuss the relationship between $\mathbb{SQB}$-$\mathbb{A}$s and $\mathbb{DSQB}$-$\mathbb{A}$s. Furthermore, we investigate the union and intersection operations of $\mathbb{DSQB}$-$\mathbb{A}$s. Secondly, we introduce the notions of a fuzzy soft quantum B-algebras (briefly, $\mathbb{FSQB}$-$\mathbb{A}$s), a fuzzy soft deductive system of $\mathbb{QB}$-$\mathbb{A}$s, and present some characterizations of $\mathbb{FSQB}$-$\mathbb{A}$s, along with several examples. Finally, we explain the basic properties of homomorphism image of $\mathbb{FSQB}$-$\mathbb{A}$s.

1. Introduction

In 1999, Molodtsov [1] introduced the notion called soft sets (briefly, $\mathbb{SS}$) (i.e., which reduce the uncertainty and vagueness of knowledge). Maji et al. [2] presented the fuzzy soft sets (briefly, $\mathbb{FSS}$). Since then, many researchers studied further on $\mathbb{SS}$ and $\mathbb{FSS}$ as in the following published articles (e.g., [3–9]).

In 2014, Rump and Yang [10] proposed the notion of $\mathbb{QB}$-$\mathbb{A}$s (i.e., a partial ordered implication algebras). Rump [11, 12] investigated many implication algebras (for example, pseudo-BCK-algebras, po-groups, BL-algebras, MV-algebras, GPE-algebras, and resituated lattices). Botur and Paseka [13] studied filters on integral $\mathbb{QB}$-$\mathbb{A}$s, and Zhang et al. [14] established the quotient structures by using q-filters in $\mathbb{QB}$-$\mathbb{A}$s and investigated the relation between basic implication algebras and $\mathbb{QB}$-$\mathbb{A}$s. Han et al. [15] constructed the unitality of $\mathbb{QB}$-$\mathbb{A}$s and explained the injective hulls of $\mathbb{QB}$-$\mathbb{A}$s in [16]. By the framework of $\mathbb{QB}$-$\mathbb{A}$s, there are many published papers on $\mathbb{QB}$-$\mathbb{A}$s (e.g., [17–23]).

Regarding these developments, as the motivation of this paper, we will combine $\mathbb{QB}$-$\mathbb{A}$s with $\mathbb{SS}$ and $\mathbb{FSS}$ (i.e., enrich the previous work on hybrid soft set and fuzzy soft set theories algebras with quantum structures). We introduce the notions of $\mathbb{SQB}$-$\mathbb{A}$s and the soft deductive system of $\mathbb{QB}$-$\mathbb{A}$s and consider the relation between $\mathbb{SQB}$-$\mathbb{A}$s and $\mathbb{DSQB}$-$\mathbb{A}$s. Furthermore, some conditions are given to ensure the operations union and intersection holds of soft deductive of $\mathbb{QB}$-$\mathbb{A}$s. Then, we investigate the homomorphism image of deductive $\mathbb{SQB}$-$\mathbb{A}$s. Lastly, we define $\mathbb{FSQB}$-$\mathbb{A}$s and fuzzy soft deductive system of $\mathbb{QB}$-$\mathbb{A}$s and give an example to illustrate its derive properties.

In the following, we have arranged the sections as follows. In Section 2, we briefly recall many notions related to $\mathbb{QB}$-$\mathbb{A}$s, $\mathbb{SS}$, and $\mathbb{FSS}$ as indicated in Definitions 1–7, which are used in the sequel. In Section 3, we propose the notions of $\mathbb{SQB}$-$\mathbb{A}$s, soft deductive system of $\mathbb{QB}$-$\mathbb{A}$s, and $\mathbb{DSQB}$-$\mathbb{A}$s. In Section 4, we present the notions of $\mathbb{FSQB}$-$\mathbb{A}$s and a fuzzy soft deductive system of $\mathbb{QB}$-$\mathbb{A}$s and discuss the homomorphism image of $\mathbb{FSQB}$-$\mathbb{A}$s. The conclusions are explained in Section 5.

2. Preliminaries

We give some basic notions of $\mathbb{QB}$-$\mathbb{A}$s, $\mathbb{SS}$, and $\mathbb{FSS}$ before defining $\mathbb{SQB}$-$\mathbb{A}$s in Section 3.

Definition 1 (cf. [10]).

(1) $\mathbb{QB}$-$\mathbb{A}$s is a partially ordered set $(\mathscr{X}, \leq)$ with two binary operations $\longrightarrow$ and $\rightsquigarrow$ which satisfy $(\forall x, y, z \in \mathscr{X})$:

$$\begin{aligned} y \longrightarrow z &\leq (x \longrightarrow y) \longrightarrow (x \longrightarrow z),\\ y \rightsquigarrow z &\leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z),\\ y \leq z &\Longrightarrow x \longrightarrow y \leq x \longrightarrow z,\\ x \leq y \longrightarrow z &\Longleftrightarrow y \leq x \rightsquigarrow z. \end{aligned} \tag{1}$$

(2) $\mathbb{QB}$-$\mathbb{A}$ is a commutative (briefly, $\mathbb{CQB}$-$\mathbb{A}$) if $x \longrightarrow y = x \rightsquigarrow y$ $(\forall x, y \in \mathscr{X})$.

(3) A subset $\mathscr{Y}$ of a $\mathbb{QB}$-$\mathbb{A}$ $\mathscr{X}$ is a subalgebra if $x \longrightarrow y, x \rightsquigarrow y \in \mathscr{Y}$ $(\forall x, y \in \mathscr{X})$.

In what follows, denote by $\mathscr{X}$ a $\mathbb{QB}$-$\mathbb{A}$ unless otherwise specified.

Definition 2 (cf. [10]). Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be two $\mathbb{QB}$-$\mathbb{A}$s. Then, $\psi\colon \mathscr{X}_1 \longrightarrow \mathscr{X}_2$ is a morphism of $\mathbb{QB}$-$\mathbb{A}$s if it satisfies $(\forall x, y \in \mathscr{X}_1)$:

$$\begin{aligned} \psi(x \longrightarrow y) &\leq \psi(x) \longrightarrow \psi(y),\\ \psi(x \rightsquigarrow y) &\leq \psi(x) \rightsquigarrow \psi(y). \end{aligned} \tag{2}$$

We say morphism ψ is exact if the inequalities become equations.

Definition 3 (cf. [1]). Assume that $\mathscr{X}$ be a set and $\mathscr{K}$ be a set of parameters. $\mathscr{S}_{\mathscr{K}}$ (called $\mathbb{SS}$) is a mapping given by $\mathscr{S}\colon \mathscr{K} \longrightarrow 2^{\mathscr{X}}$ (i.e., $2^{\mathscr{X}}$ is the power set of $\mathscr{X}$).

Definition 4 (cf. [3]). Assume that $\mathscr{S}_{\mathscr{K}_1}$ and $\mathscr{S}_{\mathscr{K}_2}$ are two $\mathbb{SS}$ over $\mathscr{X}$. $\mathscr{S}_{\mathscr{K}_1}$ is a subset of $\mathscr{S}_{\mathscr{K}_2}$ (denoted by $\mathscr{S}_{\mathscr{K}_1} \widetilde{\subset} \mathscr{S}_{\mathscr{K}_2}$) if

(1) $\mathscr{K}_1 \subset \mathscr{K}_2$

(2) For every $k \in \mathscr{K}_1$, $\mathscr{S}_{\mathscr{K}_1}(k)$ and $\mathscr{S}_{\mathscr{K}_2}(k)$ are identical approximations

Definition 5 (cf. [3]). Assume that $\mathscr{S}_{\mathscr{K}_1}, \mathscr{S}_{\mathscr{K}_2}$, and $\mathscr{S}_{\mathscr{K}_3}$ are three $\mathbb{SS}$ over $\mathscr{X}$. $\mathscr{S}_{\mathscr{K}_3}$ is the intersection of $\mathscr{S}_{\mathscr{K}_1}$ and $\mathscr{S}_{\mathscr{K}_2}$ (denoted by $\mathscr{S}_{\mathscr{K}_3} = \mathscr{S}_{\mathscr{K}_1} \widetilde{\cap} \mathscr{S}_{\mathscr{K}_2}$) if

(1) $\mathscr{K}_3 = \mathscr{K}_1 \cap \mathscr{K}_2$

(2) $\forall k \in \mathscr{K}_3$, $\mathscr{S}_{\mathscr{K}_3}(k) = \mathscr{S}_{\mathscr{K}_1}(k)$ or $\mathscr{S}_{\mathscr{K}_2}(k)$ (as both are same sets)

Definition 6 (cf. [3]). Assume that $\mathscr{S}_{\mathscr{K}_1}, \mathscr{S}_{\mathscr{K}_2}$, and $\mathscr{S}_{\mathscr{K}_3}$ are three $\mathbb{SS}$ over $\mathscr{X}$. $\mathscr{S}_{\mathscr{K}_3}$ is called the union of $\mathscr{S}_{\mathscr{K}_1}$ and $\mathscr{S}_{\mathscr{K}_2}$ (denoted by $\mathscr{S}_{\mathscr{K}_3} = \mathscr{S}_{\mathscr{K}_1} \widetilde{\cup} \mathscr{S}_{\mathscr{K}_2}$) if

(1) $\mathscr{K}_3 = \mathscr{K}_1 \cup \mathscr{K}_2$.

(2) $k \in \mathscr{K}_3$,

$$\mathscr{S}_{\mathscr{K}_3}(k) = \begin{cases} \mathscr{S}_{\mathscr{K}_1}(k), & k \in \mathscr{K}_1 \backslash \mathscr{K}_2\\ \mathscr{S}_{\mathscr{K}_2}(k), & k \in \mathscr{K}_2 \backslash \mathscr{K}_1,\\ \mathscr{S}_{\mathscr{K}_1}(k) \cup \mathscr{S}_{\mathscr{K}_2}(k), & k \in \mathscr{K}_1 \cap \mathscr{K}_2 \end{cases} \tag{3}$$

Definition 7 (cf. [2]). $\mathbb{FSS}$ (called $\mathbb{FSS}$) $\widehat{\mathscr{S}}_{\mathscr{K}}$ is a mapping given by $\widehat{\mathscr{S}}\colon \mathscr{K} \longrightarrow I^X$ (i.e., I^X is the set of all fuzzy sets [24] of $\mathscr{X}$).

3. $\mathbb{SQB}$-$\mathbb{A}$s

We define the $\mathbb{SQB}$-$\mathbb{A}$s and give several examples based on $\mathbb{SQB}$-$\mathbb{A}$s. Also, we will study the union and intersection operations between two $\mathbb{SQB}$-$\mathbb{A}$s as follows .

Definition 8. $\mathscr{S}_{\mathscr{K}}$ is a $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$ if $\mathscr{S}_{\mathscr{K}}(x)(\forall x \in \mathscr{K})$ are subalgebras of $\mathscr{X}$ (i.e., in case $\mathscr{K} = \mathscr{X}$).

Example 1

(1) Suppose $\mathscr{X}$ (i.e., $\mathscr{X} = \{k_1, k_2, k_3, 1\}$) with the order $k_2, k_3 < k_1 < 1$. Now, we show, by Table 1, the binary operation $\longrightarrow$.

Clearly, $\mathscr{X}$ is a $\mathbb{CQB}$-$\mathbb{A}$. We define $\mathscr{S}_{\mathscr{K}}(\forall x \in \mathscr{K})$ (i.e., $\mathscr{K} = \mathscr{X}$) by

$$\mathscr{S}_{\mathscr{K}}(x) = \{y \in \mathscr{X} \mid (x \longrightarrow y) \longrightarrow y \in \{k_1, 1\}\}. \tag{4}$$

From Table 1, we can get on $\mathscr{S}_{\mathscr{K}}(k_1) = \mathscr{X}$, $\mathscr{S}_{\mathscr{K}}(k_2) = \mathscr{S}_{\mathscr{K}}$, $(k_3) = \{k_1, k_3, 1\}$, and $\mathscr{S}_{\mathscr{K}}(1) = \mathscr{X}$, and then, $\mathscr{S}_{\mathscr{K}}(x)(x \in \mathscr{K})$ are all subalgebras of $\mathscr{X}$. Consequently, $\mathscr{S}_{\mathscr{K}}$ is a $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$.

(2) Suppose $\mathscr{X}$ (i.e., $\mathscr{X} = \{k_1, k_2, k_3, 1\}$) with the order $k_1 < k_2 < k_3 < 1$. Now, we show, by Table 2, the binary operation $\longrightarrow$.

Clearly, $\mathscr{X}$ is a $\mathbb{CQB}$-$\mathbb{A}$. We define $\mathscr{S}_{\mathscr{K}}(\forall x \in \mathscr{K})$ (i.e., $\mathscr{K} = \mathscr{X}$) by

$$\mathscr{S}_{\mathscr{K}}(x) = \{y \in \mathscr{X} | x\mathscr{R}y \Longleftrightarrow x \longrightarrow (x \longrightarrow y) \in \{k_3, 1\}\}. \tag{5}$$

From Table 2, we can get on (HTML translation failed), and then, $\mathscr{S}_{\mathscr{K}}(x)(x \in \mathscr{K})$ are all subalgebras of $\mathscr{X}$. Consequently, $\mathscr{S}_{\mathscr{K}}$ is a $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$.

We ensure the operations (i.e., union and intersection) are holding on $\mathbb{SQB}$-$\mathbb{A}$s by the following suggested theorem.

Theorem 1. *Assume that $\mathscr{S}_{\mathscr{K}_1}$ and $\mathscr{S}_{\mathscr{K}_2}$ are $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$. Then,*

(1) If $\mathscr{K}_3 = \mathscr{K}_1 \cap \mathscr{K}_2$, then $\mathscr{S}_{\mathscr{K}_3} = \mathscr{S}_{\mathscr{K}_1} \widetilde{\cap} \mathscr{S}_{\mathscr{K}_2}$ is called a $\mathbb{SQB}$-$\mathbb{A}$ over $\mathscr{X}$

(2) If $\mathscr{K}_1 \cap \mathscr{K}_2 = \varnothing$, then $\mathscr{S}_{\mathscr{K}_1} \widetilde{\cup} \mathscr{S}_{\mathscr{K}_2}$ is called a $\mathbb{SQB}$-$\mathbb{A}$ over $\mathscr{X}$

Table 1: The binary operation $\longrightarrow$.

$\longrightarrow$	k_1	k_2	k_3	1
k_1	1	k_1	k_1	1
k_2	1	1	k_1	1
k_3	1	1	k_1	1
1	k_1	k_2	k_3	1

Table 2: The binary operation $\longrightarrow$.

$\longrightarrow$	k_1	k_2	k_3	1
k_1	1	1	1	1
k_2	k_1	k_2	1	1
k_3	k_1	k_1	1	1
1	k_1	k_1	k_3	1

Proof

(1) If $\mathscr{K}_3 = \mathscr{K}_1 \cap \mathscr{K}_2$ and by Definition 5, we obtain $\mathscr{S}_{\mathscr{K}_3}(x) = \mathscr{S}_{\mathscr{K}_1}(x)$ or $\mathscr{S}_{\mathscr{K}_3}(x) = \mathscr{S}_{\mathscr{K}_2}(x)$, for all $x \in \mathscr{K}_3$. Since $\mathscr{S}_{\mathscr{K}_1}$ and $\mathscr{S}_{\mathscr{K}_2}$ are $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$, which implies that $\mathscr{S}_{\mathscr{K}_3}$ is a $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$, that is, $\mathscr{S}_{\mathscr{K}_3}(x) = \mathscr{S}_{\mathscr{K}_1}(x)$ or $\mathscr{S}_{\mathscr{K}_3}(x) = \mathscr{S}_{\mathscr{K}_2}(x)$ are both subalgebras of $\mathscr{X}(\in \mathscr{K}_3)$, therefore, $\mathscr{S}_{\mathscr{K}_3} = \mathscr{S}_{\mathscr{K}_1} \widetilde{\cap} \mathscr{S}_{\mathscr{K}_2}$ is a $\mathbb{SQB}$-$\mathbb{A}$ over $\mathscr{X}$.

(2) If $\mathscr{K}_3 = \mathscr{K}_1 \cup \mathscr{K}_2$ and by Definition 6, we obtain

$$\mathscr{S}_{\mathscr{K}_3}(x) = \begin{cases} \mathscr{S}_{\mathscr{K}_1}(x), & x \in \mathscr{K}_1 \backslash \mathscr{K}_2, \\ \mathscr{S}_{\mathscr{K}_2}(x), & x \in \mathscr{K}_2 \backslash \mathscr{K}_1, \\ \mathscr{S}_{\mathscr{K}_1}(x) \cup \mathscr{S}_{\mathscr{K}_2}(x), & x \in \mathscr{K}_1 \cap \mathscr{K}_2 \end{cases} \tag{6}$$

For $x \in \mathscr{K}_1 \backslash \mathscr{K}_2$ and since $\mathscr{S}_{\mathscr{K}_1}$ is a $\mathbb{SQB}$-$\mathbb{A}$, then we have $\mathscr{S}_{\mathscr{K}_3}(x) = \mathscr{S}_{\mathscr{K}_1}(x)$ is a subalgebra of $\mathscr{X}$. Similarly, for $x \in \mathscr{K}_2 \backslash \mathscr{K}_1$, then $\mathscr{S}_{\mathscr{K}_3}(x) = \mathscr{S}_{\mathscr{K}_2}(x)$ is a subalgebra of $\mathscr{X}$ due to $\mathscr{S}_{\mathscr{K}_2}$ is a $\mathbb{SQB}$-$\mathbb{A}$. Again, for $\mathscr{K}_1 \cap \mathscr{K}_2 = \varnothing$, so $x \in \mathscr{K}_1 \cap \mathscr{K}_2$ or $x \in \mathscr{K}_2 \cap \mathscr{K}_1$, for all $x \in \mathscr{K}_3$. Thus, $\mathscr{S}_{\mathscr{K}_3} = \mathscr{S}_{\mathscr{K}_1} \widetilde{\cup} \mathscr{S}_{\mathscr{K}_2}$ is a $\mathbb{SQB}$-$\mathbb{A}$ over $\mathscr{X}$. □

Remark 1. If $\mathscr{K}_1 \cap \mathscr{K}_2 \neq \varnothing$, then Theorem 1 (2) does not hold by the following example.

Example 2. Suppose $\mathscr{X}$ (i.e., $\mathscr{X} = \{0, k_1, k_2, k_3, k_4, 1\}$). Now, we show, by Tables 3 and 4, the binary operations $\longrightarrow$ and $\rightsquigarrow$, respectively.

Clearly, $\mathscr{X}$ is a $\mathbb{CQB}$-$\mathbb{A}$. Then,

(i) We define $\mathscr{S}_{\mathscr{K}_1}(\forall x \in \mathscr{K}_1)$ (i.e., $\mathscr{K}_1 = \mathscr{X}$) by

$$\mathscr{S}_{\mathscr{K}_1}(x) = \{y \in \mathscr{X} | x\mathscr{R}y \Longleftrightarrow x \longrightarrow (x \longrightarrow y)x \rightsquigarrow (x \rightsquigarrow y) \in \{k_3, k_4, 1\}\}. \tag{7}$$

From Table 3, we can get $\mathscr{S}_{\mathscr{K}_1}(0) = \mathscr{X}$ and $\mathscr{S}_{\mathscr{K}_1}(k_1) = \mathscr{S}_{\mathscr{K}_1}(k_2) = \mathscr{S}_{\mathscr{K}_1}(k_3) = \mathscr{S}_{\mathscr{K}_1}(k_4) = \mathscr{S}_{\mathscr{K}_1}(1) = \{k_3, k_4, 1\}$, and then, $\mathscr{S}_{\mathscr{K}_1}(x)(x \in \mathscr{K}_1)$ are all subalgebras of $\mathscr{X}$. Consequently, $\mathscr{S}_{\mathscr{K}_1}$ is a $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$.

Table 3: The binary operation $\longrightarrow$.

$\longrightarrow$	0	k_1	(HTML translation failed)	k_3	k_4	1
0	1	1	1	1	1	1
k_1	0	k_2	0	k_4	1	1
k_2	0	0	k_2	k_4	k_4	1
k_3	0	0	0	1	1	1
k_4	0	0	0	k_4	1	1
1	0	0	0	k_4	k_4	1

Table 4: The binary operation $\rightsquigarrow$.

$\rightsquigarrow$	0	k_1	k_2	k_3	k_4	1
0	1	1	1	1	1	1
k_1	0	0	0	1	1	1
k_2	0	k_1	k_2	k_3	k_4	1
k_3	0	0	0	1	1	1
k_4	0	0	0	1	1	1
1	0	0	0	k_3	k_4	1

(ii) We define $\mathscr{S}_{\mathscr{K}_2}(\forall x \in \mathscr{K}_2)$ (i.e., $\mathscr{K}_2 = \{k_2\}$) by

$$\mathscr{S}_{\mathscr{K}_2}(x) = \{y \in \mathscr{K}_2 | x\mathscr{R}y \Longleftrightarrow x \longrightarrow y = k_2, x \rightsquigarrow y = k_2\}. \tag{8}$$

From Table 4, we can get $\mathscr{S}_{\mathscr{K}_2}(k_2) = \{k_2\}$ is the subalgebra of $\mathscr{X}$. Consequently, $\mathscr{S}_{\mathscr{K}_2}$ is a $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$.

From (i) and (ii) and $\mathscr{K}_1 \cap \mathscr{K}_2 = \{k_2\} \neq \varnothing$, then we have $\mathscr{S}_{\mathscr{K}_3}(k_2) = \mathscr{S}_{\mathscr{K}_1}(k_2) \cup \mathscr{S}_{\mathscr{K}_2}(k_2) = \{k_3, k_4, 1\} \cup \{k_2\} = \{k_2, k_3, k_4, 1\}$ is not a subalgebra over $\mathscr{X}$. Thus, $\mathscr{S}_{\mathscr{K}_3}$ is not a $\mathbb{SQB}$-$\mathbb{A}$.

3.1. Soft Deductive Systems of $\mathbb{SQB}$-$\mathbb{A}$*s.* Based on Definition 8, we will propose the notion of soft deductive systems of $\mathbb{SQB}$-$\mathbb{A}$s as indicated below.

Definition 9. Assume that $\mathscr{X} = (\mathscr{X}, \longrightarrow, \rightsquigarrow, \leq)$ be a $\mathbb{SQB}$-$\mathbb{A}$. A nonempty subset $\mathscr{D} \subseteq \mathscr{X}$ is a deductive system of $\mathscr{X}$ if it satisfies

(1) $\forall x \in \mathscr{D}$, $x \longrightarrow x \in \mathscr{D}$, $x \rightsquigarrow x \in \mathscr{D}$

(2) $\forall x, y \in \mathscr{X}$, $x \in \mathscr{D}$, $x \longrightarrow y \in \mathscr{D} \Longrightarrow y \in \mathscr{D}$

Definition 10. Let $\mathscr{X}$ be a $\mathbb{SQB}$-$\mathbb{A}$ and $\mathscr{Y}$ a subalgebra of $\mathscr{X}$. A subset $\mathscr{D}$ of $\mathscr{X}$ is a deductive system of $\mathscr{X}$ related to $\mathscr{Y}$ (i.e., $\mathscr{Y}$-deductive system of $\mathscr{X}$), denoted by $\mathscr{D} \bowtie \mathscr{Y}$, and satisfies the following two conditions:

(1) $\forall x \in \mathscr{D}$, $x \longrightarrow x \in \mathscr{D}$, $x \rightsquigarrow x \in \mathscr{D}$

(2) $\forall y \in \mathscr{Y}$, $x \in \mathscr{D}$, $x \longrightarrow y \in \mathscr{D} \Longrightarrow y \in \mathscr{D}$

Remark 2. According to Definitions 9 and 10, we obtain that any deductive system of $\mathscr{X}$ is $\mathscr{Y}$-deductive system if $\mathscr{Y}$ is a subalgebra of $\mathscr{X}$.

The converse of Remark 2 does not hold by Example 3 (i.e., $\mathcal{Y}$ is a subalgebra of $\mathcal{X}$ and $\mathcal{Y}$-deductive system is not a deductive system).

Example 3. Suppose $\mathcal{X}$ (i.e., $\mathcal{X} = \{0, k_1, k_2, k_3, 1\}$) with partial order $0 < k_1 < k_3 < 1$ and $0 < k_1 < k_2 < 1$. Now, we show, by Tables 5 and 6, the binary operations $\longrightarrow$ and $\rightsquigarrow$, respectively.

Clearly, $\mathcal{X}$ is a $\mathbb{CQB}$-$\mathbb{A}$. Consider a subalgebra $\mathcal{Y} = \{k_1, 1\}$ and a subset $\mathcal{D} = \{k_1, k_2, 1\}$; we can see that $\mathcal{D} \bowtie \mathcal{Y}$. However, $\mathcal{D}$ is not a deductive system of $\mathcal{X}$ since $k_3 \longrightarrow 1 = 1 \in \mathcal{D}$ and $k_3 \notin \mathcal{D}$.

Definition 11. Assume that $\mathcal{S}_{\mathcal{K}}$ is a $\mathbb{SQB}$-$\mathbb{A}$ over $\mathcal{X}$. $\mathcal{S}_{\mathcal{D}}$ (i.e., $\mathbb{SS}$) over $\mathcal{X}$ is a soft deductive system of $\mathcal{S}_{\mathcal{K}}$, denoted by $\mathcal{S}_{\mathcal{D}} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}$, and satisfies the following two conditions:

(1) $\mathcal{D} \subseteq \mathcal{K}$

(2) $\forall x \in \mathcal{D}$, $\mathcal{S}_{\mathcal{D}}(x) \triangleright\triangleleft \mathcal{S}_{\mathcal{K}}(x)$

Now, we will give an example to illustrate Definition 11 as follows.

Example 4. Suppose $\mathcal{X}$ (i.e., $\mathcal{X} = \{k_1, k_2, k_3, k_4, 1\}$) with partial order $k_1 < k_2 < k_3 < k_4 < 1$. Now, we show, by Tables 7 and 8, the binary operations $\longrightarrow$ and $\rightsquigarrow$, respectively.

Clearly, $\mathcal{X}$ is a $\mathbb{CQB}$-$\mathbb{A}$. We define $\mathcal{S}_{\mathcal{K}}$ $(\forall x \in \mathcal{K})$ (i.e., $\mathcal{K} = \mathcal{X}$) by

$$\mathcal{S}_{\mathcal{K}}(x) = \{y \in \mathcal{X} | x \mathcal{R} y \Longleftrightarrow (x \longrightarrow y) \rightsquigarrow y = 1\}. \tag{9}$$

From Tables 7 and 8, we can get on $\mathcal{S}_{\mathcal{K}}(k_1) = \mathcal{S}_{\mathcal{K}}(k_2) = 1$, $\mathcal{S}_{\mathcal{K}}(k_3) = \{k_2, 1\}$, $\mathcal{S}_{\mathcal{K}}(k_4) = \{k_2, k_3, 1\}$, and $\mathcal{S}_{\mathcal{K}}(1) = \mathcal{X}$, and then, $\mathcal{S}_{\mathcal{K}}(x)$ $(x \in \mathcal{K})$ are all subalgebras of $\mathcal{X}$. Consequently, $\mathcal{S}_{\mathcal{K}}$ is a $\mathbb{SQB}$-$\mathbb{A}$s over $\mathcal{X}$.

Next, for a subset $\mathcal{D} = \{k_2, k_4\}$, we define $\mathcal{S}_{\mathcal{D}}$ $(\forall x \in \mathcal{D})$ by

$$\mathcal{S}_{\mathcal{D}}(x) = \{1\} \cup \{y \in \mathcal{X} | y \leq x\}. \tag{10}$$

Then, we obtain $\mathcal{S}_{\mathcal{D}}(k_2) = \{k_1, k_2, 1\} \triangleright\triangleleft \{1\} = \mathcal{S}_{\mathcal{K}}(k_2)$ and $\mathcal{S}_{\mathcal{D}}(k_4) = \mathcal{X} \triangleright\triangleleft \{k_2, k_3, 1\} = \mathcal{S}_{\mathcal{K}}(k_4)$. Consequently, $\mathcal{S}_{\mathcal{D}}$ is a soft deductive system of $\mathcal{S}_{\mathcal{K}}$.

Theorem 2. *Assume that $\mathcal{S}_{\mathcal{K}}$ is a $\mathbb{SQB}$-$\mathbb{A}$ over $\mathcal{X}$ and $\mathcal{S}_{\mathcal{D}_1}$ and $\mathcal{S}_{\mathcal{D}_2}$ are two $\mathbb{SS}$. Then,*

(1) If $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \varnothing$, then $\mathcal{S}_{\mathcal{D}_1} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}, \mathcal{S}_{\mathcal{D}_2} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}} \Longrightarrow \mathcal{S}_{\mathcal{D}_1} \widetilde{\cap} \mathcal{S}_{\mathcal{D}_2} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}$

(2) If $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, then $\mathcal{S}_{\mathcal{D}_1} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}, \mathcal{S}_{\mathcal{D}_2} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}} \Longrightarrow \mathcal{S}_{\mathcal{D}_1} \widetilde{\cup} \mathcal{S}_{\mathcal{D}_2} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}$

Proof

(1) Follow from Definition 5.

(2) If $\mathcal{S}_{\mathcal{D}_1} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}, \mathcal{S}_{\mathcal{D}_2} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}$, then, by Definition 6, we have $\mathcal{D}_3 = \mathcal{D}_1 \cap \mathcal{D}_2$ (i.e., $x \in \mathcal{D}_3$), $\mathcal{S}_{\mathcal{D}_1} \widetilde{\cup} \mathcal{S}_{\mathcal{D}_2} = \mathcal{S}_{\mathcal{D}_3}$, and

TABLE 5: The binary operation $\longrightarrow$.

$\longrightarrow$	0	k_1	k_2	k_3	1
0	1	1	1	1	1
k_1	0	1	k_2	1	1
k_2	k_1	k_1	1	1	1
k_3	0	k_1	k_2	1	1
1	0	k_1	k_2	k_3	1

TABLE 6: The binary operation $\rightsquigarrow$.

$\rightsquigarrow$	0	k_1	k_2	k_3	1
0	1	1	1	1	1
k_1	k_2	1	k_2	1	1
k_2	0	k_1	1	1	1
k_3	0	k_1	k_2	1	1
1	0	k_1	k_2	k_3	1

TABLE 7: The binary operation $\longrightarrow$.

$\longrightarrow$	k_1	k_2	k_3	k_4	1
k_1	1	1	1	1	1
k_2	k_3	1	1	1	1
k_3	k_2	k_2	1	1	1
k_4	k_2	k_2	k_c	1	1
1	k_1	k_2	k_3	k_4	1

TABLE 8: The binary operation $\rightsquigarrow$.

$\rightsquigarrow$	k_1	k_2	k_3	k_4	1
k_1	1	1	1	1	1
k_2	k_4	1	1	1	1
k_3	k_2	k_2	1	1	1
k_4	k_1	k_2	k_3	1	1
1	k_1	k_2	k_3	k_4	1

$$\mathcal{S}_{\mathcal{D}_3}(x) = \begin{cases} \mathcal{S}_{\mathcal{D}_1}(x), & x \in \mathcal{D}_1 \backslash \mathcal{D}_2 \\ \mathcal{S}_{\mathcal{D}_2}(x), & x \in \mathcal{D}_2 \backslash \mathcal{D}_1, \\ \mathcal{S}_{\mathcal{D}_1}(x) \cup \mathcal{S}_{\mathcal{D}_2}(x), & x \in \mathcal{D}_1 \cap \mathcal{D}_2 \end{cases} \tag{11}$$

Since $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, we obtain either $x \in \mathcal{D}_1 \backslash \mathcal{D}_2$ or $x \in \mathcal{D}_2 \backslash \mathcal{D}_1$. Then, we have the following:

Case 1: if $x \in \mathcal{D}_1 \backslash \mathcal{D}_2$, since $\mathcal{S}_{\mathcal{D}_1} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}$, then $\mathcal{S}_{\mathcal{D}_3}(x) = \mathcal{S}_{\mathcal{D}_1}(x) \triangleright\triangleleft \mathcal{S}_{\mathcal{K}}(x)$

Case 2: if $x \in \mathcal{D}_2 \backslash \mathcal{D}_1$ and $\mathcal{S}_{\mathcal{D}_2} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}$, then $\mathcal{S}_{\mathcal{D}_3}(x) = \mathcal{S}_{\mathcal{D}_2}(x) \triangleright\triangleleft \mathcal{S}_{\mathcal{K}}(x)$

Consequently, for all $x \in \mathcal{D}_3$, we have $\mathcal{S}_{\mathcal{D}_3}(x) \triangleright\triangleleft \mathcal{S}_{\mathcal{K}}(x)$, which implies that $\mathcal{S}_{\mathcal{D}_1} \widetilde{\cup} \mathcal{S}_{\mathcal{D}_2} = \mathcal{S}_{\mathcal{D}_3} \widetilde{\triangleright\triangleleft} \mathcal{S}_{\mathcal{K}}$. □

Remark 3. If $\mathcal{K}_1 \cap \mathcal{K}_2 \neq \varnothing$, then Theorem 2 (2) does not hold by the following example.

Example 5. Suppose $\mathcal{X}$ (i.e., $\mathcal{X} = \{0, k_1, k_2, k_3, k_4, 1\}$). Now, we show, by Table 9, the binary operations $\longrightarrow$.

Table 9: The binary operation $\longrightarrow$.

$\longrightarrow$	0	k_1	k_2	k_3	k_4	1
0	1	1	1	1	1	1
k_1	k_3	1	k_2	k_3	k_2	1
k_2	k_4	k_1	1	k_2	k_1	1
k_3	k_1	k_1	1	1	k_1	1
k_4	k_2	1	1	k_2	1	1
1	0	k_1	k_2	k_3	k_4	1

Clearly, $\mathscr{X}$ is a $\mathbb{CQB}$-$\mathbb{A}$. Then,

(i) We define $\mathscr{S}_{\mathscr{K}}(\forall x \in \mathscr{K})$ (i.e., $\mathscr{K} = \mathscr{X}$) by

$$\mathscr{S}_{\mathscr{K}}(x) = \{y \in \mathscr{X} | x\mathscr{R}y \Longleftrightarrow (x \longrightarrow y) \longrightarrow y \in \{k_1, k_2, 1\}\}. \quad (12)$$

From Table 9, we can get $\mathscr{S}_{\mathscr{K}}(0) = \{k_1, k_2, 1\}$, $\mathscr{S}_{\mathscr{K}}(k_1) = \mathscr{X}$, $\mathscr{S}_{\mathscr{K}}(k_2) = \mathscr{S}_{\mathscr{K}}(k_3) = \{k_1, 1\}$, and $\mathscr{S}_{\mathscr{K}}(k_4) = \{k_1, k_2, k_3, 1\}$, $\mathscr{S}_{\mathscr{K}}(1) = \mathscr{X}$, and then, $\mathscr{S}_{\mathscr{K}}(x)(\forall x \in \mathscr{K})$ are all subalgebras of $\mathscr{X}$. Consequently, $\mathscr{S}_{\mathscr{K}}$ is a $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$.

(ii) We define $\mathscr{S}_{\mathscr{K}_1}(\forall x \in \mathscr{K}_1)$ (i.e., $\mathscr{K}_1 = \{k_1, k_2, k_3\}$) by

$$\mathscr{S}_{\mathscr{K}_1}(x) = \{y \in \mathscr{X} | x\mathscr{R}y \Longleftrightarrow x \longrightarrow y = 1\}. \quad (13)$$

Then, we can get $\mathscr{S}_{\mathscr{K}_1}(k_1) = \{k_1, 1\} \rhd\lhd X = \mathscr{S}_{\mathscr{K}}(k_1)$, $\mathscr{S}_{\mathscr{K}_1}(k_2) = \{k_2, 1\} \rhd\lhd \{a, 1\} = \mathscr{S}_{\mathscr{K}}(k_2)$, and $\mathscr{S}_{\mathscr{K}_1}(k_3) = \{k_2, k_3, 1\} \rhd\lhd \{k_1, 1\}$. Therefore, $\mathscr{S}_{\mathscr{K}_1}$ is a soft deductive system over $\mathscr{S}_{\mathscr{K}}$.

(iii) We define $\mathscr{S}_{\mathscr{K}_2}(\forall x \in \mathscr{K}_2)$ (i.e., $\mathscr{K}_2 = \{k_1\}$) by

$$\mathscr{S}_{\mathscr{K}_2}(x) = \{y \in \mathscr{X} | x\mathscr{R}y \Longleftrightarrow y \longrightarrow x = k_1\}. \quad (14)$$

Then, we can get $\mathscr{S}_{\mathscr{K}_2}(k_1) = \{k_2, k_3, 1\} \rhd\lhd X = \mathscr{S}_{\mathscr{K}_2}(k_1)$. Therefore, $\mathscr{S}_{\mathscr{K}_2}$ is a soft deductive system over $\mathscr{S}_{\mathscr{K}}$.

From (i)–(iii), we have $\mathscr{S}_{\mathscr{K}_3} = \mathscr{S}_{\mathscr{K}_1} \widetilde{\cup} \mathscr{S}_{\mathscr{K}_2}$ which is not a soft deductive system of $\mathscr{S}_{\mathscr{K}}$, where $\mathscr{S}_{\mathscr{K}_3}(k_1) = \mathscr{S}_{\mathscr{K}_1}(k_1) \cup \mathscr{S}_{\mathscr{K}_2}(k_1) = \{k_1, k_2, k_3, 1\}$ is not a $\mathscr{S}_{\mathscr{K}}(a)$-deductive system because $k_2 \longrightarrow k_4 = k_1 \in \{k_1, k_2, k_3, 1\}$ and $k_4 \notin \{k_1, k_2, k_3, 1\}$.

3.2. $\mathbb{DSQB}$-$\mathbb{A}$s. We will give the notion of $\mathbb{DSQB}$-$\mathbb{A}$s and investigate homomorphism image of $\mathbb{DSQB}$-$\mathbb{A}$s as indicated below.

Definition 12. Assume that $\mathscr{S}_{\mathscr{K}}$ is a $\mathbb{SQB}$-$\mathbb{A}$ over $\mathscr{X}$. If $\mathscr{S}_{\mathscr{K}}(x)(\forall x \in \mathscr{K})$ is a deductive system of $\mathscr{X}$, then $\mathscr{S}_{\mathscr{K}}$ is called a $\mathbb{DSQB}$-$\mathbb{A}$ over X.

Example 6 (continued from Example 1 (2)). Clearly, $\mathscr{S}_{\mathscr{K}}$ is $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathscr{X}$.

Definition 13

(1) Suppose $\mathscr{X}$ be a $\mathbb{QB}$-$\mathbb{A}$ with the greatest element 1 (i.e., $\mathscr{X}$ just only a poset); for any $x \in \mathscr{X}$, the order of element x is defined as

$$\mathscr{O}(x) = \min\left\{p, q \in N | x \xrightarrow{p} x = 1, x \overset{q}{\rightsquigarrow} x = 1\right\}, \quad (i), \quad (15)$$

where N is a natural number and $x \longrightarrow^p x = (((x \longrightarrow x) \longrightarrow \cdots) \longrightarrow x)$, $x \overset{q}{\rightsquigarrow} x = (((x \rightsquigarrow x) \rightsquigarrow \cdots) \rightsquigarrow x)$.

(2) If $p, q \in N$ does not exist to satisfy the above condition (i), then $x(\forall x \in \mathscr{X})$ is called infinite order.

Remark 4. Assume that $\mathscr{S}_{\mathscr{K}}$ and $\mathscr{S}_{\mathscr{K}_1}$ be two $\mathbb{SQB}$-$\mathbb{A}$s over $\mathscr{X}$ such that $\mathscr{K}_1 \subseteq \mathscr{K} \subseteq \mathscr{X}$. If $\mathscr{S}_{\mathscr{K}}$ is a $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathscr{X}$, then $\mathscr{S}_{\mathscr{K}_1}$ is a $\mathbb{DSQB}$-$\mathbb{A}$.

The converse of Remark 4 does not hold by the following Example 7.

Example 7 (continued from Example 2). We define $\mathscr{S}_{\mathscr{K}}(\forall x \in \mathscr{K})$ (i.e., $\mathscr{K} = \mathscr{X}$) by

$$\mathscr{S}_{\mathscr{K}}(x) = \{y \in \mathscr{X} | \mathscr{O}(x) = \mathscr{O}(y)\}. \quad (16)$$

Then, we get on $\mathscr{S}_{\mathscr{K}}(0) = \mathscr{S}_{\mathscr{K}}(k_3) = \mathscr{S}_{\mathscr{K}}(k_4) = \mathscr{S}_{\mathscr{K}}(k_1) = \{0, k_3, k_4, 1\}$, $\mathscr{S}_{\mathscr{K}}(k_1) = \{k_1\}$, and $\mathscr{S}_{\mathscr{K}}(k_2) = \{k_2\}$. However, $k_3 \longrightarrow k_1 = 0 \in \{0, k_3, k_4, 1\}$ and $k_1 \notin \{0, k_3, k_4, 1\}$ imply that $\mathscr{S}_{\mathscr{K}}$ is not $\mathbb{DSQB}$-$\mathbb{A}$. If we take $\mathscr{K}_1 = \{k_3, k_4, 1\} \subseteq \mathscr{K}$ and we define $\mathscr{S}_{\mathscr{K}_1} = \{y \in \mathscr{X} | \mathscr{O}(x) = \mathscr{O}(y)\}(\forall x \in \mathscr{K}_1)$, then $\mathscr{S}_{\mathscr{K}_1}$ is $\mathbb{DSQB}$-$\mathbb{A}$.

Definition 14. Assume that $\mathscr{S}_{\mathscr{K}}$ is $\mathbb{SQB}$-$\mathbb{A}$ over $\mathscr{X}$ with the greatest element 1. If $\mathscr{S}_{\mathscr{K}}(x) = \mathscr{X}(\forall x \in \mathscr{K})$, then $\mathscr{S}_{\mathscr{K}}$ is called whole $\mathbb{DSQB}$-$\mathbb{A}$.

Example 8. Suppose $\mathscr{X}$ (i.e., $\mathscr{X} = \{0, k_1, k_2, 1\}$) with partial order $0 < k_1 < k_2 < 1$. Now, we show, by Tables 10 and 11, the binary operations $\longrightarrow$ and $\rightsquigarrow$, respectively.

Clearly, $\mathscr{X}$ is a $\mathbb{CQB}$-$\mathbb{A}$. We define $\mathscr{S}_{\mathscr{K}}(\forall x \in \mathscr{K})$ (i.e., $\mathscr{K} = \mathscr{X}$) by

$$\mathscr{S}_{\mathscr{K}}(x) = \{y \in \mathscr{X} | \mathscr{O}(x) = \mathscr{O}(y)\}. \quad (17)$$

From Tables 10 and 11, we can get on $\mathscr{S}_{\mathscr{K}}(x) = \mathscr{X}(\forall x \in \mathscr{K})$. Thus, $\mathscr{S}_{\mathscr{K}}$ is a whole $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathscr{X}$.

Now, we will study homomorphism image of $\mathbb{DSQB}$-$\mathbb{A}$s by the following two theorems.

Theorem 3. *Assume that ψ: $\mathscr{X} \longrightarrow \mathscr{Y}$ be a surjective exact morphism of $\mathbb{QB}$-$\mathbb{A}$ and $\mathscr{X}$ is a $\mathbb{QB}$-$\mathbb{A}$s. If $\mathscr{S}_{\mathscr{K}}$ is a $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathscr{X}$, then $\psi(\mathscr{S}_{\mathscr{K}})$ is also$\mathbb{DSQB}$-$\mathbb{A}$ over $\mathscr{Y}$.*

Proof. Since $\mathscr{S}_{\mathscr{K}}(x)(x \in \mathscr{K})$ is a deductive system of $\mathscr{X}$ and ψ is surjective, then $\psi(\mathscr{S}_{\mathscr{K}})(x) = \psi(\mathscr{S}_{\mathscr{K}}(x))$ is a deductive system of $\mathscr{Y}$ which implies that $\psi(\mathscr{S}_{\mathscr{K}})$ is a $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathscr{X}$. □

Theorem 4. *Assume that ψ: $X \longrightarrow Y$ be a surjective exact morphism of $\mathbb{QB}$-$\mathbb{A}$ and $\mathscr{S}_{\mathscr{K}}$ a $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathscr{X}$. Then,*

Table 10: The binary operation $\longrightarrow$.

$\longrightarrow$	0	k_1	k_2	1
0	1	1	1	1
k_1	k_1	1	1	1
k_2	k_1	k_1	1	1
1	0	k_1	k_2	1

Table 11: The binary operation $\rightsquigarrow$.

$\rightsquigarrow$	0	k_1	k_2	1
0	1	1	1	1
k_1	k_2	1	1	1
k_2	0	k_1	1	1
1	0	k_1	k_2	1

(1) If $\mathcal{S}_{\mathcal{K}}(x) = ker(\psi)$, for all $x \in \mathcal{K}$, then $\psi(\mathcal{S}_{\mathcal{K}})$ is the whole $\mathbb{DSQB}$-$\mathbb{A}$ *over $\mathcal{Y}$*

(2) If is whole $\mathbb{DSQB}$-$\mathbb{A}$ *over $\mathcal{X}$, then $\psi(\mathcal{S}_{\mathcal{K}})$ is the whole* $\mathbb{DSQB}$-$\mathbb{A}$ *over $\mathcal{Y}$*

Proof

(1) Assume that $\mathcal{S}_{\mathcal{K}}(x) = \ker(\psi)$, where $\ker(\psi) = \{x \in \mathcal{X} | \psi(x) = x \longrightarrow x, \quad \psi(x) = x \rightsquigarrow x\}$. Since ψ is surjective, then, from Theorem 3, we have $\psi(\mathcal{S}_{\mathcal{K}})(x) = \psi(\mathcal{S}_{\mathcal{K}}(x)) = \psi(\mathcal{X}) \quad = \mathcal{Y}(x \in \mathcal{K})$. Thus, $\psi(\mathcal{S}_{\mathcal{K}})$ is the whole $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathcal{Y}$.

(2) Clearly, $\mathcal{S}_{\mathcal{K}}(x) = \mathcal{X}$ since $\mathcal{S}_{\mathcal{K}}$ is whole $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathcal{X}(x \in \mathcal{K})$. Thus, $\psi(\mathcal{S}_{\mathcal{K}})(x) = \psi(\mathcal{S}_{\mathcal{K}}(x)) = \psi(\mathcal{X}) = \mathcal{Y}(x \in \mathcal{K})$. By Theorem 3, we have $\psi(\mathcal{S}_{\mathcal{K}})$ is the whole $\mathbb{DSQB}$-$\mathbb{A}$ over $\mathcal{Y}$ □

4. $\mathbb{FSQB}$-$\mathbb{A}$s

We give the definition of $\mathbb{FSQB}$-$\mathbb{A}$s; a concrete example is given to illustrate its derive properties. Furthermore, we study the homomorphism image and preimage of $\mathbb{FSQB}$-$\mathbb{A}$s. Now, we first propose the definition of fuzzy quantum B-algebra (briefly, $\mathbb{FQB}$-$\mathbb{A}$) as indicated below.

Definition 15. We call $\mathbb{FQB}$-$\mathbb{A}$ (or a fuzzy set $\widehat{\mu}$ in $\mathbb{QB}$-$\mathbb{A}$) if it satisfies $(\forall x, y \in \mathcal{X}, \mathcal{X}$ is $\mathbb{QB}$-$\mathbb{A})$:

$$\begin{aligned}\widehat{\mu}(x \longrightarrow y) &\geq \min\{\widehat{\mu}(x), \widehat{\mu}(y)\},\\ \widehat{\mu}(x \rightsquigarrow y) &\geq \min\{\widehat{\mu}(x), \widehat{\mu}(y)\}.\end{aligned} \tag{18}$$

Definition 16. We call $\widehat{\mu}$ is a fuzzy deductive system of $\mathcal{X}$ if it satisfies $(\forall x, y \in \mathcal{X})$:

$$\begin{aligned}\widehat{\mu}(x \longrightarrow x) &\geq \widehat{\mu}(x),\\ \widehat{\mu}(x \rightsquigarrow x) &\geq \widehat{\mu}(x),\\ \widehat{\mu}(y) &\geq \min\{\widehat{\mu}(x \longrightarrow y), \widehat{\mu}(x)\}.\end{aligned} \tag{19}$$

Definition 17. Assume that $\widehat{\mathcal{S}}_{\mathcal{K}}$ be a $\mathbb{FSS}$ over $\mathcal{X}$. Then,

(1) If there exists $\widehat{\mu} \in \mathcal{K}$ such that $\widehat{\mathcal{S}}_{\mathcal{K}}[\mu]$ is a $\mathbb{FQB}$-$\mathbb{A}$ (i.e., fuzzy deductive system) in a $\mathbb{QB}$-$\mathbb{A}$ over $\mathcal{X}$, then $\widehat{\mathcal{S}}_{\mathcal{K}}$ is called a $\widehat{\mathcal{S}}_{\mathcal{K}}$-$\mathbb{A}$ (i.e., fuzzy soft deductive system $\mathbb{FSDS}$) which depends on a parameter set $\widehat{\mu}$ over $\mathcal{X}$

(2) If $\widehat{\mathcal{S}}_{\mathcal{K}}[\mu]$ is a $\mathbb{FQB}$-$\mathbb{A}$ (i.e., fuzzy deductive system) of $\mathcal{X}$ based on all parameters, then we say that $\widehat{\mathcal{S}}_{\mathcal{K}}$ is a $\mathbb{FSQB}$-$\mathbb{A}$ (i.e., $\mathbb{FSDS}$) of $\mathcal{X}$

In the following, a concrete example is given to illustrate Definition 17.

Example 9. Suppose that there are five-class cars:

$$X = \{\text{BMW, Audi, Toyota, Jeep, Cadilac}\}. \tag{20}$$

Let $\oplus$ and $\otimes$ be two soft machines to characterize two cars, defined by the following manner.

BMW$\oplus x$ = Cadilac, forall $x \in \mathcal{X}$,

$$\begin{aligned}
&\text{Audi}\oplus y = \begin{cases}\text{Jeep}, & y = \text{BMW},\\ \text{Cadilac}, & y \in \{\text{Audi, Toyouta, Jeep, Cadilac}\},\end{cases}\\
&\text{Toyota}\oplus z = \begin{cases}\text{Toyota}, & z = \text{BMW},\\ \text{Jeep}, & z = \text{Audi},\\ \text{Cadilac}, & z \in \{\text{Toyota, Jeep, Cadilac}\},\end{cases}\\
&\text{Jeep}\oplus s = \begin{cases}\text{Toyota}, s = \text{BMW},\\ \text{Jeep}, s \in \{\text{Audi, Toyoya}\},\\ \text{Cadilac}, s \in \{\text{Jeep, Cadilac}\},\end{cases}\\
&\text{Cadilac}\oplus t = \begin{cases}\text{BMW}, t = \text{BMW},\\ \text{Audi}, t = \text{Audi},\\ \text{Toyota}, t = \text{Toyota},\\ \text{Jeep}, t = \text{Jeep},\\ \text{Cadilac}, t = \text{Cadilac},\end{cases}
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\text{BMW} \otimes x = \text{Cadilac forall } x \in \mathcal{X},\\
&\text{Audi} \otimes y = \begin{cases}\text{Jeep}, \ y = \text{BMW},\\ \text{Cadilac}, \ y \in \{\text{Audi, Toyouta, Jeep, Cadilac}\},\end{cases}\\
&\text{Toyota} \otimes z = \begin{cases}\text{Jeep}, \ z \in \{\text{BMW, Audi}\},\\ \text{Cadilac}, \ z \in \{\text{Toyouta, Jeep, Cadilac}\},\end{cases}\\
&\text{Jeep} \otimes s = \begin{cases}\text{Audi}, \ s = \text{BMW},\\ \text{Jeep}, \ s \in \{\text{Audi, Toyouta}\},\\ \text{Cadilac}, \ s \in \{\text{Jeep, Cadilac}\},\end{cases}\\
&\text{Cadilac} \otimes t = \begin{cases}\text{BMW}, \ t = \text{BMW},\\ \text{Audi}, \ t = \text{Audi},\\ \text{Toyota}, \ t = \text{Toyota},\\ \text{Jeep}, \ t = \text{Jeep},\\ \text{Cadilac}, \ t = \text{Cadilac}.\end{cases}
\end{aligned} \tag{22}$$

Then, $(\mathscr{X}, \oplus, \otimes, \leq)$ is a $\mathbb{QB}$-$\mathbb{A}$. Now, we consider a set of parameters: $\widehat{\mu}$ =(Excellent, Good, Moderate)$\in\in\mathscr{K}$. Then, we have the following:

(1) We define $\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}]$ over $\mathscr{X}$ (i.e., $\widehat{\mathcal{S}}_{\mathscr{K}}$[Excellent], $\widehat{\mathcal{S}}_{\mathscr{K}}$[Good], and $\widehat{\mathcal{S}}_{\mathscr{K}}$[Moderate] are fuzzy sets) by Table 12.

Therefore, we can see that $\widehat{\mathcal{S}}_{\mathscr{K}}$[Excellent], $\widehat{\mathcal{S}}_{\mathscr{K}}$[Good], and $\widehat{\mathcal{S}}_{\mathscr{K}}$[Moderate] are all $\mathbb{FSQB}$-$\mathbb{A}$s based on parameters "Excellent," "Good," and "Moderate" over $\mathscr{X}$. Thus, $\widehat{\mathcal{S}}_{\mathscr{K}}$ is a $\mathbb{FSQB}$-$\mathbb{A}$ over $\mathscr{X}$.

(2) We define $\widehat{\mathcal{S}}_{\mathscr{K}_1}[\widehat{\mu}]$ over $\mathscr{X}$ (i.e., $\widehat{\mathcal{S}}_{\mathscr{K}_1}$[Excellent], $\widehat{\mathcal{S}}_{\mathscr{K}_1}$[Good], and $\widehat{\mathcal{S}}_{\mathscr{K}_1}$[Moderate] are fuzzy sets) by Table 13.

However, $\widehat{\mathcal{S}}_{\mathscr{K}_1}[\widehat{\mu}]$ is not a $\mathbb{FSQB}$-$\mathbb{A}$ based on a parameter "Excellent" over (HTML translation failed), where $\widehat{\mathcal{S}}_{\mathscr{K}_1}$[Excellent] (Toyota$\oplus$BMW) = $\widehat{\mathcal{S}}_{\mathscr{K}_1}$[Excellent] (Toyota) = $0.1 \ngeq 0.2 = \min\{0.2, 0.4\}$ = $\min\{\widehat{\mathcal{S}}_{\mathscr{K}_1}$ [Excellent] (Audi), $\widehat{\mathcal{S}}_{\mathscr{K}_1}$[Excellent] (BMW)$\}$. Also, we obtain that $\widehat{\mathcal{S}}_{\mathscr{K}_1}[\widehat{\mu}]$ is a $\mathbb{FSQB}$-$\mathbb{A}$ based on both the parameter "Good" and "Moderate" over $\mathscr{X}$.

(3) We define $\widehat{\mathcal{S}}_{\mathscr{K}_2}[\widehat{\mu}]$ over $\mathscr{X}$ (i.e., $\widehat{\mathcal{S}}_{\mathscr{K}_2}$[Excellent] and $\widehat{\mathcal{S}}_{\mathscr{K}_2}$ [Good] are fuzzy sets) by Table 14.

Then, $\widehat{\mathcal{S}}_{\mathscr{K}_2}[\widehat{\mu}]$ is a $\mathbb{FSDS}$ on parameters "Excellent." However, $\widehat{\mathcal{S}}_{\mathscr{K}_2}[\mu]$ is not a fuzzy deductive system of $\mathscr{X}$ based on parameter "Good," where $\widehat{\mathcal{S}}_{\mathscr{K}_2}$[Good](Toyota)= $0.3 < 0.5$ = $\min\{\widehat{\mathcal{S}}_{\mathscr{K}_2}$ [Good] (Jeep$\oplus$Toyota), $\widehat{\mathcal{S}}_{\mathscr{K}_2}$[Good] (Jeep)$\}$.

(4) We define $\widehat{\mathcal{S}}_{\mathscr{K}_3}[\widehat{\mu}]$ over $\mathscr{X}$ (i.e., $\widehat{\mathcal{S}}_{\mathscr{K}_3}$[Excellent] and $\widehat{\mathcal{S}}_{\mathscr{K}_2}$[Moderate] are fuzzy sets) by Table 15.

Then, $\widehat{\mathcal{S}}_{\mathscr{K}_3}[\widehat{\mu}]$ is a $\mathbb{FSDS}$ of $\mathscr{X}$.

Now, we will present several characterizations of $\mathbb{FSQB}$-$\mathbb{A}$s.

By Definition 17, if $\widehat{\mathcal{S}}_{\mathscr{K}}$ is a $\mathbb{FSQB}$-$\mathbb{A}$ of $\mathbb{QB}$-$\mathbb{A}$ over $\mathscr{X}$ based on all parameters, then we say that $\widehat{\mathcal{S}}_{\mathscr{K}}$ is a $\mathbb{FSQB}$-$\mathbb{A}$ of $\mathscr{X}$, that is,

$$\begin{aligned}\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x \longrightarrow y) &\geq \min\left\{\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x), \widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](y)\right\},\\ \widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x \rightsquigarrow y) &\geq \min\left\{\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x), \widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](y)\right\}.\end{aligned} \tag{23}$$

Proposition 1. *Assume $\mathscr{X}$ be a $\mathbb{QB}$-$\mathbb{A}$. If $\widehat{\mathcal{S}}_{\mathscr{K}}$ is $\mathbb{FSQB}$-$\mathbb{A}$ over $\mathscr{X}$, then, for all $t \in [0,1]$, $(\widehat{\mathcal{S}}_{\mathscr{K}})_t \neq \varnothing$ is the subalgebra of $\mathscr{X}$, in which*

$$\left(\widehat{\mathcal{S}}_{\mathscr{K}}\right)_t = \left\{\left(\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}]\right)_t | \widehat{\mu} \in \mathscr{K}\right\}. \tag{24}$$

Proof. Let $(\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}])_t \neq \varnothing$. Then, $\forall x, y \in (\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}])_t$; since $\widehat{\mathcal{S}}_{\mathscr{K}}$ is a $\mathbb{FSQB}$-$\mathbb{A}$, then $\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x) \geq t$, $\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](y) \geq t$. So,

$$\begin{aligned}\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x \longrightarrow y) &\geq \min\left\{\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x \longrightarrow y), \widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x \rightsquigarrow y)\right\}\\ &\geq \min\left\{\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x), \widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](y)\right\} \geq t.\end{aligned} \tag{25}$$

Similarly, we have $\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}](x \rightsquigarrow y) \geq t$. Therefore, $x \longrightarrow y, x \rightsquigarrow y \in (\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}])_t$. This implies that $(\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}])_t$ is the subalgebra of $\mathscr{X}$.

Analogously, we can get Proposition 2 as follows. □

Table 12: Fuzzy sets $\widehat{\mathcal{S}}_{\mathscr{K}}[\widehat{\mu}]$ over $\mathscr{X}$.

$\widehat{\mathcal{S}}_{\mathscr{K}}$	BMW	Audi	Toyota	Jeep	Cadilac
Excellent	0.2	0.2	0.5	0.6	0.8
Good	0.1	0.2	0.3	0.5	0.7
Moderate	0.1	0.1	0.4	0.4	0.6

Table 13: Fuzzy sets $\widehat{\mathcal{S}}_{\mathscr{K}_1}[\widehat{\mu}]$ over $\mathscr{X}$.

$\widehat{\mathcal{S}}_{\mathscr{K}_1}$	BMW	Audi	Toyota	Jeep	Cadilac
Excellent	0.4	0.2	0.1	0.6	0.8
Good	0.2	0.2	0.3	0.5	0.7
Moderate	0.1	0.1	0.4	0.5	0.9

Table 14: Fuzzy sets $\widehat{\mathcal{S}}_{\mathscr{K}_2}[\widehat{\mu}]$ over $\mathscr{X}$.

$\widehat{\mathcal{S}}_{\mathscr{K}_2}$	BMW	Audi	Toyota	Jeep	Cadilac
Excellent	0.2	0.2	0.2	0.2	0.6
Good	0.2	0.2	0.3	0.5	0.7

Table 15: Fuzzy sets $\widehat{\mathcal{S}}_{\mathscr{K}_3}[\widehat{\mu}]$.

$\widehat{\mathcal{S}}_{\mathscr{K}_3}$	BMW	Audi	Toyota	Jeep	Cadilac
Excellent	0.3	0.3	0.3	0.3	0.3
Moderate	0.1	0.1	0.1	0.1	0.7

Proposition 2. *Assume that $\mathcal{S}_{\mathscr{K}_1}$ and $\mathcal{S}_{\mathscr{K}_2}$ are two $\mathbb{FSQB}$-$\mathbb{A}$ over $\mathscr{X}$. Then, $\mathcal{S}_{\mathscr{K}_1} \widetilde{\cap} \mathcal{S}_{\mathscr{K}_2}$ and $\mathcal{S}_{\mathscr{K}_1} \widetilde{\cup} \mathcal{S}_{\mathscr{K}_2}$ are $\mathbb{FSQB}$-$\mathbb{A}$s over $\mathscr{X}$.*

Definition 18. Let (α, β) be a fuzzy soft map from $\mathbb{QB}$-$\mathbb{A}$ over $\mathscr{X}$ to $\mathbb{QB}$-$\mathbb{A}$ over $\mathscr{Y}$. Then,

(1) If α is an exact morphism from $\mathscr{X}$ to $\mathscr{Y}$, then (α, β) is called a $\mathbb{FSQB}$-$\mathbb{A}$ exact morphism from $\mathscr{X}$ to $\mathscr{Y}$

(2) If α is an isomorphism from $\mathscr{X}$ to $\mathscr{Y}$ and β is a bijective from $\mathscr{K}_1$ to $\mathscr{K}_2$, then (α, β) is a called an isomorphism between $\mathbb{FSQB}$-$\mathbb{A}$s

Proposition 3. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two $\mathbb{QB}$-$\mathbb{A}$s. $\mathcal{S}_{\mathscr{K}}$ is a $\mathbb{FSQB}$-$\mathbb{A}$ over $\mathscr{Y}$ and (α, β) a $\mathbb{FSQB}$-$\mathbb{A}$ exact morphism from $\mathscr{X}$ to $\mathscr{Y}$; then, $(\alpha, \beta)^{-1}\mathcal{S}_{\mathscr{K}}$ is $\mathbb{FSQB}$-$\mathbb{A}$ over $\mathscr{X}$.*

Proof. For $\widehat{\mu} \in \beta^{-1}(\mathscr{K})$,

$$\begin{aligned} &\alpha^{-1}(\mathcal{S}_{\mathscr{K}})[\widehat{\mu}](x \longrightarrow y) \\ &\quad = \mathcal{S}_{\mathscr{K}}\beta[\widehat{\mu}](\alpha(x \longrightarrow y)) \\ &\quad = \mathcal{S}_{\mathscr{K}}\beta[\widehat{\mu}](\alpha(x) \longrightarrow \alpha(y)) \\ &\quad \geq \min\{\mathcal{S}_{\mathscr{K}}\beta[\widehat{\mu}]\alpha(x), \mathcal{S}_{\mathscr{K}}\beta[\widehat{\mu}]\alpha(y)\} \\ &\quad = \min\{\alpha^{-1}(\mathcal{S}_{\mathscr{K}})[\widehat{\mu}](x), \alpha^{-1}(\mathcal{S}_{\mathscr{K}})[\widehat{\mu}](x)\}. \end{aligned} \tag{26}$$

Consequently, $(\alpha,\beta)^{-1}\mathcal{S}_{\mathscr{K}}$ is a $\mathbb{FSQB}$-$\mathbb{A}$ over $\mathscr{X}$.
Similarly, we can get Proposition 4 as follows. □

Proposition 4. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two $\mathbb{QB}$-$\mathbb{A}$s. $\mathcal{S}_{\mathscr{K}}$ is a $\mathbb{FSQB}$-$\mathbb{A}$s over $\mathscr{X}$ and (α,β) a $\mathbb{FSQB}$-$\mathbb{A}$s isomorphism from $\mathscr{X}$ to $\mathscr{Y}$; then, $(\alpha,\beta)\mathcal{S}_{\mathscr{K}}$ is the $\mathbb{FSQB}$-$\mathbb{A}$s over $\mathscr{Y}$.*

5. Conclusions

In this paper, we introduce the concept of $\mathbb{SQB}$-$\mathbb{A}$s, and some examples are given to illustrate this definition. Also, we investigate the union and intersection operations between two $\mathbb{SQB}$-$\mathbb{A}$s and give some conditions for the operation holds. With the help of the definition of $\mathbb{SQB}$-$\mathbb{A}$s, we define soft deductive systems of $\mathbb{SQB}$-$\mathbb{A}$s and then investigate the relation between them. As a further step, we define $\mathbb{DSQB}$-$\mathbb{A}$s and investigate the homomorphism image of $\mathbb{DSQB}$-$\mathbb{A}$s. Moreover, we define $\mathbb{FSQB}$-$\mathbb{A}$s. Finally, a concrete example is given to illustrate its derive properties; besides, homomorphism image and preimage of $\mathbb{FSQB}$-$\mathbb{A}$s are discussed.

As a future work, it makes sense to apply $\mathbb{SQB}$-$\mathbb{A}$s to medical diagnosis (for example, [25, 26]) in practice. Furthermore, it would be interesting if we study hybrid soft lattice-ordered quantum B-algebras.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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