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Research Article

Some Inequalities of Generalized p-Convex Functions concerning Raina's Fractional Integral Operators

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Convex functions play an important role in pure and applied mathematics specially in optimization theory. In this paper, we will deal with well-known class of convex functions named as generalized p-convex functions. We develop Hermite–Hadamard-type inequalities for this class of convex function via Raina's fractional integral operator.

1. Introduction

The subject of fractional calculus got rapid development in the last few decades. As a matter of fact, fractional calculus give more accuracy to model applied problems in engineering and other sciences then classical calculus. In order to model recent complicated problems, scientists are using fractional inequalities and fractional equations. For more on this, we refer the books [1, 2]. The models with fractional calculus have been applied successfully in ecology, aerodynamics, physics, biochemistry, environmental science, and many other branches. For more about fractional calculus and models, we refer [3–5].

Fractional integral inequalities are considered one of the important tools to study the behavior and properties of solutions of various fractional problems [6–14]. There are many interesting generalization of fractional derivatives as per need of practical problems or some theoretical approach, for example, Raina's fractional integral operator, Caputo-Fabrizio fractional integral, and extended Caputo-Fabrizio fractional integral. For recent work on it, we refer [15–20].

Convex functions also play an important role in pure and applied mathematics specially in optimization theory. Classical convexity does not fulfil needs of modern mathematics; therefore, several generalizations of convex functions are presented in literature. s-convex function [21], M-convex functions [22], and h-convex function [23] are some examples of generalized convex functions. It is always interesting to study properties of some generalized convex function in the setting of fractional integral operators. This paper is an effort in this direction. In this paper, we study the p-convex functions and present some of its properties in the setting of Raina's fractional integral operators.

The paper is organized as follows. In Section 2, we present some basic definition and properties of Raina's fractional integral operator. Section 3 is devoted for Hermite–Hadamard type inequalities for generalized p-convex functions in terms of Raina's fractional integral operators.

2. Preliminaries

Here, we present some basic definitions and known results.

Definition 1 (convex function). A function $\phi: I \longrightarrow R$ is said to be convex function if the following inequality holds:

$$\phi(\vartheta x + (1-\vartheta)y) \leq \vartheta\phi(x) + (1-\vartheta)\phi(y), \tag{1}$$

for $\forall x, y \in I$ and $\vartheta \in [0,1]$.

One of the novel generalization of convexity is η-convexity introduced by M. R. Delavar and S. S. Dragomir in [24].

Definition 2. A function $\phi: I \longrightarrow R$ is said to be generalized convex function with respect to $\eta: A \times A \longrightarrow B$ for appropriate $A, B \subseteq R$ if

$$\phi(\vartheta x + (1-\vartheta)y) \leq \phi(y) + \vartheta\eta(\phi(x), \phi(y)), \tag{2}$$

for $\forall x, y \in I$ and $\vartheta \in [0,1]$

In [25], Zhang and Wan gave definition of p-convex function as follows.

Definition 3. Let I be a p-convex set. A function $f: I \longrightarrow \mathbb{R}$ is said to be p-convex function if

$$\phi\left(\left[\vartheta x^p + (1-\vartheta)y^p\right]^{1/p}\right) \leq \vartheta\phi(x) + (1-\vartheta)\phi(y), \tag{3}$$

holds, for all $x, y \in I$ and $\vartheta \in [0,1]$.

In [26], the authors gave the definition of the generalized p-convex function as follows.

Definition 4. A function $\phi: I \longrightarrow R$ is said to be generalized p-convex function with respect to $\eta: A \times A \longrightarrow B$ for appropriate $A, B \subseteq R$ if

$$\phi\left(\vartheta x^p + (1-\vartheta)y^p\right)^{1/p} \leq \phi(y) + \vartheta\eta(\phi(x), \phi(y), \tag{4}$$

for $\forall x, y \in I$, $p > 0$ and $\vartheta \in [0,1]$.

For some important properties and results about generalized p-convexity, see [26]. Moreover, in [26], the following Hermite–Hadamard type inequality for p-convex functions can be found.

Theorem 1. *Let $\phi: I \longrightarrow \mathbb{R}$ be generalized p-convex function for $\xi_1, \xi_2 \in I$ with condition $\xi_1 < \xi_2$; then, we obtain the inequality*

$$\begin{aligned}
&\phi\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{1/p} - \frac{p}{2\left(\xi_2^p - \xi_1^p\right)}\int_{\xi_1}^{\xi_2} x^{p-1}\eta\left(\phi\left(\xi_1^p + \xi_2^p - x^p\right)^{1/p}, \phi(x)\right)\mathrm{d}x \\
&\leq \frac{p}{\xi_2^p - \xi_1^p}\int_{\xi_1}^{\xi_2} x^{p-1}\phi(x)\mathrm{d}x \leq \frac{\phi(\xi_1) + \phi(\xi_2)}{2} + \frac{1}{4}\left[\eta(\phi(\xi_1), \phi(\xi_2)) + \eta(\phi(\xi_2), \phi(\xi_1))\right].
\end{aligned} \tag{5}$$

In [27], the author introduced a class of functions defined formally by

$$\mathscr{F}_{\rho,\lambda}^{\sigma} z = \mathscr{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\cdots} z = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} z^k, \tag{6}$$

where $\rho, \lambda > 0$, $|z| < \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), and $\sigma = (\sigma(1), \ldots, \sigma(k), \ldots)$ is a bounded sequence of positive real numbers.

Using (6), in [28], the authors defined the following left-sided and right-sided fractional integral operators, respectively:

$$\begin{aligned}
\left(\mathscr{J}_{\rho,\lambda,\xi_1^+;w}^{\sigma}\phi\right)(z) &= \int_{\xi_1}^{z} (z-\vartheta)^{(\lambda-1)} \mathscr{F}_{\rho,\lambda}^{\sigma} z\left[w(z-\vartheta)^{\rho}\right]\phi(\vartheta)\mathrm{d}\vartheta, \quad (z > \xi_1), \\
\left(\mathscr{J}_{\rho,\lambda,b^-;w}^{\sigma}\phi\right)(z) &= \int_{\xi_1}^{z} (\vartheta-z)^{(\lambda-1)} \mathscr{F}_{\rho,\lambda}^{\sigma} z\left[w(\vartheta-z)^{\rho}\right]\phi(\vartheta)\mathrm{d}\vartheta, \quad (z < \xi_2),
\end{aligned} \tag{7}$$

where $\rho, \lambda > 0, \omega \in \mathbb{R}$, and ϕ is such that the integral on the right side exits.

It is easy to verify that $(\mathscr{J}_{\rho,\lambda,\xi_1^+;w}^{\sigma}\phi)(z)$ and $(\mathscr{J}_{\rho,\lambda,\xi_2^-;w}^{\sigma}\phi)(z)$ are bounded integral operators on $L_p(\xi_1, \xi_2)$, $(1 \leq p \leq \infty)$ if

$$\mathfrak{M} := \mathscr{F}_{\rho,\lambda+1}^{\sigma}\left[w(\xi_2 - \xi_1)^{\rho}\right] \leq \infty. \tag{8}$$

In fact, for $\phi \in L_p(\xi_1, \xi_2)$, we have

$$\begin{aligned}
\left\|\left(\mathscr{J}_{\rho,\lambda,\xi_1^+;w}^{\sigma}\phi\right)\right\|_p &\leq \mathfrak{M}\|\phi\|_p, \\
\left\|\left(\mathscr{J}_{\rho,\lambda,\xi_2^-;w}^{\sigma}\phi\right)\right\|_p &\leq \mathfrak{M}\|\phi\|_p,
\end{aligned} \tag{9}$$

where

$$\|\phi\|_p = \left(\int_{\xi_1}^{\xi_2} |\phi(z)|^p \mathrm{d}z\right)^{1/p}. \tag{10}$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Let $\phi \in L[\xi_1, \xi_2]$. The right-hand side and left-hand side Riemann–Liouville fractional integral of order $\alpha > 0$ with $\xi_2 > \xi_1 > 0$ are defined by

$$J^{\alpha}_{\xi_1+}\phi(z)=\frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{z}(z-k)^{\alpha-1}\phi(k)\mathrm{d}k,\quad x>\xi_1,$$
$$J^{\alpha}_{\xi_2-}\phi(z)=\frac{1}{\Gamma(\alpha)}\int_{z}^{\xi_2}(k-z)^{\alpha-1}\phi(k)\mathrm{d}k,\quad z<\xi_2, \tag{11}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined as $\Gamma(\alpha)=\int_0^{\infty}e^{-k}k^{\alpha-1}\mathrm{d}k$.

Lemma 1 (see [29, 30]). *Let $\lambda,\rho>0,\omega\in\mathbb{R}$, and σ be a sequence of nonnegative real numbers. Let ϕ: $[\xi_1,\xi_2]\longrightarrow\mathbb{R}$ be a differentiable mapping on (ξ_1,ξ_2) with $\xi_1<\xi_2$ and $\lambda>0$. If $\phi'\in L[\xi_1;\xi_2]$, the following equality for the fractional integral operator holds:*

$$\begin{aligned}&\frac{\phi(\xi_1)+\phi(\xi_2)}{2}-\frac{1}{2(\xi_2-\xi_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(\xi_2-\xi_1)^{\rho}]}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]\\&=\frac{\xi_2-\xi_1}{2(\xi_2-\xi_1)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(\xi_2-\xi_1)^{\rho}]}\left[\int_0^1(1-\vartheta)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(\xi_2-\xi_1)^{\rho}(1-\vartheta)^{\rho}]\phi'(\vartheta\xi_1+(1-\vartheta)\xi_2)\mathrm{d}\vartheta\right.\\&\left.-\int_0^1(\vartheta)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(\xi_2-\xi_1)^{\rho}(\vartheta)^{\rho}]\phi'(\vartheta\xi_1+(1-\vartheta)\xi_2)\mathrm{d}\vartheta.\right.\end{aligned} \tag{12}$$

3. Main Results

In this section, we establish new Hermite–Hadamard type inequalities for generalized p-convex functions in terms of Raina's fractional integral operators.

Theorem 2. *Let ϕ: $I\longrightarrow\mathbb{R}$ be generalized p-convex function and provided $\eta(.,.)$ is bounded from above on $\phi(I)\times\phi(I)$ and $\phi\in L[\xi_1,\xi_2]$ with $\xi_1<\xi_2$ and $p>0$. Then, following fractional integral inequality holds:*

$$\begin{aligned}\phi\left(\left[\frac{\xi_1^p+\xi_2^p}{2}\right]^{1/p}\right)-N_{\eta}&\le\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]},\\&\le\frac{\phi(\xi_1)+\phi(\xi_2)}{2}+\frac{\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}M_{\eta},\end{aligned} \tag{13}$$

where $\sigma_1(k)=\sigma(k)(k\rho+\lambda)$, for all $k=0,1,2,\ldots$ and N_{η} and M_{η} are bounds of ϕ.

Proof. From inequality (6), we have

$$\phi\left(\frac{x^p+y^p}{2}\right)^{1/p}-\frac{N_{\eta}}{2}\le\frac{\phi(x)+\phi(y)}{2}+\frac{N_{\eta}}{2}, \tag{14}$$

where N_{η} are bounds of ϕ. Substitute $x^P=\vartheta\xi_1^p+(1-\vartheta)\xi_2^p$ and $y^p=(1-\vartheta)\xi_1^p+\vartheta\xi_2^p$; then, (6) can be written as

$$2\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p}-N_{\eta}\le\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}+\phi\left((1-\vartheta)\xi_1^p+\vartheta\xi_2^p\right)^{1/p}+N_{\eta}. \tag{15}$$

Multiplying both sides by $\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(\xi_2^p-\xi_1^p)^{\rho}\vartheta^{\rho}]$, we obtain

$$\begin{aligned}&\left[2\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p}-N_{\eta}\right]\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\le\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}\\&+\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left((1-\vartheta)\xi_1^p+\vartheta\xi_2^p\right)^{1/p}+\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]N_{\eta}.\end{aligned} \tag{16}$$

Integrate over $\vartheta \in [0,1]$, we obtain

$$\left[2\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p}-N_\eta\right]\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]\le\int_0^1\vartheta^{\lambda-1}\mathcal{F}^\sigma_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}\mathrm{d}\vartheta$$
$$+\int_0^1\vartheta^{\lambda-1}\mathcal{F}^\sigma_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]\phi\left((1-\vartheta)\xi_1^p+\vartheta\xi_2^p\right)^{1/p}\mathrm{d}\vartheta+\int_0^1\vartheta^{\lambda-1}\mathcal{F}^\sigma_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]N_\eta\mathrm{d}\vartheta. \tag{17}$$

With the convenient change of the variable, we can observe that

$$\begin{aligned}
&\int_0^1\vartheta^{\lambda-1}\mathcal{F}^\sigma_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}\mathrm{d}\vartheta\\
&=\frac{-p}{\xi_2^p-\xi_1^p}\int_{\xi_2}^{\xi_1}\left(\frac{\xi_2^p-x^p}{\xi_2^p-\xi_1^p}\right)^{\lambda-1}\mathcal{F}^\sigma_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\left(\frac{\xi_2^p-x^p}{\xi_2^p-\xi_1^p}\right)^\rho\right]\phi(x)x^{p-1}\mathrm{d}x\\
&=\frac{p}{\left(\xi_2^p-\xi_1^p\right)^\lambda}\int_{\xi_1}^{\xi_2}\left(\xi_2^p-x^p\right)^{\lambda-1}\mathcal{F}^\sigma_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\left(\frac{\xi_2^p-x^p}{\xi_2^p-\xi_1^p}\right)^\rho\right]\phi(x)x^{p-1}\mathrm{d}x\\
&=\frac{p}{\left(\xi_2^p-\xi_1^p\right)^\lambda}\left(\mathcal{J}^\sigma_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2).
\end{aligned} \tag{18}$$

Similarly, the second integral can be written as

$$\int_0^1\vartheta^{\lambda-1}\mathcal{F}^\sigma_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\vartheta^\rho\right]\phi\left((1-\vartheta)\xi_1^p+\vartheta\xi_2^p\right)^{1/p}\mathrm{d}\vartheta=\frac{p}{\left(\xi_2^p-\xi_1^p\right)^\lambda}\left(\mathcal{J}^\sigma_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1). \tag{19}$$

Now, equation (17) becomes

$$\left[2\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p}-N_\eta\right]\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]\le\frac{p}{\left(\xi_2^p-\xi_1^p\right)^\lambda}\left[\left(\mathcal{J}^\sigma_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2)+\left(\mathcal{J}^\sigma_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1)\right]+\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]N_\eta$$

$$\phi\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{1/p}-N_\eta\le\frac{p}{2\left(\xi_2^p-\xi_1^p\right)^\lambda\mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^\rho\right]}\left[\left(\mathcal{J}^\sigma_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2)+\left(\mathcal{J}^\sigma_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1)\right], \tag{20}$$

which is the left-hand side of inequality (13). To prove right-hand side of (13), using the Definition 4 of generalized p-convex function,

$$\begin{aligned}\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}&\le\phi(\xi_2)+\vartheta\eta(\phi(\xi_1),\phi(\xi_2)),\\ \phi\left(\vartheta\xi_2^p+(1-\vartheta)\xi_1^p\right)^{1/p}&\le\phi(\xi_1)+\vartheta\eta(\phi(\xi_2),\phi(\xi_1)).\end{aligned}\tag{21}$$

Multiplying both inequalities by $\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(\xi_2^p-\xi_1^p)^{\rho}\vartheta^{\rho}]$ and then adding, we obtain

$$\begin{aligned}&\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left(\vartheta\xi_1^p+(1-\vartheta)\xi_2^p\right)^{1/p}+\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\phi\left(\vartheta\xi_2^p+(1-\vartheta)\xi_1^p\right)^{1/p}\\ &\le\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\left[\phi(\xi_2)+\vartheta\eta(\phi(\xi_1),\phi(\xi_2))\right]+\vartheta^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\vartheta^{\rho}\right]\left[\phi(\xi_1)+\vartheta\eta(\phi(\xi_2),\phi(\xi_1))\right].\end{aligned}\tag{22}$$

Integrate over $\vartheta\in[0,1]$, we obtain

$$\begin{aligned}&\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1)\right]}{\left(\xi_2^p-\xi_1^p\right)^{\lambda}}\le[\phi(a)+\phi(\xi_2)]\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]\\ &+\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]\left[\eta(\phi(\xi_1),\phi(\xi_2))+\eta(\phi(\xi_2),\phi(\xi_1))\right]\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\\ &\le\frac{[\phi(\xi_1)+\phi(\xi_2)]}{2}+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}{\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}M_{\eta},\end{aligned}\tag{23}$$

where $\sigma_1(k)=\sigma(k)(k\rho+\lambda)$. This completes the proof.

Corollary 1. *Let ϕ: $I\longrightarrow\mathbb{R}$ be generalized p-convex function and provided $\eta(.,.)$ is bounded from above on $\phi(I)\times\phi(I)$ and $\phi\in L[a,\xi_2]$ with $a<\xi_2$ and $p>0$. Then, the following inequality holds:*

$$\begin{aligned}\phi\left(\left[\frac{\xi_1^p+\xi_2^p}{2}\right]^{1/p}\right)-N_{\eta}&\le\frac{p\Gamma(\alpha+1)\left[\left(I^{\alpha}_{\xi_1^+}\phi\right)(\xi_2)+\left(I^{\alpha}_{\xi_2^-}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\alpha}},\\ &\le\frac{\phi(\xi_1)+\phi(\xi_2)}{2}+\frac{\alpha}{\alpha+1}M_{\eta},\end{aligned}\tag{24}$$

where N_{η} and M_{η} are bounds of ϕ.

Proof. By taking $\lambda=\alpha$, $\sigma=(1,0,0,\ldots)$, $w=0$, and $p=1$, we obtain

$$\begin{aligned}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]&=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}=\frac{1}{\Gamma(\alpha+1)},\\ \mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]&=\frac{\alpha}{\Gamma(\alpha+2)},\\ \left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(x)&=I^{\alpha}_{\xi_1^+}\phi x,\\ \left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(x)&=I^{\alpha}_{\xi_2^-}\phi x.\end{aligned}\tag{25}$$

Making the substitution in (13), we obtain (26).

Remark 1. In Corollary 1, if we take $\eta(x,y)=x-y$, $\lambda=\alpha$, $\sigma=(1,0,0,\ldots)$, $w=0$, and $p=1$, then we get Theorem 1.4 of [29, 31].

Theorem 3. *Let ϕ: $[\xi_1,\xi_2]\longrightarrow\mathbb{R}$ be a differentiable function on (ξ_1,ξ_2) with $\xi_1<\xi_2$. If $|\phi'|$ is a generalized p-convex function on $[\xi_1,\xi_2]$, then the following inequality for fractional integral operator holds:*

$$\begin{aligned}&\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2}-\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\right|\\ &\le\frac{\left(\xi_2^p-\xi_1^p\right)\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\left[2|\phi'(\xi_2)|+\eta\left(|\phi'(\xi_1)|,|\phi'(\xi_2)|\right)\right],\end{aligned}\tag{26}$$

where $\sigma_1(k) = \sigma(k) = (1 - (1/2)^{k\rho+\lambda})$.

Proof. Using Lemma 1 and definition of generalized p-convexity of $|\phi'|$, we have

$$\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2} - \frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p - \xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\right| = \frac{\left(\xi_2^p - \xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\Bigg|\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p - \xi_1^p\right)^{k\rho}\right]$$
$$\int_0^1 (1-\vartheta)^{k\rho+\lambda}\phi'\left(\vartheta\xi_1^p + (1-\vartheta)\xi_2^p\right)^{1/2}\mathrm{d}\vartheta - \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p - \xi_1^p\right)^{k\rho}\right]\int_0^1 \vartheta^{k\rho+\lambda}\phi'\left(\vartheta\xi_1^p + (1-\vartheta)\xi_2^p\right)^{1/p}\mathrm{d}\vartheta\Bigg|$$
$$= \frac{\left(\xi_2^p - \xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p - \xi_1^p\right)^{k\rho}\right] \times \left|\int_0^1\left((1-\vartheta)^{k\rho+\lambda} - \vartheta^{k\rho+\lambda}\right)\left(\phi'\left(\vartheta\xi_1^p + (1-\vartheta)\xi_2^p\right)\right|\right. \tag{27}$$
$$\le \frac{\left(\xi_2^p - \xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p - \xi_1^p\right)^{k\rho}\right] \times \left[\int_0^{1/2}\left((1-\vartheta)^{k\rho+\lambda} - \vartheta^{k\rho+\lambda}\right)\left[\left|\phi'(\xi_2)\right| + \vartheta\eta\left(\left|\phi'(\xi_1)\right|, \left|\phi'(\xi_2)\right|\right)\right]\mathrm{d}\vartheta\right]$$
$$+\left[\int_{1/2}^1\left(\vartheta^{k\rho+\lambda} - (1-\vartheta)^{k\rho+\lambda}\right)\left[\left|\phi'(\xi_2)\right| + \vartheta\eta\left(\left|\phi'(\xi_1)\right|, \left|\phi'(\xi_2)\right|\right)\right]\mathrm{d}\vartheta\right].$$

It is easy to verify that

$$\int_0^{1/2}\left((1-\vartheta)^{k\rho+\lambda} - \vartheta^{k\rho+\lambda}\right)\mathrm{d}\vartheta = \frac{1-(1/2)^{k\rho+\lambda}}{k\rho+\lambda+1},$$
$$\int_0^{1/2}\left((1-\vartheta)^{k\rho+\lambda} - \vartheta^{k\rho+\lambda}\right)\vartheta\mathrm{d}\vartheta = \frac{1/2}{k\rho+\lambda+1} - \frac{1}{k\rho+\lambda+2}, \tag{28}$$
$$\int_{1/2}^1\left(\vartheta^{k\rho+\lambda+1} - \vartheta(1-\vartheta)^{k\rho+\lambda}\right)\mathrm{d}\vartheta = \frac{1}{k\rho+\lambda+2} - \frac{1}{2(k\rho+\lambda+1)}.$$

Equation (27) becomes

$$\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2} - \frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p - \xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\right| \le \frac{\left(\xi_2^p - \xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p - \xi_1^p\right)^{k\rho}\right]$$
$$\times\left[2\left|\phi'(\xi_2)\right|\frac{1-(1/2)^{k\rho+\lambda}}{k\rho+\lambda+1} + \eta\left(\left|\phi'(\xi_1)\right|, \left|\phi'(\xi_2)\right|\right)\frac{1-(1/2)^{k\rho+\lambda}}{k\rho+\lambda+1}\right]. \tag{29}$$

Finally, we can write it as

$$\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2} - \frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p - \xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\right| \le, \frac{\left(\xi_2^p - \xi_1^p\right)\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p - \xi_1^p\right)^{\rho}\right]}\left[2\left|\phi'(\xi_2)\right| + \eta\left(\left|\phi'(\xi_1)\right|, \left|\phi'(\xi_2)\right|\right)\right], \tag{30}$$

where $\sigma_1(k) = \sigma(k) = (1 - (1/2)^{k\rho+\lambda})$, which is our required result.

Corollary 2. *Let ϕ: $[\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) with $\xi_1 < \xi_2$. If $|\phi'|$ is a generalized p-convex function on $[\xi_1, \xi_2]$; then, the following inequality for fractional integral operator holds:*

$$\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2} - \frac{p\Gamma(\alpha+1)\left[\left(I^{\alpha}_{\xi_2^-}\phi\right)(\xi_1)+\left(I^{\alpha}_{\xi_1^+}\phi\right)(\xi_2)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\alpha}}\right| \leq \frac{\left(\xi_2^p-\xi_1^p\right)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[2|\phi'(\xi_2)| + \eta\left(|\phi'(\xi_1)|, |\phi'(\xi_2)|\right)\right]. \tag{31}$$

Proof. By taking $\lambda = \alpha$, $\sigma = (1, 0, 0, \ldots)$, $w = 0$, and $p = 1$, we obtain

$$\begin{aligned}
\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right] &= \frac{1}{\Gamma(\alpha+1)},\\
\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right] &= \frac{1-1/2^{\alpha}}{\Gamma(\alpha+2)},\\
\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(x) &= I^{\alpha}_{\xi_1^+}\phi x,\\
\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w}\phi\right)(x) &= I^{\alpha}_{\xi_2^-}\phi x.
\end{aligned} \tag{32}$$

Making the substitution in (35), we obtain (31).

Remark 2. In Corollary 2, if we take $\eta(x, y) = x - y$, $\lambda = \alpha$, $\sigma = (1, 0, 0, \ldots)$, $w = 0$, and $p = 1$, then we obtain Theorem 1.5 in [29, 31].

Theorem 4. *Let ϕ: $[\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) with $\xi_1 < \xi_2$. If $|\phi'|^{q_1}$ is a generalized p-convex function on $[\xi_1, \xi_2]$ with $q_1 = p_1/p_1 + 1$ for some fixed $p_1 > 0$, then the following inequality for fractional integral operator holds:*

$$\begin{aligned}
&\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2} - \frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\right|\\
&\leq \frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma_1(k)}{\Gamma(k\rho_1+\lambda_1+1)}\left[w^k\left(\left(\xi_2^p-\xi_1^p\right)^{1/p}\right)^{k\rho_1}\right]\\
&\times\left(\left[\frac{|\phi'(\xi_2)|^{q_1}}{2} + \vartheta\frac{\eta\left(|\phi'(\xi_1)|^{q_1}, |\phi'(\xi_2)|^{q_1}\right)}{8}\right]d\vartheta\right)^{1/q_1}\\
&+\left(\left[\frac{|\phi'(\xi_2)|^{q_1}}{2} + \vartheta\frac{\eta\left(|\phi'(\xi_1)|^{q_1}, |\phi'(\xi_2)|^{q_1}\right)}{8}\right]d\vartheta\right)^{1/q_1},
\end{aligned} \tag{33}$$

where $\rho_1 = \rho p$, $\lambda_1 = \lambda p$, and

$$\sigma_1(k) = \sigma(k)\left(\frac{1-(1/2)^{(k\rho+\lambda)p_1+1}}{(k\rho+\lambda+1)p_1+1}\right)^{1/p_1}, \tag{34}$$

for all $k = 0, 1, 2 \ldots$.

Proof. Using Lemma 1, definition of generalized p-convexity of ϕ, and Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{\phi(\xi_1)+\phi(\xi_2)}{2}-\frac{p\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_2^-;w}\phi\right)(\xi_2)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\xi_1^+;w}\phi\right)(\xi_1)\right]}{2\left(\xi_2^p-\xi_1^p\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\right| \\
&\le\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right] \\
&\quad\times\left[\left(\int_0^{1/2}\left((1-\vartheta)^{k\rho+\lambda}-\vartheta^{k\rho+\lambda}\right)^{p_1}\mathrm{d}\vartheta\right)^{1/p_1}\left(\int_0^{1/2}\left[\left|\phi'(\xi_2)\right|^{q_1}+\vartheta\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)\right]\mathrm{d}\vartheta\right)^{1/q_1}\right] \\
&\quad+\left(\left[\int_{\frac{1}{2}}^1\left(\vartheta^{k\rho+\lambda}-(1-\vartheta)^{k\rho+\lambda}\right)^{p_1}\mathrm{d}\vartheta\right)^{1/p_1}\left(\int_{1/2}^1\left[\left|\phi'(\xi_2)\right|^{q_1}+\vartheta\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)\right]\mathrm{d}\vartheta\right)^{1/q_1}\right] \\
&\le\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right] \\
&\quad\times\left[\left(\int_0^{1/2}\left((1-\vartheta)^{(k\rho+\lambda)p_1}-\vartheta^{(k\rho+\lambda)p_1}\right)\mathrm{d}\vartheta\right)^{1/p_1}\left(\int_0^{1/2}\left[\left|\phi'(\xi_2)\right|^{q_1}+\vartheta\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)\right]\mathrm{d}\vartheta\right)^{1/q_1}\right] \\
&\quad+\left(\left[\int_{1/2}^1\left(\vartheta^{(k\rho+\lambda)p_1}-(1-\vartheta)^{(k\rho+\lambda)p_1}\right)d\vartheta\right)^{1/p_1}\left(\int_{1/2}^1\left[\left|\phi'(\xi_2)\right|^{q_1}+\vartheta\eta\left(\left|\phi'(a)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)\right]\mathrm{d}\vartheta\right)^{1/q_1}\right] \\
&\le\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}\left[w^k\left(\xi_2^p-\xi_1^p\right)^{k\rho}\right]\left(\frac{1-(1/2)^{(k\rho+\lambda)p_1+1}}{(k\rho+\lambda+1)p_1+1}\right)^{1/p_1} \\
&\quad\times\left(\left[\frac{\left|\phi'(\xi_2)\right|^{q_1}}{2}+\vartheta\frac{\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)}{8}\right]\mathrm{d}\vartheta\right)^{1/q_1}+\left(\left[\frac{\left|\phi'(\xi_2)\right|^{q_1}}{2}+\vartheta\frac{\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)}{8}\right]\mathrm{d}\vartheta\right)^{1/q_1} \\
&\le\frac{\left(\xi_2^p-\xi_1^p\right)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\xi_2^p-\xi_1^p\right)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma_1(k)}{\Gamma(k\rho_1+\lambda_1+1)}\left[w^k\left(\left(\xi_2^p-\xi_1^p\right)^{1/p}\right)^{k\rho_1}\right] \\
&\quad\times\left(\left[\frac{\left|\phi'(\xi_2)\right|^{q_1}}{2}+\vartheta\frac{\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)}{8}\right]\mathrm{d}\vartheta\right)^{1/q_1}+\left(\left[\frac{\left|\phi'(\xi_2)\right|^{q_1}}{2}+\vartheta\frac{\eta\left(\left|\phi'(\xi_1)\right|^{q_1},\left|\phi'(\xi_2)\right|^{q_1}\right)}{8}\right]\mathrm{d}\vartheta\right)^{1/q_1},
\end{aligned}
\tag{35}
$$

which completes the proof.

Data Availability

All data required for this research are included within the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Changyue Chen wrote the final version of this paper, verified the results, and arranged the funding for this paper, Muhammad Shoaib Saleem proposed the problem, proved the results, and supervised the work, and Muhammad Sajid Zahoor wrote the first version of the paper.

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