Research Article

Principal Primitive Ideals in Quadratic Orders and Pell's Equations

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In this paper, we introduce a method of determining whether the primitive ideal is principal in a real quadratic order, depending on the solvability of Pell's equation.

1. Introduction

Let $d > 1$ be square-free integer and n a nonzero integer. Then, the equation $x^2 - dy^2 = n$ is known as Generalized Pell's equation. This equation has been studied by many authors (cf. [2, 4, 6–8]). In [1, 3, 5], this equation has been applied to study the class numbers of real cyclotomic fields.

In this work, we give necessary and sufficient condition to solve $|x^2 - dy^2| = n$, depending on the ideals in orders of the real quadratic fields. We also introduce the formula of the opposite ideal for every solution of this equation, in special cases of d and n .

2. Preliminaries

A nonsquare integer $\Delta \in \mathbb{Z}$ is called a quadratic discriminant if $\Delta \equiv 0$ or $1 \pmod{4}$, we set $\omega_\Delta = (\sigma_\Delta + \sqrt{\Delta})/2$, where $\sigma_\Delta =$

$$\begin{cases} 0, & \text{if } \Delta \equiv 0 \pmod{4} \\ 1, & \text{if } \Delta \equiv 1 \pmod{4} \end{cases} \text{ and } \mathcal{O}_\Delta = [1, \omega_\Delta] = \mathbb{Z}[\omega_\Delta] = \left\{ \frac{u + v\sqrt{\Delta}}{2} \mid u, v \in \mathbb{Z}, \quad u \equiv v\Delta \pmod{2} \right\}. \quad (1)$$

We call ω_Δ the basic number and $\mathcal{O}_\Delta$ the order of the quadratic discriminant Δ .

Now, we investigate ideals. Let Δ be a quadratic discriminant. An ideal $I \subset \mathcal{O}_\Delta$ is called $\mathcal{O}_\Delta$ -primitive if $e^{-1}I \not\subset \mathcal{O}_\Delta$ for all $e \in \mathbb{N}_{\geq 2}$. For a nonzero ideal I of $\mathcal{O}_\Delta$, we call $N(I) = (\mathcal{O}_\Delta : I)$ the absolute norm of I . If α is a nonzero of $\mathcal{O}_\Delta$, then $N(\langle \alpha \rangle) = |N(\alpha)|$.

3. Main Results

In this section, we will prove our main results. As a start, we record the following theorem.

Theorem 1. Let $\Delta = 4d \in \mathbb{N}$ be a quadratic discriminant. Suppose that $c = 1$ if $d \equiv 3 \pmod{4}$, and $c = 2$ if $d \equiv 0 \pmod{8}$, $d = c^2ab$, where $a, b \in \mathbb{N}$, and a is odd and square-free. Then,

- (1) $I = \langle 2ca, ca + \sqrt{\Delta} \rangle$ is an $\mathcal{O}_\Delta$ -primitive ideal satisfying $N(I) = 2ca$. In addition, the following assertions are equivalent:
 - (a) I is a principal ideal of $\mathcal{O}_\Delta$.
 - (b) There exist $x, y \in \mathbb{Z}$ such that $|x^2 - dy^2| = 2ca$ and $\gcd(c, (x/c)y) = 1$.
- (2) Let $x, y \in \mathbb{Z}$ be such that $|x^2 - dy^2| = 2ca$ and $\gcd(c, (x/c)y) = 1$. Then,

$$I = \langle x + y\sqrt{d} \rangle. \quad (2)$$

Proof (1). I is an ideal of $\mathcal{O}_\Delta$ satisfying $N(I) = 2ca$. Let $e^{-1}I \subset \mathcal{O}_\Delta$ with $e \in \mathbb{N}$ which implies $e = 1$. Therefore, I is an $\mathcal{O}_\Delta$ -primitive ideal.

(a) $\implies$ (b) Let I be principal. Then, there exists $\alpha = (x_1 + y_1\sqrt{\Delta})/2 \in \mathcal{O}_\Delta$, where $I = \langle \alpha \rangle = \alpha\mathcal{O}_\Delta$. We set $u = (x_1/2)$, $v = y_1$. Then, $I = \langle u + v\sqrt{d} \rangle$. Since I is an $\mathcal{O}_\Delta$ -primitive ideal, $\gcd(u, v) = 1$.

If $c = 1$, $I = \langle 2a, a + \sqrt{d} \rangle = \langle u + v\sqrt{d} \rangle$, then

$$2a = N(I) = |N(u + v\sqrt{d})| = |u^2 - dv^2|. \quad (3)$$

On the other hand, if $c = 2$, $I = \langle 4a, 2a + \sqrt{d} \rangle = \langle u + v\sqrt{d} \rangle$, since $\alpha \in I$, there exist $m, Y \in \mathbb{Z}$ such that

$$u + v\sqrt{d} = 4am + (2a + \sqrt{d})Y = 2a(2m + Y) + Y\sqrt{d}. \quad (4)$$

If $X = 2m + Y$, then $u = 2aX$ and $v = Y$, so $I = (2aX + Y\sqrt{d})\mathcal{O}_\Delta$ and $\gcd(2, aXY) = 1$. On the other hand, $4a = N(I) = |N(2aX + Y\sqrt{d})| = |4a^2X^2 - dY^2|$. Let $x = 2aX$ and $y = Y$; hence, $|x^2 - dy^2| = 4a$ and $(2, (x/2)y) = 1$.

$$(b) \implies (a). \quad (5)$$

If $c = 1$, let $x, y \in \mathbb{Z}$ such that $|x^2 - dy^2| = 2a$, which implies $x = ax_1$, for some $x_1 \in \mathbb{Z}$ and $\gcd(2, xy) = 1$; then, $\gcd(2, x_1) = 1$ and $x_1 - y = 2w$, for some $w \in \mathbb{Z}$. Now, we obtain $x + y\sqrt{d} = 2aw + y(a + \sqrt{d}) \in I$; hence, $\langle x + y\sqrt{d} \rangle \subset I$, and equality holds since

$$N(\langle x + y\sqrt{d} \rangle) = |N(x + y\sqrt{d})| = |x^2 - dy^2| = 2a = N(I). \quad (6)$$

If $c = 2$, let $x, y \in \mathbb{Z}$ such that $|x^2 - dy^2| = 4a$ and $(2, (x/2)y) = 1$; then, $|((x/2)^2 - aby^2)| = a$, which implies $(x/2) = ax_1$, for some $x_1 \in \mathbb{Z}$; then, x_1 is odd and $x_1 - y = 2w$, for some $w \in \mathbb{Z}$. Now, we obtain $x + y\sqrt{d} = 4aw + y(2a + \sqrt{d}) \in I$. Hence, $\langle x + y\sqrt{d} \rangle \subset I$, and equality holds since

$$N(\langle x + y\sqrt{d} \rangle) = |N(x + y\sqrt{d})| = 4a = N(I). \quad (7) \quad \square$$

(2) By (b) $\implies$ (a). Now, we will be devoted to numerical examples illustrating Theorem 1.

Example 1. Let $\Delta = 364 = 4d$ and $d = 91 = 7 \times 13 = ab \equiv 3 \pmod{4}$. Since $(105, 11)$ is the solution of $x^2 - 91y^2 = 14$; hence, $\langle 14, 7 + \sqrt{d} \rangle = \langle 105 + 11\sqrt{d} \rangle$.

Example 2. Let $\Delta = 1248 = 4d$ and $d = 312 = 4 \times 3 \times 26 = 4ab \equiv 0 \pmod{8}$. Since $(18, 1)$ is the solution of $x^2 - 312y^2 = 12$; hence, $\langle 12, 6 + \sqrt{d} \rangle = \langle 18 + \sqrt{d} \rangle$.

Theorem 2. Let $\Delta = 4d \in \mathbb{N}$ be a quadratic discriminant. Suppose that $d = 4ab$, where $a, b \in \mathbb{N}$, and a is square-free. Then,

(1) $I = \langle a, \sqrt{d} \rangle$ is an $\mathcal{O}_\Delta$ -primitive ideal satisfying $N(I) = a$. In addition, the following assertions are equivalent:

(a) I is a principal ideal of $\mathcal{O}_\Delta$.

(b) There exist $x, y \in \mathbb{Z}$ such that $|x^2 - dy^2| = 4a$ and $\gcd(x, y) = 2$.

(2) Let $x, y \in \mathbb{Z}$ be such that $|x^2 - dy^2| = 4a$ and $\gcd(x, y) = 2$. Then,

$$I = \left\langle \frac{x}{2} + \frac{y}{2}\sqrt{d} \right\rangle. \quad (8)$$

Proof (1). I is an ideal of $\mathcal{O}_\Delta$ satisfying $N(I) = a$. Let $e^{-1}I \subset \mathcal{O}_\Delta$ with $e \in \mathbb{N}$ which implies $e = 1$. Therefore, I is $\mathcal{O}_\Delta$ -primitive.

(a) $\implies$ (b) Let I be principal. Then, there exists $\alpha = (x_1 + y_1\sqrt{\Delta})/2 \in \mathcal{O}_\Delta$, where $I = \langle \alpha \rangle = \alpha\mathcal{O}_\Delta$. We set $u = (x_1/2)$, $v = y_1$. Then, $I = \langle u + v\sqrt{d} \rangle$. Since I is an $\mathcal{O}_\Delta$ -primitive ideal, $\gcd(u, v) = 1$.

On the other hand, $a = N(I) = |N(u + v\sqrt{d})| = |u^2 - dv^2|$. Let $x = 2u$ and $y = 2v$; then, $|x^2 - dy^2| = 4a$ and $\gcd(x, y) = 2$.

(2) (b) $\implies$ (a) Let $x, y \in \mathbb{Z}$ such that $|x^2 - dy^2| = 4a$ and $\gcd(x, y) = 2$. Then, $|((x/2)^2 - aby^2)| = a$, which implies $(x/2) = ax_1$, for some $x_1 \in \mathbb{Z}$. $(x/2) + (y/2)\sqrt{d} = ax_1 + (y/2)\sqrt{d} \in I$. Hence, $((x/2) + (y/2)\sqrt{d})\mathcal{O}_\Delta \subset I$, and equality holds since

$$N\left(\left(\frac{x}{2} + \frac{y}{2}\sqrt{d}\right)\mathcal{O}_\Delta\right) = \left|N\left(\frac{x}{2} + \frac{y}{2}\sqrt{d}\right)\right| = \left|\frac{x^2 - dy^2}{4}\right| = a = N(I). \quad (9) \quad \square$$

(2) By (b) $\implies$ (a). Now, we will be devoted to numerical examples illustrating Theorem 2.

Example 3. Let $\Delta = 1488 = 4d$ and $d = 372 = 4 \times 3 \times 31 = 4ab$. Since $(270, 14)$ is the solution of $x^2 - 372y^2 = -12$; hence, $\langle 3, \sqrt{d} \rangle = \langle 135 + 7\sqrt{d} \rangle$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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