Research Article

On a Diophantine Inequality with s Primes

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Let $2 < c < \delta$. In this study, for prime numbers $p_1, \dots, p_s$ and a sufficiently large real number N , we prove the Diophantine inequality $|p_1^c + \dots + p_s^c - N| < N^{-(9/10c)(\delta-c)}$, where $\delta = (294 - 210s)/(123 - 97s)$ and $s \geq 5$. When $s = 5$, this result improves a previous result.

1. Introduction

Suppose that $k \geq 1$ is an integer and $c > 1$ is not an integer. Let ε be a small positive number. The Waring–Goldbach problem is to study the solvability of the Diophantine equality:

$$N = p_1^k + \dots + p_s^k, \quad (1)$$

in prime numbers $p_1, \dots, p_s$. In [1], the author studied the analog of the Waring–Goldbach problem. For any sufficiently large real number N , let $H(c)$ denote the smallest natural number s such that the Diophantine inequality,

$$|p_1^c + \dots + p_s^c - N| < \varepsilon, \quad (2)$$

is solvable in prime numbers $p_1, \dots, p_s$. In [1], the author proved that

$$\limsup_{c \rightarrow \infty} \frac{H(c)}{c \log c} \leq 4. \quad (3)$$

In [1], the author also obtained that $H(c) \leq 5$, for $1 < c < (3/2)$. Later, the result was improved in [2–6]. Now, the best result for $H(c) \leq 5$ is $2 < c < (52/25)$ by Li and Cai [3].

In [4], the authors first proved that $H(c) \leq 4$, for $1 < c < (81/68)$. Later, the result was improved in [7]. Now, the best result for $H(c) \leq 4$ is $1 < c < (1193/889)$ by Zhang and Li [8]. When $s = 3$ in inequality (2), Tolev [9] obtained

the result $1 < c < (27/26)$. Afterwards, the range of c was enlarged by several authors in [10–16]. Now, the best result for $s = 3$ is $1 < c < (43/36)$ by Cai [12]. When $s = 2$, in inequality (2), Laporta obtained $1 < c < (15/14)$ in [17]. Laporta's result was improved by some authors in [4, 18, 19]. Now, the best result for $r = 2$ is $1 < c < (59/44)$ by Li and Cai [19].

In this paper, we focus on the Diophantine inequality (2) and prove the following result.

Theorem 1. Let $2 < c < \delta$. For any sufficiently large real number N , let $\varepsilon = N^{-(9/10c)(\delta-c)}$ and let $B_0(N, s)$ denote the number of solutions of the Diophantine inequality:

$$|p_1^c + p_2^c + \dots + p_s^c - N| < \varepsilon, \quad (4)$$

in prime numbers $p_1, p_2, \dots, p_s$, where $\delta = (294 - 210s)/(123 - 97s)$ and $s \geq 5$. We can obtain

$$B_0(N, s) \gg \frac{\varepsilon N^{(s/c)-1}}{\log^s N}. \quad (5)$$

For $s = 5$ in Theorem 1, we can get better result than Li and Cai [3].

Corollary 2. Under the notations of Theorem 1, for $2 < c < (378/181)$, we have

$$B_0(N, 5) \gg \frac{\varepsilon N^{(5/c)-1}}{\log^5 N}. \quad (6)$$

For comparison, $2 < c < (378/181) = 2.0883977, \dots$, $(52/25) = 2.08$.

Notation 1. In this paper, let $\delta = (294 - 210s)/(123 - 97s)$ and $\delta_1 = (131 - 97s)/(123 - 97s)$, $s \geq 5$. We also assume

$$\begin{aligned} 2 < c < \delta, \\ X &= \left(\frac{3N}{10}\right)^{1/c}, \\ \eta &= \frac{1}{1000}(\delta - c), \\ \kappa &= X^{1-c-\eta}, \\ \varepsilon &= N^{-(9/10c)(\delta-c)}, \\ \chi &= X^{\delta-c}, \\ a &= \frac{9\varepsilon}{10}, \\ b &= \frac{\varepsilon}{10}, \\ S(x) &= \sum_{(X/2) < p \leq X} e(xp^c) \log p, \\ I(x) &= \int_{X/2}^X e(xt^c) dt. \end{aligned} \quad (7)$$

Let $\Lambda(n)$ denote von Mangoldt's function. We write $\omega \sim \Omega$ if the range of ω is $0 < \omega \leq 2\Omega$. $a(\omega) \ll b(\omega)$ means that $a(\omega) = O(b(\omega))$.

2. Some Lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 1. *Let r be a positive integer. There exists a function $\phi(y)$ which is $r = [\log X]$ times continuously differentiable and satisfies*

$$\begin{aligned} \phi(y) &= 1, \text{ for } |y| \leq a - b, \\ 0 < \phi(y) &< 1, \text{ for } a - b < |y| < a + b, \\ \phi(y) &= 0, \text{ for } |y| \geq a + b, \end{aligned} \quad (8)$$

and its Fourier transformation,

$$\Phi(x) = \int_{-\infty}^{+\infty} e(-xy) \phi(y) dy, \quad (9)$$

satisfies

$$|\Phi(x)| \leq \min\left(2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left(\frac{r}{2\pi|x|b}\right)^r\right). \quad (10)$$

Proof. This can be found in Piatetski-Shapiro [1]. $\square$

Lemma 2. *Let $a(l)$ be a sequence of complex numbers; then, for $L, Q \geq 1$, we have*

$$\left| \sum_{L < l \leq 2L} a(l) \right|^2 \leq \left(2 + \frac{L}{Q}\right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q}\right) \left(\sum_{L \leq l+q, l-q \leq 2L} a(l+q) \overline{a(l-q)} \right), \quad (11)$$

where $\bar{z}$ denotes the conjugate of the complex number z .

Proof. This is Lemma 2 of Fouvry and Iwaniec [20]. $\square$

Lemma 3. *We have*

$$\int_{-\infty}^{\infty} I^s(x) e(-xN) \Phi(x) dx \gg \varepsilon X^{s-c}. \quad (12)$$

Proof. Similar to Lemma 6 of [16], we have

$$\begin{aligned} H &:= \int_{-\infty}^{\infty} I^s(x) e(-xN) \Phi(x) dx \\ &= \int_{X/2}^X \cdots \int_{X/2}^X \int_{-\infty}^{\infty} e(x(t_1^c + \cdots + t_s^c - N)) \Phi(x) dx dt_1 \cdots dt_s \\ &= \int_{X/2}^X \cdots \int_{X/2}^X \Phi(t_1^c + \cdots + t_s^c - N) dt_1 \cdots dt_s \\ &\geq \int_{X/2}^X \cdots \int_{X/2}^X (*) dt_1 \cdots dt_s \\ &\geq \int_{\mu X}^{\lambda X} dt_1 \cdots \int_{\mu X}^{\lambda X} dt_{s-1} \int_{\mathcal{M}} dt_s, \end{aligned} \quad (13)$$

where $*$ means that $|t_1^c + \cdots + t_s^c - N| < (4\varepsilon/5)$, and λ and μ satisfy

$$\frac{1}{2} < \left(\frac{1}{2}\right)^{1/c} < \mu < \lambda < \left(\frac{1}{2}\left(2 - \frac{1}{2^c}\right)\right)^{1/c} < 1,$$

$$\mathcal{M} = \left[\frac{X}{2}, X\right] \cap \left[\left(N - \frac{4\epsilon}{5} - t_1^c - \dots - t_{s-1}^c\right)^{1/c}, \left(N + \frac{4\epsilon}{5} - t_1^c - \dots - t_{s-1}^c\right)^{1/c}\right]. \quad (14)$$

We have

$$H \gg \epsilon \int_{\mu X}^{\lambda X} \dots \int_{\mu X}^{\lambda X} (\xi_{t_1, \dots, t_{s-1}})^{(1/c)-1} dt_1 \dots dt_{s-1}, \quad (15)$$

by the mean-value theorem, where $\xi_{t_1, \dots, t_{s-1}} \asymp X^c$. Therefore, we can obtain

$$H \gg \epsilon X^{s-c}, \quad (16)$$

which proves Lemma 3. $\square$

Lemma 4

$$(i) \int_{-\kappa}^{\kappa} |S(x)|^2 dx \ll X^{2-c} \log^3 X$$

$$(ii) \int_{-\kappa}^{\kappa} |I(x)|^2 dx \ll X^{2-c} \log^3 X$$

Proof. This is Lemma 7 of Tolev [16]. $\square$

Lemma 5. Let $0 \leq k \leq (1/2) \leq l \leq 1$ and let $|B| > 0$. For any exponent pair (k, l) and $\Omega \leq \Omega' \leq 10\Omega$, we have

$$\sum_{\Omega \leq \omega \leq \Omega'} e(B\omega^c) \ll (|B|\Omega^c)^k \Omega^{l-k} + \frac{\Omega}{|B|\Omega^c}. \quad (17)$$

Proof. This is Lemma 3 of Li and Cai [3]. $\square$

Lemma 6. For $|x| < \kappa$, we have

$$S(x) = I(x) + O\left(X \exp\left(-\log^{(1/5)} X\right)\right). \quad (18)$$

Proof. This is Lemma 14 of Tolev [16]. $\square$

Lemma 7. For $\kappa < |x| < \chi$, we have

$$\int_{\kappa < |x| < \chi} |S^4(x)| |\Phi(x)| dx \ll (X^{4-c} + X^2) X^{3\eta}. \quad (19)$$

Proof. This is (50) of Zhai and Cao [5]. $\square$

Lemma 8. Let α and β be real numbers such that $\alpha \neq 0, 1, 2$ and $\beta \neq 0, 1, 2, 3$. Set

$$T(\Omega, \Gamma) = \sum_{\Omega < \omega \leq 2\Omega} \sum_{\Gamma < \gamma \leq 2\Gamma} a(\omega) b(\gamma) e\left(\frac{F\omega^\alpha \gamma^\beta}{\Omega^\alpha \Gamma^\beta}\right), \quad (20)$$

where $F \gg \Omega \Gamma^2$, $\Gamma \geq \Omega \geq 1$, and $|a(\omega)|, |b(\gamma)| \leq 1$. Then, we have

$$T(\Omega, \Gamma) \ll \left(F^{1/16} \Omega^{7/8} \Gamma^{13/16} + F^{1/26} \Omega^{93/104} \Gamma^{23/26} + F^{-(1/128)} \Omega^{467/512} \Gamma^{65/64} + \Omega^{65/72} \Gamma\right) (\Omega \Gamma)^\eta. \quad (21)$$

Proof. This is Theorem 1 of Baker and Weingartner [6]. $\square$

Lemma 9. Let $3 < Q < K < W < X$. Suppose that $W - (1/2) \in \mathbb{N}$, $X \geq 64W^2 UQ$, $W \geq 4Q^2$, $K^3 \geq 32X$. Assume further that $F(n)$ is a complex-valued function such that $|F(n)| \leq 1$. Then, the sum,

$$\sum_{X < n \leq 2X} \Lambda(n) F(n), \quad (22)$$

can be written in $O(\log^{10} X)$ sums.

Proof. This is Lemma 3 of Heath-Brown [21]. $\square$

3. The Estimate of $S(x)$

In this section, we draw our attention to the estimate of exponential sums, which also has lots of applications (e.g., see [22–35]). Suppose that $|a(\omega)| \ll \omega^\eta$ and $|b(\gamma)| \ll \gamma^\eta$; then, we estimate the exponential sums in the following two forms. Type I:

$$\sum_{\substack{\omega \sim \Omega, \gamma \sim \Gamma \\ \Omega \Gamma \sim X}} a(\omega) F(\omega \gamma), \quad (23)$$

and Type II:

$$\sum_{\substack{\omega \sim \Omega, n \sim L \\ \Omega \Gamma \sim X}} a(\omega) b(\gamma) F(\omega \gamma). \quad (24)$$

Lemma 10. For complex number sequences $a(\omega)$ and $b(\gamma)$, suppose that $|a(\omega)| \ll 1$, $|b(n)| \ll 1$, $\kappa \leq |x| \leq \chi$, $X^{16/(97s-123)} \ll \Gamma \ll X^{(438-178s)/(369-291s)}$, $\Omega \Gamma \asymp X$; then, we have

$$S_{II} = \sum_{\omega \sim \Omega} \sum_{\gamma \sim \Gamma} a(\omega) b(\gamma) e(x(\omega \gamma)^c) \ll X^{\delta_1 + \eta}. \quad (25)$$

Proof. By Cauchy's inequality and Lemma 2 with $M = X^{(16/(97s-123))-\eta}$, we obtain

$$\begin{aligned}
|S_{II}|^2 &\ll \sum_{\gamma \sim \Gamma} |b(\gamma)|^2 \sum_{\gamma \sim \Gamma} \left| \sum_{\omega \sim \Omega} a(\omega) e(x(\omega\gamma)^c) \right|^2 \\
&\ll \Gamma \sum_{\gamma \sim \Gamma} \frac{\Omega}{M} \sum_{0 \leq m < M} \left(1 - \frac{m}{M} \right) \\
&\quad \times \sum_{\omega \sim \Omega} a(\omega + m) \overline{a(\omega - m)} e(x\omega^c((\gamma + m)^c - (\gamma - m)^c)) \\
&\ll \Gamma \frac{\Omega}{M} \sum_{\gamma \sim \Gamma} \left(\Omega^{1+\eta} + \sum_{1 \leq m < M} \left(1 - \frac{m}{M} \right) \right. \\
&\quad \left. \times \sum_{\omega \sim \Omega} a(\omega + m) \overline{a(\omega - m)} e(x\omega^c((\gamma + m)^c - (\gamma - m)^c)) \right) \\
&\ll \frac{X^2}{M} + \frac{X}{M} \sum_{1 \leq m \leq M} \sum_{\gamma \sim \Gamma} \left| \sum_{\omega \sim \Omega} e(x\omega^c((\gamma + m)^c - (\gamma - m)^c)) \right| \\
&\ll \frac{X^2}{M} + \frac{X}{M} |S^*|,
\end{aligned} \tag{26}$$

where

$$S^* = \sum_{1 \leq m \leq M} \sum_{\gamma \sim \Gamma} \left| \sum_{\omega \sim \Omega} e(f(\omega, \gamma, m)) \right|, \tag{27}$$

with $f(\omega, \gamma, m) = x\omega^c((\gamma + m)^c - (\gamma - m)^c)$. By Lemma 5, we choose the exponent pair $((1/14), (11/14))$. Then, we can obtain

$$\begin{aligned}
S^* &\ll \sum_{1 \leq m \leq M} \sum_{\gamma \sim \Gamma} \left((m|x|X^{c-1})^{1/14} \left(\frac{X}{\Gamma} \right)^{11/14} + \frac{1}{m|x|X^{c-1}} \right) \\
&\ll \Gamma M^{15/14} \chi^{1/14} X^{(c-1)/14} X^{11/14} \Gamma^{-(11/14)} + \Gamma \kappa^{-1} X^{1-c} \log M \\
&\ll X.
\end{aligned} \tag{28}$$

Now, Lemma 10 follows from (26) and (28). $\square$

Lemma 11. Let $a(\omega)$ be a sequence of complex numbers. Suppose that $|a(\omega)| \ll 1$, $\kappa \leq |x| \leq \chi$, and $\Omega \ll X^{(6029-2231s)/(3075-2425s)}$; then, we have

$$S_I = \sum_{\omega \sim \Omega} \sum_{\gamma \sim \Gamma} a(\omega) e(x(\omega\gamma)^c) \ll X^{\delta_1+\eta}. \tag{29}$$

Proof. If $\Omega \ll X^{(41-27s)/(123-97s)}$, by Lemma 5, we choose the exponent pair $((1/6), (2/3))$. Then, we obtain

$$S_I \ll \sum_{\omega \leq 2\Omega} \left((|x|X^c)^{1/6} \left(\frac{X}{\omega} \right)^{1/2} + \frac{X}{|x|X^c\omega} \right) \ll X^{\delta_1+\eta}. \tag{30}$$

If $X^{(41-27s)/(123-97s)} \ll \Omega \ll X^{(438-178s)/(369-291s)}$, then, by Lemma 10, we have

$$S_I \ll X^{\delta_1+\eta}. \tag{31}$$

If $X^{(438-178s)/(369-291s)} \ll \Omega \ll X^{(6029-2231s)/(3075-2425s)}$, by Lemma 8 with $(\omega, \gamma) = (\omega, \gamma)$ or $(\omega, \gamma) = (\gamma, \omega)$, we have

$$S(x) \ll X^{\delta_1+\eta}. \tag{32}$$

Now, the proof of Lemma 11 is completed. $\square$

Lemma 12. For $\kappa \leq |x| \leq \chi$, we have

$$S(x) \ll X^{\delta_1+4\eta}. \tag{33}$$

Proof. We can obtain

$$S^{**} = \sum_{n \sim X} \Lambda(n) e(xn^c) \ll X^{\delta_1+3\eta}. \tag{34}$$

Let

$$\begin{aligned}
Q &= X^{16/(97s-123)}, \\
K &= X^{(438-178s)/(369-291s)},
\end{aligned} \tag{35}$$

$$W = \left[X^{(194s+2954)/(2425s-3075)} \right] + \frac{1}{2}.$$

Let $F(\gamma) = e(x\gamma^c)$ in Lemma 9; we reduce the estimation of S^{**} to the estimations of type I sums:

$$\sum_{\omega \sim \Omega, \gamma \sim \Gamma} \sum_{\Omega\Gamma \sim X} a(\omega) F(\omega\gamma), \quad \Gamma > W, \tag{36}$$

and type II sums:

$$\sum_{\omega \sim \Omega, \gamma \sim \Gamma} \sum_{\Omega\Gamma \sim X} a(\omega) F(\omega\gamma), \quad Q < \Gamma < K, \tag{37}$$

and estimate (34) follows from Lemmas 10 and 11. $\square$

4. Proof of the theorem

Let

$$\begin{aligned}
B(N, s) &= \sum_{\substack{(X/2) < p_1, \dots, p_s \leq X \\ |p_1^c + \dots + p_s^c - N| < \varepsilon}} (\log p_1) \cdots (\log p_s), \\
B_1(N, s) &= \sum_{(X/2) < p_1, \dots, p_s \leq X} (\log p_1) \cdots (\log p_s) \phi(p_1^c + \dots + p_s^c - N).
\end{aligned} \tag{38}$$

By the definition of $\phi(y)$ in Lemma 1, we have

$$B(N, s) \geq B_1(N, s). \quad (39)$$

By the inverse Fourier transformation formula, we obtain

$$\begin{aligned} B_1(N, s) &= \int_{-\infty}^{\infty} S^s(x) e(-xN) \Phi(x) dx \\ &= D_1(N) + D_2(N) + D_3(N), \end{aligned} \quad (40)$$

where

$$\begin{aligned} D_1(N) &= \int_{-\kappa}^{\kappa} S^s(x) e(-xN) \Phi(x) dx, \\ D_2(N) &= \int_{\kappa \nless x \leq \chi} S^s(x) e(-xN) \Phi(x) dx, \\ D_3(N) &= \int_{|x| > \chi} S^s(x) e(-xN) \Phi(x) dx. \end{aligned} \quad (41)$$

Let

$$\begin{aligned} H_1(N) &= \int_{-\kappa}^{\kappa} I^s(x) e(-xN) \Phi(x) dx, \\ H(N) &= \int_{-\infty}^{\infty} I^s(x) e(-xN) \Phi(x) dx. \end{aligned} \quad (42)$$

Then, we have

$$D_1(N) = H(N) + (H_1(N) - H(N)) + (D_1(N) - H_1(N)). \quad (43)$$

From Lemma 10 in [36], we have $I(x) \ll X^{1-c} |x|^{-1}$. Thus, by Lemma 1, we have

$$\begin{aligned} |H_1(N) - H(N)| &\ll \int_{|x| > \kappa} |I(x)|^s |\Phi(x)| dx \\ &\ll X^{s-sc} \int_{|x| > \kappa} |\Phi(x)| |x|^{-s} dx \\ &\ll \varepsilon X^{s-sc} \kappa^{-s+1} \\ &\ll \frac{\varepsilon X^{s-c}}{\log X}. \end{aligned} \quad (44)$$

It follows from Lemmas 1, 4, and 6 that

$$\begin{aligned} |D_1(N) - H_1(N)| &\ll \int_{-\kappa}^{\kappa} |S^s(x) - I^s(x)| |\Phi(x)| dx \\ &\ll \varepsilon \max_{|x| \leq \kappa} |S(x) - I(x)| \int_{-\kappa}^{\kappa} (|S^{s-1}(x)| + |I^{s-1}(x)|) dx \\ &\ll \varepsilon X^{s-3} \max_{|x| \leq \kappa} |S(x) - I(x)| \int_{-\kappa}^{\kappa} (|S^2(x)| + |I^2(x)|) dx \\ &\ll \varepsilon X^{s-1-c} (\log^3 X) \max_{|x| \leq \kappa} |S(x) - I(x)| \\ &\ll \varepsilon X^{s-c} \exp(-\log^{1/5} X). \end{aligned} \quad (45)$$

By Lemma 3 and (43)–(45), we obtain

$$D_1(N) \gg \varepsilon X^{s-c}. \quad (46)$$

Let

$$U(x) = \sum_{(X/2) \leq n \leq X} e(xn^c). \quad (47)$$

We have

$$\begin{aligned} |D_2(N)| &= \left| \sum_{(X/2) \leq p \leq X} (\log p) \int_{\kappa \nless x \leq \chi} e(xp^c) S^{s-1}(x) \Phi(x) e(-xN) dx \right| \\ &\leq \sum_{(X/2) \leq p \leq X} (\log p) \left| \int_{\kappa \nless x \leq \chi} e(xp^c) S^{s-1}(x) \Phi(x) e(-xN) dx \right| \\ &\leq (\log X) \sum_{(X/2) \leq n \leq X} \left| \int_{\kappa \nless x \leq \chi} e(xn^c) S^{s-1}(x) \Phi(x) e(-xN) dx \right|. \end{aligned} \quad (48)$$

It follows from (48) and Cauchy's inequality that

$$\begin{aligned}
|D_2(N)|^2 &\leq X(\log^2 X) \sum_{(X/2) \leq n \leq X} \left| \int_{\kappa \nmid x| \leq \chi} e(xn^c) S^{s-1}(x) \Phi(x) e(-xN) dx \right|^2 \\
&\leq X(\log^2 X) \int_{\kappa \nmid y| \leq \chi} \overline{S^{s-1}(y) \Phi(y) e(-yN)} dy \\
&\quad \times \int_{\kappa \nmid x| \leq \chi} S^{s-1}(x) \Phi(x) e(-xN) U(x-y) dx \\
&\leq X^{(-194s^2+1135s-1187)/(123-97s)} (\log^2 X) \int_{\kappa \nmid y| \leq \chi} |S^4(y)| |\Phi(y)| dy \\
&\quad \times \int_{\kappa \nmid x| \leq \chi} |S^4(x)| |\Phi(x)| U(x-y) dx.
\end{aligned} \tag{49}$$

By Lemma 5 with the exponent pair $((13/84), (55/84))$ (see in [37]), we obtain

It follows from (50) that

$$U(x) \ll \min \left((|x|X^c)^{13/84} X^{1/2} + \frac{X}{|x|X^c}, X \right). \tag{50}$$

$$\begin{aligned}
&\cdot \int_{\kappa \nmid x| \leq \chi} |S(x)|^4 |\Phi(x)| U(x-y) dx \\
&\ll \int_{\kappa \nmid x| \leq \chi} |S(x)|^4 |\Phi(x)| U(x-y) dx \\
&\quad |x-y| \leq X^{-c} \\
&\quad + \int_{\kappa \nmid x| \leq \chi} |S(x)|^4 |\Phi(x)| U(x-y) dx \\
&\quad X^{-c} < |x-y| \leq 2\chi \\
&\ll X \int_{\kappa \nmid x| \leq \chi} |S(x)|^4 |\Phi(x)| dx \\
&\quad |x-y| \leq X^{-c} \\
&\quad + \int_{\kappa \nmid x| \leq \chi} |S(x)|^4 |\Phi(x)| \left((|x-y|X^c)^{13/84} X^{1/2} + \frac{X}{|x-y|X^c} \right) dx \\
&\quad X^{-c} < |x-y| \leq 2\chi \\
&\ll \varepsilon X^{((647-485s)/(123-97s))+16\eta} \int_{|x-y| \leq X^{-c}} dx + (\chi X^c)^{13/84} X^{1/2} \int_{\kappa \nmid x| \leq \chi} |S(x)|^4 |\Phi(x)| dx \\
&\quad + \varepsilon X^{((647-485s)/(123-97s))-c+16\eta} \int_{X^{-c} < |x-y| \leq 2\chi} \frac{1}{|x-y|} dx \\
&\ll \varepsilon X^{((647-485s)/(123-97s))-c+17\eta} + X^{(107-81s)/(123-97s)} \int_{\kappa \nmid x| \leq \chi} |S(x)|^4 |\Phi(x)| dx,
\end{aligned} \tag{51}$$

where Lemma 12 is used.

Now, by (49) and (51), we obtain

$$\begin{aligned}
|D_2(N)|^2 &\ll X^{(-194s^2+1135s-1187)/(123-97s)} (\log^2 X) \int_{\kappa \leq x \leq \chi} |S(y)|^4 |\Phi(y)| dy \\
&\times \left(\varepsilon X^{((647-485s)/(123-97s)) - c + 17\eta} + X^{(107-81s)/(123-97s)} \int_{\kappa \leq x \leq \chi} |S(x)|^4 |\Phi(x)| dx \right) \\
&\ll \varepsilon X^{((-194s^2+650s-540)/(123-97s)) - c + 18\eta} \int_{\kappa \leq x \leq \chi} |S(y)|^4 |\Phi(y)| dy \\
&\quad + X^{(-194s^2+1054s-1080)/(123-97s)} \left(\int_{\kappa \leq x \leq \chi} |S(x)|^4 |\Phi(x)| dx \right)^2 \\
&\ll \varepsilon X^{((-194s^2+456s-294)/(123-97s)) - c + 19\eta} + X^{((-194s^2+666s-588)/(123-97s)) + 4\eta},
\end{aligned} \tag{52}$$

where Lemma 7 is used. From (52), we obtain

$$\begin{aligned}
D_2(N) &\ll \varepsilon^{1/2} X^{((-97s^2+228s-147)/(123-97s)) - (c/2) + 10\eta} \\
&\quad + X^{((-97s^2+333s-294)/(123-97s)) + 2\eta} \ll \frac{\varepsilon X^{s-c}}{\log X}.
\end{aligned} \tag{53}$$

It follows from Lemma 1 that

$$\begin{aligned}
D_3(N) &\ll \int_{|x| > \chi} |S^s(x)| |\Phi(x)| dx \\
&\ll X^s \int_{|x| > \chi} \frac{1}{|x|} \left(\frac{r}{2\pi|x|b} \right)^r dx \\
&\ll X^s \left(\frac{r}{2\pi b \chi} \right)^r \ll 1.
\end{aligned} \tag{54}$$

By (40), (46), (53), and (54),

$$B_1(N, s) \gg \varepsilon X^{s-c}. \tag{55}$$

It follows from (39) and (55) that

$$B_0(N, s) \geq \frac{B(N, s)}{\log^s X} \geq \frac{B_1(N, s)}{\log^s X} \gg \frac{\varepsilon N^{(s/c)-1}}{\log^s N}. \tag{56}$$

Now, by (56), the proof of the theorem is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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