Research Article

Global Existence of Solution for the Fisher Equation via Faedo–Galerkin’s Method

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In this study, we consider the Fisher equation in bounded domains. By Faedo–Galerkin’s method and with a homogeneous Dirichlet conditions, the existence of a global solution is proved.

1. Introduction and Preliminaries

The Fisher equation arises in abundance in many fields, including chemistry, biology, and the environment [1–15]. It also has a common name, the Fisher–Kolmogorov–Petrovsky–Piskunov equation (KPP), where it describes the following equation by the process of population progress in space (F.KPP equation) [16]:

$$\Psi_t - D\Psi_{xx} = \mu\Psi(K - \Psi) \text{ in }]0, T[\times]0, K[, \quad (1)$$

where (x, t) denotes the position and the time, respectively, as $\Psi(x, t)$ is the population density, D is the propagation constant, and K is the maximum density, with the homogeneous boundary Dirichlet conditions:

$$\Psi(t, 0) = \Psi(t, K) = 0, \quad t \in]0, T[, \quad (2)$$

$$\Psi(0, x) = \Psi_0(x), \quad x \in]0, K[. \quad (3)$$

Also, this equation is closely related to biology, applied mathematics, parasites, bacteria, and genes. For more detail,

we refer the reader to the following research papers, see, for example, [17–21].

The simplest version of the FK equation is

$$\Psi_t - \Delta\Psi = \mu\Psi(1 - \Psi) \text{ in }]0, T[\times \Omega. \quad (4)$$

Based on the previous work, we will shed light on problem (4), which is a multidimensional model of Fisher’s equation, under a Dirichlet boundary condition:

$$\begin{cases} \Psi_t - \Delta\Psi = \mu\Psi(1 - \Psi) \text{ in }]0, T[\times \Omega \\ \Psi = 0 \text{ on } \Gamma =]0, T[\times \partial\Omega \\ \Psi(0, x) = \Psi_0(x) \text{ in } \Omega. \end{cases} \quad (5)$$

Our paper is divided into several sections. In Section 2, the existence of local solution is proved. In Section 3, the maximum principle under suitable condition on Ψ_0 is established. In Section 4, the existence and uniqueness of solution are proved. Finally, we give some concluding remarks in Section 5.

Firstly, we define the solution of (5) as a solution of the following weak formulation: $\Psi \in W(0, T)$ and verify

$$\begin{aligned} \langle \Psi'(t), \Phi \rangle + \langle \nabla \Psi(t), \nabla \Phi \rangle &= \langle \mu \Psi(t)(1 - \Psi(t)), \Phi \rangle \forall \Phi \\ &\in H_0^1(\Omega), t \in]0, T[, \end{aligned} \quad (6)$$

$$\Psi(0, x) = \Psi_0(x). \quad (7)$$

2. Local Existence

In this section, we state and prove the local existence result of our problem.

Theorem 1. Suppose that $(b_0, t_0) \in L^2(\Omega) \times [0, T[$. Then, $\exists t_1 = t_1(t_0, b_0) \in]t_0, T[$, and there exists a weak solution $\Psi \in W(t_0, t_1)$ of problem (5), satisfying (6) and (7).

Proof. To reach our goal, we shall use the so-called Faedo–Galerkin method. $\square$

Step 1. solution of the approximate problem:

Since $H_0^1(\Omega)$ is separable, $\exists \{h_i\}_{i=0}^\infty$ is a basis for $H_0^1(\Omega)$. For all m , the approximate solution Ψ_m of (6) given by

$$\Psi_m = \sum_{i=1}^m f_i^m h_i, \quad (8)$$

which satisfies

$$\begin{aligned} \langle \Psi_m'(t), h_i \rangle + \langle \nabla \Psi_m(t), \nabla h_i \rangle &= \langle \mu \Psi_m(t)(1 - \Psi_m(t)), h_i \rangle t \\ &\in]t_0, t_1[\quad [i = 1, 2, \dots, m, \end{aligned} \quad (9)$$

$$\Psi_m(t_0) = b_0^{(m)}, \quad (10)$$

where $b_0^{(m)} = \sum_{j=1}^m \xi_j^m h_j$ is a $L^2(\Omega)$ projection of b_0 onto the span of $\{h_1, h_2, \dots, h_m\}$.

Properties of projection operators imply

$$\|b_0^{(m)}\|_{L^2(\Omega)} \leq \|b_0\|_{L^2(\Omega)}. \quad (11)$$

From $H_0^1(\Omega)$ which is dense in $L^2(\Omega)$ and $\{h_i\}$ which is a basis for $H_0^1(\Omega)$, we obtain

$$b_0^{(m)} \longrightarrow b_0 \text{ in } L^2(\Omega) \text{ as } m \longrightarrow \infty. \quad (12)$$

Systems (9) and (10) write

$$\begin{aligned} \sum_{j=1}^m (h_j, h_i) \frac{d}{dt} f_j^{(m)}(t) + \sum_{j=1}^m (\nabla h_j, \nabla h_i) f_j^{(m)}(t) \\ = \mu \left\langle \sum_{j=1}^m f_j^{(m)}(t) h_j - \sum_{j,k=1}^m f_j^{(m)}(t) f_k^{(m)}(t) h_j h_k, h_i \right\rangle, \\ i = 1, \dots, m, \end{aligned} \quad (13)$$

$$\sum_{j=1}^m (h_j, h_i) f_j^{(m)}(t_0) = (b_0, h_i), \quad i = 1, \dots, m. \quad (14)$$

Since the functions $\{h_i\}_{i=1}^m$ are linearly independent, this means that the matrix with entries (h_j, h_i) is nonsingular, to use the inverse of this matrix to reduce (13) and (14) to the following system:

$$\begin{aligned} \frac{d}{dt} f_i^{(m)}(t) - \mu f_i^{(m)}(t) + \sum_{j=1}^m \alpha_{ij} f_j^{(m)}(t) \\ + \sum_{j,k=1}^m \beta_{ijk} f_j^{(m)}(t) f_k^{(m)}(t) = 0, \quad i = 1, \dots, m, \\ f_i^{(m)}(t_0) = \xi_i, \quad i = 1, \dots, m, \end{aligned} \quad (15)$$

for $i = 1, 2, \dots, m$, where $\alpha_{ij}, \beta_{ijk}, \xi_i \in \mathbb{R}$, and they depend on $\{h_i\}_{i=1}^m$. Systems (15) and (16) have a solution defined on a maximal right interval $[t_0, \tau^{(m)})$. Or equivalently, systems (9) and (10) have a solution $\Psi_m(t)$ defined on a maximal interval (see, e.g., [22]).

Step 2. a priori estimates for $\Psi_m(t)$.

For $t \in [t_0, \tau^{(m)})$, multiplying (9) by $f_i^{(m)}(t)$, $i = 1, \dots, m$, and adding these equations up, we obtain

$$\begin{aligned} \langle \Psi_m'(t), \Psi_m(t) \rangle + \|\nabla \Psi_m(t)\|_{L^2(\Omega)}^2 \\ = \langle \mu \Psi_m(t)[1 - \Psi_m(t)], \Psi_m(t) \rangle. \end{aligned} \quad (17)$$

Hence, by using Young's inequality, we obtain

$$\begin{aligned} \frac{d}{dt} \|\Psi_m(t)\|_{L^2(\Omega)}^2 + \|\nabla \Psi_m(t)\|_{L^2(\Omega)}^2 \\ \leq \mu \|\Psi_m(t)\|_{L^2(\Omega)}^2 + \mu \|\Psi_m(t)\|_{L^2(\Omega)}^3. \end{aligned} \quad (18)$$

Furthermore, from $H^{1/2}(\Omega) \longrightarrow L^3(\Omega)$, using the interpolation between $L^2(\Omega)$ and $H_0^1(\Omega)$, Young's and Poincare's inequalities, we obtain

$$\begin{aligned} \mu \|\Psi_m(t)\|_{L^3(\Omega)}^3 &\leq \mu \left(C \|\Psi_m(t)\|_{H^{1/2}(\Omega)}^2 \right)^3 \leq \mu \left(\tilde{C} \|\Psi_m(t)\|_{L^2(\Omega)}^{1/2} \|\Psi_m(t)\|_{H_0^1(\Omega)}^{1/2} \right)^3 \\ &\leq \mu \left(\tilde{C} \|\Psi_m(t)\|_{L^2(\Omega)}^{3/2} \|\nabla \Psi_m(t)\|_{L^2(\Omega)}^{3/2} \right) \\ &\leq C_2 \|\Psi_m(t)\|_{L^2(\Omega)}^6 + \epsilon \|\nabla \Psi_m(t)\|_{L^2(\Omega)}^2, \\ &\leq C_2 \|\Psi_m(t)\|_{L^2(\Omega)}^6 + \|\nabla \Psi_m(t)\|_{L^2(\Omega)}^2. \end{aligned} \quad (19)$$

By adding up (19) and then applying the resulting estimate, we find

$$\begin{aligned} \frac{d}{dt} \|\Psi_m(t)\|_{L^2(\Omega)}^2 + \|\nabla \Psi_m(t)\|_{L^2(\Omega)}^2 &\leq \mu \|\Psi_m(t)\|_{L^2(\Omega)}^2 \\ &\leq C_1 \|\Psi_m(t)\|_{L^2(\Omega)}^2 \\ &\quad + C_2 \|\Psi_m(t)\|_{L^2(\Omega)}^6 \\ &\quad + \|\nabla \Psi_m(t)\|_{L^2(\Omega)}^2, \end{aligned} \quad (20)$$

$$\Rightarrow \frac{d}{dt} \|\Psi_m(t)\|_{L^2(\Omega)}^2 \leq C_1 \|\Psi_m(t)\|_{L^2(\Omega)}^2 + C_2 \|\Psi_m(t)\|_{L^2(\Omega)}^6. \quad (21)$$

Integrating (21) over (t_0, t) , where $t \in]t_0, \tau^{(m)}[$, we obtain

$$\begin{aligned} \|\Psi_m(t)\|_{L^2(\Omega)}^2 - \|\Psi_m(t_0)\|_{L^2(\Omega)}^2 &\leq \int_{t_0}^t [2C_1 \|\Psi_m(s)\|_{L^2(\Omega)}^2 \\ &\quad + 2C_2 \|\Psi_m(s)\|_{L^2(\Omega)}^6] ds. \end{aligned} \quad (22)$$

Setting $z_m(t) = \|\Psi_m(t)\|_{L^2(\Omega)}^2$, we obtain

$$z_m(t) \leq z_m(t_0) + \int_{t_0}^t f(\sigma, z_m(\sigma)) d\sigma, \quad (23)$$

where $f(t, z) = C_1 z + C_2 z^3$, and from it, $f \in C([0, T] \times \mathbb{R}; \mathbb{R})$ and f is increasing in z , $\forall t$.

Set $z(t)$ as a maximal solution of the following equation:

$$\frac{dy}{dt} = 2C_1 z + 2C_2 z^3, \quad (24)$$

with

$$z_0(t) = \|z(t_0)\|_{L^2(\Omega)}^2. \quad (25)$$

There exist $J = [t_0, t_1]$ with $t_1 \in]t_0, T[$ (we have (24) and (25) and t_1 are independent of m).

Inequality (11) implies $z_m(0) \leq \bar{z}(t_0)$. Then, by setting $a = \tau^{(m)} - t_0$, we obtain

$$\begin{aligned} \|\Psi_m(t)\|_{L^2(\Omega)}^2 &= z_m(t) \leq \bar{z}(t) \leq \max_{t \in [t_0, \tau]} \bar{z}(t) \\ &= C(t_1) \forall t \in]t_0, \tau^{(m)}[\end{aligned} \quad (26)$$

Since $[t_0, \tau^{(m)}[$ is the maximal interval of existence for (23) and (26), we deduce that $\tau^{(m)} = t_1$, and the existence interval is $[t_0, t_1]$. Furthermore, by (26), we find

$$\sup_{t \in [t_0, \tau]} \|\Psi_m(t)\|_{L^2(\Omega)}^2 \leq C(t_1) \forall t \in]t_0, t_1[\quad m = 1, 2, \dots \quad (27)$$

Hence, we get $\exists t_1 \in]t_0, T[$ such that

$$\{\Psi_m\}_{m=1}^\infty \text{ belongs to a bounded set of } L^\infty((t_0, t_1); L^2(\Omega)). \quad (28)$$

By (22), (11), and (28), we obtain

$$\{\Psi_m\}_{m=1}^\infty \text{ belongs to a bounded set of } L^\infty((t_0, t_1); H_0^1(\Omega)). \quad (29)$$

Step 3. passage to limits.

A priori estimates (28) and (29) allow us to draw a subsequence of $\{\Psi_m\}$ such that

$$\Psi_m \rightharpoonup \Psi \text{ weakly star in } L^\infty((t_0, t_1); L^2(\Omega)) \quad (30)$$

$$\Psi_m \rightharpoonup \Psi \text{ weakly in } L^2((t_0, t_1); H_0^1(\Omega)), \quad (31)$$

for some Ψ in $L^\infty((t_0, t_1); L^2(\Omega)) \cap L^2((t_0, t_1); H_0^1(\Omega))$. Hence, we obtain

$$\Psi_m \rightarrow \Psi \text{ strongly in } L^2((t_0, t_1); L^2(\Omega)). \quad (32)$$

Let $\chi(t) \in C^1([t_0, t_1]; \mathbb{R})$ with $\chi(t_1) = 0$.

Multiplying (9) by $\chi(t)$ and integrating by parts, we obtain

$$\begin{aligned} & -(b_0^{(m)}, h_i)\chi(t_0) - \int_{t_0}^{t_1} \langle \Psi_m(t), \chi'(t)h_i \rangle dt \\ & + \int_{t_0}^{t_1} \langle \nabla \Psi_m(t), \chi(t)\nabla h_i \rangle dt, \\ & = \int_{t_0}^{t_1} \langle \mu \Psi_m(t)(1 - \Psi_m(t)), \chi(t)h_i \rangle dt. \end{aligned} \quad (33)$$

According to (30)–(32), as $m \rightarrow \infty$, we obtain

$$\int_{t_0}^{t_1} \langle \Psi_m(t), \chi'(t)h_i \rangle dt \rightarrow \int_{t_0}^{t_1} \langle \Psi(t), \chi'(t)h_i \rangle dt, \quad (34)$$

$$\int_{t_0}^{t_1} \langle \nabla \Psi_m(t), \chi(t)\nabla h_i \rangle dt \rightarrow \int_{t_0}^{t_1} \langle \nabla \Psi(t), \chi(t)\nabla h_i \rangle dt, \quad (35)$$

$$\int_{t_0}^{t_1} \langle \Psi_m(t), \chi(t)h_i \rangle dt \rightarrow \int_{t_0}^{t_1} \langle \Psi(t), \chi(t)h_i \rangle dt, \quad (36)$$

$$(b_0^{(m)}, h_i)\chi(t_0) \rightarrow (b_0, h_i)\chi(t_0). \quad (37)$$

By using Hölder's inequality and $H_0^1(\Omega) \rightarrow L^4(\Omega)$, we obtain

$$\begin{aligned} & \left| \int_{t_0}^{t_1} \langle \Psi_m^2(t) - \Psi^2(t), \chi(t)h_i \rangle dt \right| \\ & \leq \int_{t_0}^{t_1} |\chi(t)| \int_{\Omega} |\Psi_m(t, x) - \Psi(t, x)| \cdot |\Psi_m(t, x) + \Psi(t, x)| \cdot |h_i(x)| dx dt, \\ & \leq \int_{t_0}^{t_1} |\chi(t)| \|\Psi_m(t, x) - \Psi(t, x)\|_{L^2(\Omega)} \|\Psi_m(t, x) + \Psi(t, x)\|_{L^4(\Omega)} \|h_i\|_{L^4(\Omega)} dt, \\ & \leq C \|\chi(t)\|_{L^\infty(t_0, t_1)} \|h_i\|_{H_0^1(\Omega)} \int_{t_0}^{t_1} \|\Psi_m(t) - \Psi(t)\|_{L^2(\Omega)} \|\Psi_m(t, x) + \Psi(t, x)\|_{H_0^1(\Omega)} dt, \\ & \leq C \|\chi(t)\|_{L^\infty(t_0, t_1)} \|h_i\|_{H_0^1(\Omega)} \|\Psi_m - \Psi\|_{L^2(t_0, t_1, L^2(\Omega))} \|\Psi_m + \Psi\|_{L^2(t_0, t_1, H_0^1(\Omega))}. \end{aligned} \quad (38)$$

According (29), (32), and (38), we get, as $m \rightarrow \infty$,

$$\int_{t_0}^{t_1} \langle \Psi_m^2(t), \chi(t)h_i \rangle dt \rightarrow \int_{t_0}^{t_1} \langle \Psi^2(t), \chi(t)h_i \rangle dt. \quad (39)$$

Relations (34), (35), and (37) allow us to pass to the limits in (9) to find

$$\begin{aligned} & -(b_0, h_i)\chi(t_0) - \int_{t_0}^{t_1} \langle \Psi(t), \chi'(t)h_i \rangle dt \\ & + \int_{t_0}^{t_1} \langle \nabla \Psi(t), \chi(t)\nabla h_i \rangle dt, \\ & = \int_{t_0}^{t_1} \langle \mu \Psi(t)(1 - \Psi(t)), \chi(t)h_i \rangle dt, \end{aligned} \quad (40)$$

$\forall i = 1, 2, \dots$ We use the linearity in h_i of (40) and $\{h_i\}$ as total in $H_0^1(\Omega)$, we obtain

$$\begin{aligned} & -(b_0, \psi)\chi(t_0) - \int_{t_0}^{t_1} \langle \Psi(t), \chi'(t)\psi \rangle dt \\ & + \int_{t_0}^{t_1} \langle \nabla \Psi(t), \chi(t)\nabla \psi \rangle dt, \\ & = \int_{t_0}^{t_1} \langle \mu \Psi(t)(1 - \Psi(t)), \chi(t)\psi \rangle dt \quad \forall \psi \in H_0^1(\Omega). \end{aligned} \quad (41)$$

Since, (41) holds for any $\psi \in \mathcal{D}(0, T)$ such that Ψ satisfies

$$\begin{aligned} \langle \Psi'(t), \psi \rangle + \langle \nabla \Psi(t), \nabla \psi \rangle & = \langle \mu \Psi(t)(1 - \Psi(t)), \psi \rangle dt \\ \forall \psi \in H_0^1. \end{aligned} \quad (42)$$

Using [23], we get $\Psi' \in L^2(0, T, H^{-1}(\Omega))$ in the sense of distributions (in time).

Finally, it rests to show that Ψ verifies the initial condition $\Psi(t_0) = b_0$.

Multiplying (42) by $\chi(t)$ and integrating by parts, we find

$$\begin{aligned} & -(\Psi(t_0), \psi)\chi(t_0) - \int_{t_0}^{t_1} \langle \Psi(t), \chi'(t)\psi \rangle dt \\ & + \int_{t_0}^{t_1} \langle \nabla \Psi(t), \chi(t)\nabla \psi \rangle dt, \\ & = \int_{t_0}^{t_1} \langle \mu \Psi(t)(1 - \Psi(t)), \chi(t)\psi \rangle dt \quad \forall \psi \in H_0^1(\Omega). \end{aligned} \quad (43)$$

A comparison of (41) and (43) yields $(b_0 - \Psi(t_0), \psi)\chi(t_0) = 0 \forall \psi \in H_0^1(\Omega)$, and we pick χ with $\chi(t_0) = 1$.

$(b_0 - \Psi(t_0), \psi) = 0 \forall \psi \in H_0^1(\Omega)$, i.e., $b_0 = \Psi(t_0)$ a.e. Ω . This is end of the proof of Theorem 1.

3. Maximum Principles

Consider Ψ a solution of (5) satisfying (6) and (7), $\exists]t_0, t_1[$, in Section 5. Under suitable hypothesis on Ψ_0 , we prove that the solution Ψ verifies $0 \leq \Psi \leq 1$. This result proves the global existence in Section 4.

Theorem 2. Suppose that $b_0 \in L^2(\Omega)$ and $b_0(x) \geq 0$ a.e in Ω . If $\Psi(t, x)$ is a solution of problem (5) satisfying (6) and (7), where $t_0 \in [0, T[$ and $t_1 \in]t_0, T[$, then $\Psi \geq 0$ a.e in $]t_0, t_1[\times \Omega$.

Proof. For a.e $t \in [t_0, t_1]$, we let $\psi(x) = \Psi_-(t, x) = \max\{-\Psi(t, x), 0\}$ in (6); we have

$$\begin{aligned} \frac{d}{dt} \|\Psi_-(t)\|_{L^2(\Omega)}^2 + \|\nabla \Psi_-(t)\|_{L^2(\Omega)}^2 &= \int_{\Omega} [\Psi_-(t)]^2 [1 - \Psi_-(t)] dx \\ &\leq \mu \|\Psi_-(t)\|_{L^2(\Omega)}^2 \\ &\quad + \mu \|\Psi_m(t)\|_{L^3(\Omega)}^3. \end{aligned} \quad (44)$$

By (21), we have

$$\frac{d}{dt} \|\Psi_-(t)\|_{L^2(\Omega)}^2 \leq 2\mu \|\Psi_-(t)\|_{L^2(\Omega)}^2 + 2C\mu \|\Psi_m(t)\|_{L^2(\Omega)}^6. \quad (45)$$

Integration over t_0 to t gives

$$\begin{aligned} \|\Psi_-(t)\|_{L^2(\Omega)}^2 &\leq \|\Psi_-(t_0)\|_{L^2(\Omega)}^2 + \int_{t_0}^t 2\mu \left(1 + \|\Psi_-(s)\|_{L^2(\Omega)}^4\right) \\ &\quad \cdot \|\Psi_m(s)\|_{L^2(\Omega)}^2 ds, \end{aligned} \quad (46)$$

where $t \in [t_0, t_1]$. We have

$$\Psi_- = \max\{-\Psi, 0\} = \frac{|\Psi| - \Psi}{2}, \quad (47)$$

and $\Psi \in C([t_0, t_1]; L^2(\Omega))$ deduce that $\Psi_- \in C([t_0, t_1]; L^2(\Omega))$.

Setting $K = \|\Psi_-(t_0)\|_{L^2(\Omega)}^2 = 0$, $\mathcal{Y} = \|\Psi_-(s)\|_{L^2(\Omega)}^2$, and $\mathcal{G} = 2\mu((1 + \|\Psi_-(s)\|_{L^2(\Omega)}^4))$, from (46), we find

$$\mathcal{Y}(t) \leq K + \int_{t_0}^t \mathcal{Y}(s) \mathcal{G}(s) ds \quad \forall t \in [t_0, t_1]. \quad (48)$$

Furthermore, by Gronwall's inequality, we get $\mathcal{Y}(t) \equiv 0$, on $[t_0, t_1]$, i.e., $\|\Psi_-(t)\|_{L^2(\Omega)}^2 \equiv 0$ and $\Psi_-(t, x) = 0$ a.e in $]t_0, t_1[\times \Omega$. $\square$

Theorem 3. Suppose that the suppositions of Theorem 2 and $b_0(x) \leq 1$ a.e in Ω hold. Then, $\Psi(t, x)$ is the local solution of (6) and (7) which satisfies $\Psi \leq 1$ a.e in $]t_0, t_1[\times \Omega$.

Proof. For a.e. $t \in [t_0, t_1]$, we let $\psi(x) = (\Psi - 1)_+(t, x) = \max\{(\Psi(t, x) - 1), 0\}$ in (6) to find

$$\begin{aligned} \langle \Psi'(t), (\Psi - 1)_+ \rangle + \int_{\Omega} \nabla \Psi \nabla (\Psi - 1)_+ dx \\ = \int_{\Omega} \Psi(1 - \Psi)(\Psi - 1)_+ dx. \end{aligned} \quad (49)$$

Equation (49) can be written as

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(\Psi - 1)_+\|_{L^2(\Omega)}^2 + \|\nabla (\Psi - 1)_+\|_{L^2(\Omega)}^2 \\ = - \int_{\Omega} \Psi [(\Psi - 1)_+]^2 dx \leq 0. \end{aligned} \quad (50)$$

Then, $(\Psi - 1)_+(t, x) = 0$ a.e in $]t_0, t_1[\times \Omega$. $\square$

4. Global Existence

In this section, we will show the global existence and uniqueness.

By the result (Theorems 1–3), under suitable hypothesis on $\Psi_0(x)$, we deduce that there exists a solution Ψ for our problem (5), satisfying (6) and (7) on some interval $[t_0, t_1]$ and $0 \leq \Psi \leq 1$ a.e in $]t_0, t_1[\times \Omega$. Hence, we expect a global solution to exist on any interval $[0, T]$.

On the contrary, the first global existence theorem below is a consequence of the local existence theorem and maximum principles; for the theorem of the second global existence, it requires further work.

For this purpose, we give now the following result.

Theorem 4. Let $T \in]0, \infty[$. Suppose that $0 \leq \Psi_0(x) \leq 1$ a.e. $x \in \Omega$. Then, $\exists ! \Psi \in W(0, T)$ such that Ψ is a solution of (6) and (7) and $0 \leq \Psi \leq 1$ a.e. in $]0, T[\times \Omega$.

Proof. Using Theorem 1, $\exists t_1 > 0$ such that $\exists \Psi \in W(0, t_1)$ as a solution of (6) and (7) on $[0, t_1]$. Applying Theorems 2 and 3, we obtain

$$0 \leq \Psi(t, x) \leq 1, \text{ a.e. } (t, x) \in]0, t_1[\times \Omega. \quad (51)$$

We let $\tilde{t} = \sup\{\tilde{t} : \exists \Psi \in W(0, \tilde{t}) \text{ satisfying (1.3) and (1.4) hold, for a.e. } t \in]0, \tilde{t}[\}$. Hence, $\tilde{t}$ must equal T . Furthermore, Theorem 1 would allow us to continue the solution beyond $\tilde{t}$, and this would contradict the maximality supposition of $\tilde{t}$ (if $\Psi \in W(0, \tilde{t})$ and $\Psi \in W(\tilde{t}, \tilde{t} + \delta)$ for some $\delta > 0$; then, $\Psi \in W(0, \tilde{t} + \delta)$).

To show the solution is a unique, we assume Ψ_1 and Ψ_2 are two solutions of (6) and (7) and set $\Gamma = \Psi_1 - \Psi_2$. Hence, Γ satisfies

$$\begin{aligned} \langle \Gamma'(t), \phi \rangle + \langle \nabla \Gamma(t), \nabla \phi \rangle &= \langle \mu \Gamma(t)(1 - \Gamma(t)), \phi \rangle \quad \forall \phi \in H_0^1(\Omega), \\ &\text{a.e. } t \in]0, T[, \end{aligned} \quad (52)$$

$$\Gamma(0, x) = 0. \quad (53)$$

By letting $\phi = \Gamma(t, \cdot)$ in (52), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Gamma(t)\|_{L^2(\Omega)}^2 + \|\nabla \Gamma(t)\|_{L^2(\Omega)}^2 &= \mu \|\Gamma(t)\|_{L^2(\Omega)}^2 - \mu \langle \Gamma^2, \Gamma \rangle, \\ a.e. t \in]0, T[, \end{aligned} \quad (54)$$

so that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Gamma(t)\|_{L^2(\Omega)}^2 + \|\nabla \Gamma(t)\|_{L^2(\Omega)}^2 &\leq \mu \|\Gamma(t)\|_{L^2(\Omega)}^2, a.e. t \in]0, T[, \\ \Rightarrow \frac{1}{2} \frac{d}{dt} \|\Gamma(t)\|_{L^2(\Omega)}^2 &\leq \mu \|\Gamma(t)\|_{L^2(\Omega)}^2, a.e. t \in]0, T[. \end{aligned} \quad (55)$$

Hence, by Gronwall's inequality and (53), we find

$$\|\Gamma(t)\|_{L^2(\Omega)}^2 = 0, a.e. t \in]0, T[. \quad (56)$$

We obtain $\Gamma = 0$ a.e. in $]0, T[\times \Omega$. The proof is complete. $\square$

Now, we present a new a priori estimate:

Lemma 1. Suppose that $0 \leq b_0(x) \leq 1$ a.e. $x \in \Omega$. From the solution Ψ of (6) and (7),

$$\begin{aligned} \|\Psi(t)\|_{L^2(\Omega)}^2 + \|\Psi\|_{L^2(0,T,H_0^1(\Omega))}^2 + \|\Psi'\|_{L^2(0,T,H^{-1}(\Omega))}^2 \\ \leq \|\Psi(t_0)\|_{L^2(\Omega)}^2 + C_1 T. \end{aligned} \quad (57)$$

Proof. Using Theorems 2 and 3, we get $0 \leq \Psi(t, x) \leq 1$ a.e. in $]0, T[\times \Omega$. Hence, for a.e. $t \in [0, T]$, upon letting $\psi(x) = \Psi(t, x)$ in (6), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Psi(t)\|_{L^2(\Omega)}^2 + \|\nabla \Psi(t)\|_{L^2(\Omega)}^2 &= \int_{\Omega} \mu [\Psi(t, x)]^2 \\ &\quad \cdot [1 - \Psi(t, x)] dx, \\ &\leq \int_{\Omega} \mu dx \leq \mu C_1(\mu, \Omega), \end{aligned} \quad (58)$$

where $C_1 > 0$ is constant. Integrating over $(0, t)$, we obtain

$$\|\Psi(t)\|_{L^2(\Omega)}^2 + \|\Psi\|_{L^2(0,T,H_0^1(\Omega))}^2 \leq \|\Psi(t_0)\|_{L^2(\Omega)}^2 + C_1 T. \quad (59)$$

From (6), we find

$$\begin{aligned} \langle \Psi'(t), \psi \rangle + \int_{\Omega} \nabla \Psi(t, x) \nabla \psi(x) dx &= \int_{\Omega} \mu [\Psi(t, x)]^2 [1 - \Psi(t, x)] \psi(x) dx, \\ &\leq \|\Psi(t)\|_{H_0^1(\Omega)} + \|\psi\|_{H_0^1(\Omega)} + \mu C \int_{\Omega} |\psi(x)| dx, \\ &\leq \|\Psi(t)\|_{H_0^1(\Omega)} + \|\psi\|_{H_0^1(\Omega)} + \mu C(\mu, \Omega) \|\psi(x)\|, \end{aligned} \quad (60)$$

for all $\psi \in H_0^1(\Omega)$ and a.e. $t \in]0, T[$.

We pick the supremum over $\psi \in H_0^1(\Omega)$, and we obtain

$$\|\Psi'(t)\|_{H^{-1}(\Omega)}^2 \leq \|\Psi\|_{H_0^1(0,T,H_0^1(\Omega))}^2 + C(\mu, \Omega), \quad (61)$$

for a.e. $t \in]0, T[$. According (59) and (61), we obtain (57). $\square$

Theorem 5. Let $T \in]0, \infty[$. $0 \leq \Psi_0(x) \leq 1$ a.e. $x \in \Omega$. Then, $\exists! \Psi \in W(0, T)$ such that Ψ is a solution of (5) satisfying (6) and (7), $0 \leq \Psi \leq 1$, a.e. in $]0, T[\times \Omega$, and

$$\begin{aligned} \|\Psi(t)\|_{L^2(\Omega)}^2 + \|\Psi\|_{L^2(0,T,H_0^1(\Omega))}^2 + \|\Psi'\|_{L^2(0,T,H^{-1}(\Omega))}^2 \\ \leq \|\Psi(t_0)\|_{L^2(\Omega)}^2 + C_1 T. \end{aligned} \quad (62)$$

Proof. For any m , Theorem 4 deduces the existence of $\Psi_m \in W(0, T)$ so that

$$\begin{aligned} \langle \Psi'_m(t), \psi \rangle + (\nabla \Psi_m(t), \nabla \psi) &= \mu (\Psi_m(t) (1 - \Psi_m(t), \psi)) \forall \psi \\ &\in H_0^1(\Omega), a.e. t \in]0, T[, \end{aligned} \quad (63)$$

$$\Psi(0, x) = \Psi_0. \quad (64)$$

Furthermore, we have that $0 \leq \Psi_n(x, t) \leq 1$ a.e. $]t, x[\in]0, T[\times \Omega$. By Lemma 1, we obtain

$$\|\Psi_m\|_{L^2(\Omega)}^2 + \|\Psi'_m\|_{L^2(0,T,H_0^1(\Omega))}^2 \leq \|\Psi(t_0)\|_{L^2(\Omega)}^2 + C_1 T. \quad (65)$$

Inequality (65) allows us to extract a subsequence of $\{\Psi_m\}$ so that

$$\Psi_m \rightharpoonup \Psi \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \quad (66)$$

$$\Psi'_m \rightharpoonup \Psi' \text{ weakly in } L^\infty(0, T; H^{-1}(\Omega)), \quad (67)$$

$$\Psi_m \rightarrow \Psi \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (68)$$

Similar to Step 3 in Theorem 1 and passing to the limits in (63), we obtain

$$\langle \Psi', \psi(t) \rangle + (\nabla \Psi(t), \nabla \psi) = \mu(\Psi(1 - \Psi), \psi) \forall \psi \in H_0^1(\Omega), a.e. t \in]0, T[. \quad (69)$$

We get (62) from (65). Theorems 2 and 3 imply $0 \leq \Psi \leq 1$ a.e. in $]0, T[\times \Omega$, and by repeating the proof of Theorem 4, the uniqueness of solution is obtained.

This is end of the proof. $\square$

5. Conclusion

The objective of this work is the study of the Fisher equation in bounded domains. By Faedo–Galerkin’s method and with a homogeneous Dirichlet conditions, we establish the existence of a global solution. This type of problem is frequently found in many fields, including chemistry, biology, and the environment.

In the next work, we will try to using the same method with same problem but by adding other conditions and damping.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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