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Research Article

Edge Metric Dimension of Some Classes of Toeplitz Networks

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Toeplitz networks are used as interconnection networks due to their smaller diameter, symmetry, simpler routing, high connectivity, and reliability. The edge metric dimension of a network is recently introduced, and its applications can be seen in several areas including robot navigation, intelligent systems, network designing, and image processing. For a vertex s and an edge $g = s_1 s_2$ of a connected graph G, the minimum number from distances of s with s_1 and s_2 is called the distance between s and g. If for every two distinct edges $s_1, s_2 \in E(G)$, there always exists $w_1 \varepsilon W_E \subseteq V(G)$, such that $d(s_1, w_1) \neq d(s_2, w_1)$; then, W_E is named as an edge metric generator. The minimum number of vertices in W_E is known as the edge metric dimension of G. In this study, we consider four families of Toeplitz networks $T_n(1,2)$, $T_n(1,3)$, $T_n(1,4)$, and $T_n(1,2,3)$ and studied their edge metric dimension. We prove that for all $n \geq 4$, $\text{edim}(T_n(1,2)) = 4$, for $n \geq 5$, $\text{edim}(T_n(1,3)) = 3$, and for $n \geq 6$, $\text{edim}(T_n(1,4)) = 3$. We further prove that for all $n \geq 5$, $\text{edim}(T_n(1,2,3)) \leq 6$, and hence, it is bounded.

1. Introduction and Preliminaries

Computer networking provides a technique of communication between a number of processors connected in a network. An interconnection network is a structure of links that joins one or more computers to each other for communication purposes. In the framework of computer networking, an interconnection network is used mainly to attach processors to processors or to permit several processors to access one or more common memory disks. Often, they are used to attach processors with nearby attached memories to each other. The approach in which these processors/memories are attached to each other have a major effect on the cost, applicability, consistency, scalability, and performance of a computer networking. It is always desirable for an interconnection network to have a smaller diameter, alternate links among the processors, a higher level of symmetry, and simpler routing. Toeplitz networks are the finest example of such an interconnection network [1].

Let $G = (V(G), E(G))$ be a connected and undirected graph. Let d_v represents the degree of a vertex v which is the total number of vertices adjacent to $v \in V(G)$. Also, the minimum degree of graph G is $\delta = \delta(G)$, and the maximum degree of G is represented by $\Delta = \Delta(G)$. The number of edges on a shortest path from vertex x_1 to vertex x_2 is called the distance between them, which is denoted by $d(x_1, x_2)$. Let $e_1 = y_1 y_2$ be an arbitrary edge of graph G and x_1 belongs to $V(G)$; then, the distance between them is represented and defined by $d(x_1, e_1) = \min\{d(x_1, y_1), d(x_1, y_2)\}$.

Metric dimension introduced by Slater introduced the metric dimension in [2], and he used it to address the challenge of locating an intruder in a network. Slater worked on the application of robot navigation and coast guard loran in [2, 3]. Melter and Harary introduced the term resolving set by expanding Slater's concept in [4]. Melter and Tomescu studied the metric dimension's role in pattern recognition and image processing issues [5]. Sebo and Tannier studied the metric dimension in combinatorial optimization in [6]. Caceres et al. worked on the mastermind and coin weighing games through metric dimension in [7]. Chartrand et al. computed the resolvability of graphs in [8]. Khuller et al. studied the application of metric dimension in navigation systems [9]. Salman et al. calculated the metric dimension of circulant graphs in [10]. A vertex x_1 distinguishes two vertices v_1 and v_2 if $d(x_1, v_1) \neq d(x_1, v_2)$. We assume $W \subseteq V(G)$ is a metric generator of graph G, if each pair of elements of $V(G)$ can be distinguished by some vertex of W. The metric dimension $\dim(G)$ of graph G is the smallest cardinality of the metric generator of G.

Kelenc et al. in [11] introduced the idea of edge metric dimension as follows. A vertex x_1 distinguishes any two edges f_1 and f_2, if $d(f_1, x_1) \neq d(f_2, x_1)$. We assume $W_E \subseteq V(G)$ is an edge metric generator of graph G, if each pair of elements of $E(G)$ can be distinguished by some vertex of W_E. The edge metric dimension $e\dim(G)$ of graph G is the smallest cardinality of the edge metric generator of graph G. The smallest edge metric generator is called the edge basis (edge metric basis). Furthermore, Kelenc et al. in [11] compared the metric dimension with the edge metric dimension and also discussed some useful results for paths P_n, cycles C_n, complete graphs K_n, and wheel graphs. Zubrilina computed the edge metric dimension of a graph with relation to the total number of vertices of graph G in [12]. Filipovi et al. computed $e\dim(GP(n,k))$ for $k = 1, 2$ and found the lower bound for all other values of k in [13]. Mufti et al. calculated $e\dim(BS(Cay(Z_n \oplus Z_2)))$ in [14]. Ahsan et al. computed $e\dim C_n(1,k)$ for $k = 2, 3$ in [15]. Fang et al. discussed the application of networks in electrical engineering in [16]. Chen et al. studied the application in chemical graphs in [17]. Yang et al. calculated the edge dimension of some families of wheel-related graphs in [18]. Wei et al. studied the edge dimension of some complex convex polytopes in [19]. Deng et al. computed the edge dimension of triangular, square, and hexagonal Mobius ladder networks in [20]. Ahmad et al. calculated the edge dimension of the benzenoid tripod structure in [21]. Moreover, Ahsan et al. calculated the edge dimension of convex polytopes in [22]. Xing et al. computed the vertex edge resolvability of the wheel graphs in [23]. Some useful lemmas are given.

Lemma 1 (See [11]). *For any $n \geq 2$, $e\dim(P_n) = \dim(P_n) = 1$, $e\dim(C_n) = \dim(C_n) = 2$, and $e\dim(K_n) = \dim(K_n) = n-1$. Moreover, $e\dim(G)$ is $1 \Leftrightarrow G$ is path.*

Lemma 2 (See [11]). *For a simple, connected graph G,*

(i) $e\dim(G) \geq \log_2(\Delta(G))$.

(ii) $e\dim(G) \geq 1 + \lceil \log_2 \delta(G) \rceil$.

The rest of the study is organized as follows. The exact edge metric dimension of the families of Toeplitz networks $T_n(1,2)$, $T_n(1,3)$, and $T_n(1,4)$ are computed in Sections 2, 3, and 4, respectively. In Section 5, we will calculate the upper bound of the family of Toeplitz networks $T_n(1,2,3)$. Last, the conclusion of the article is given.

2. Edge Metric Dimension of Toeplitz Networks $T_n(1,2)$

In this section, we will find $e\dim(T_n(1,2))$. It has $V(T_n(1,2)) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(T_n(1,2)) = \{v_\xi v_{\xi+1}\colon 1 \leq \xi \leq n-1\} \cup \{v_\xi v_{\xi+2}\colon 1 \leq \xi \leq n-2\}$.

The Toeplitz network for $n = 10$ is shown in Figure 1. The metric dimension of $T_n(1,2)$ is given.

Theorem 1 (See [24, 25]). *If $T_n(1,2)$ be a graph of the Toeplitz network with $n \geq 4$, then $\dim(T_n(1,2)) = 2$.*

In the next theorem, we will find $e\dim(T_n(1,2))$.

Theorem 2. *Let $T_n(1,2)$ be the Toeplitz network. Then, $e\dim(T_n(1,2)) = 4$, where $n \geq 4$.*

Proof. We have the following cases in order to determine $e\dim(T_n(1,2))$.

Case (i): let $n = 2\rho$, $\rho \geq 2$, and $W_E = \{v_1, v_2, v_{n-1}, v_n\} \subset V(T_n(1,2))$; we will prove that W_E is an edge basis of $T_n(1,2)$. Now, representations of each edge of $T_n(1,2)$ are given by

$$
\begin{aligned}
r\left(v_{2\xi-1}v_{2\xi+1} \mid W_E\right) &= \begin{cases} (0,1,\rho-2,\rho-1), & \text{if } \xi = 1, \\ (\xi-1,\xi-1,\rho-\xi-1,\rho-\xi), & \text{if } 2 \leq \xi \leq \rho-1, \end{cases} \\
r\left(v_{2\xi}v_{2\xi+2} \mid W_E\right) &= \begin{cases} (\xi,\xi-1,\rho-\xi-1,\rho-\xi-1), & \text{if } 1 \leq \xi \leq \rho-2, \\ (\rho-1,\rho-2,1,0), & \text{if } \xi = \rho-1, \end{cases} \\
r\left(v_{2\xi-1}v_{2\xi} \mid W_E\right) &= (\xi-1,\xi-1,\rho-\xi,\rho-\xi), \quad \text{for } 1 \leq \xi \leq \rho, \\
r\left(v_{2\xi}v_{2\xi+1} \mid W_E\right) &= (\xi,\xi-1,\rho-\xi-1,\rho-\xi), \quad \text{for } 1 \leq \xi \leq \rho-1.
\end{aligned} \tag{1}
$$

Since representations of every two edges are different, it shows that $e\dim(T_n(1,2)) \leq 4$.

Now, we will prove that the edge metric generator of cardinality three does not exist. Suppose contrarily that $e\dim(T_n(1,2)) = 3$ and let $W_E = \{v_1, v_\alpha, v_\beta\}$. Now, Table 1 provides conditions on α, β, and all edges (e_1, e_2) for which $r(e_1 \mid W_E) = r(e_2 \mid W_E)$.

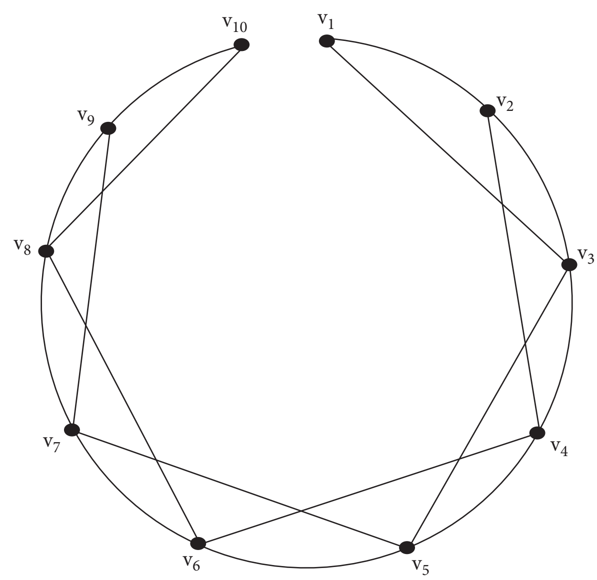

Figure 1: Toeplitz network $T_{10}(1,2)$.

Table 1: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on α and β	(e_1, e_2)
$2 \le \alpha \le n-3, 3 \le \beta \le n-2$	$(v_{n-2}v_{n-1}, v_{n-2}v_n)$
$\alpha = n-2, \beta = n-1$	$(v_{n-3}v_{n-2}, v_{n-2}v_{n-4})$
$\alpha = n-1, \beta = n$	(v_3v_4, v_2v_4)

As a result, there is no generator with three vertices, showing that $edim(T_n(1,2)) = 4$ for $n = 2\rho$, $\rho \ge 2$.

Case (ii): let $n = 2\rho + 1$, $\rho \ge 2$, and $W_E = \{v_1, v_2, v_{n-1}, v_n\} \subset V(T_n(1,2))$; we will prove that W_E is an edge basis of $T_n(1,2)$. Now, representations of each edge of $T_n(1,2)$ are given by

$$\begin{aligned}
r\left(v_{2\xi-1}v_{2\xi+1} \mid W_E\right) &= \begin{cases} (0, 1, \rho-1, \rho-1), & \text{if } \xi = 1, \\ (\xi-1, \xi-1, \rho-\xi, \rho-\xi), & \text{if } 2 \le \xi \le \rho-1, \\ (\rho-1, \rho-1, 1, 0), & \text{if } \xi = \rho, \end{cases} \\
r\left(v_{2\xi-1}v_{2\xi} \mid W_E\right) &= (\xi-1, \xi-1, \rho-\xi, \rho-\xi+1), \quad \text{for } 1 \le \xi \le \rho, \\
r\left(v_{2\xi}v_{2\xi+1} \mid W_E\right) &= (\xi, \xi-1, \rho-\xi, \rho-\xi), \quad \text{for } 1 \le \xi \le \rho, \\
r\left(v_{2\xi}v_{2\xi+2} \mid W_E\right) &= (\xi, \xi-1, \rho-\xi-1, \rho-\xi), \quad \text{for } 1 \le \xi \le \rho-1.
\end{aligned} \tag{2}$$

Since representations of every two edges are different, it shows that $edim(T_n(1,2)) \le 4$.

Now, we will prove that the edge metric generator of cardinality three does not exist. Suppose contrarily that $edim(T_n(1,2)) = 3$ and let $W_E = \{v_1, v_\alpha, v_\beta\}$. Now, Table 2 provides conditions on α, β, and all edges (e_1, e_2) for which $r(e_1 \mid W_E) = r(e_2 \mid W_E)$.

Hence, there is no generator having three vertices which prove that $edim(T_n(1,2)) = 4$ for $n = 2\rho + 1$, $\rho \ge 2$. □

3. Edge Metric Dimension of Toeplitz Networks $T_n(1,3)$

Now, we will find $edim(T_n(1,3))$. It has $V(T_n(1,3)) = \{v_1, v_2, \ldots, v_n\}$ and $E(T_n(1,3)) = \{v_\xi v_{\xi+1}: 1 \le \xi \le n-1\} \cup \{v_\xi v_{\xi+3}: 1 \le \xi \le n-3\}$. Figure 2 shows the Toeplitz network for $T_{18}(1,3)$. The metric dimension of $T_n(1,3)$ is given.

Table 2: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on α and β	(e_1, e_2)
$2 \le \alpha \le n-3, 3 \le \beta \le n-2$	$(v_{n-2}v_{n-1}, v_{n-2}v_n)$
$\alpha = n-2, \beta = n-1$	$(v_{n-3}v_{n-4}, v_{n-3}v_{n-5})$
$\alpha = n-1, \beta = n$	(v_3v_4, v_2v_4)

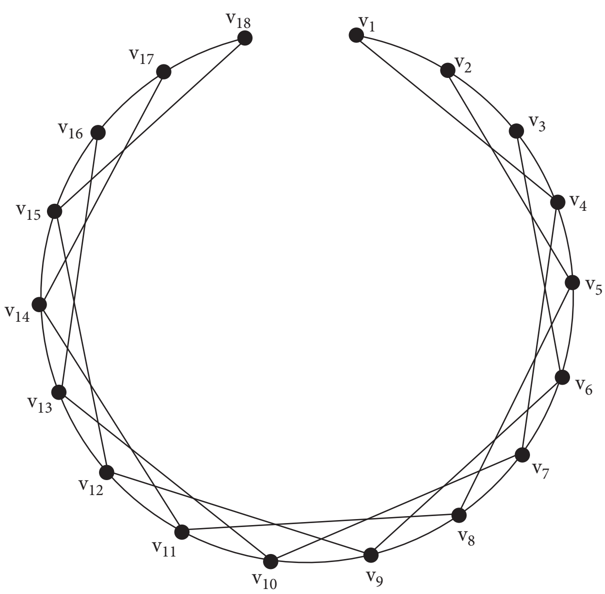

Figure 2: Toeplitz network $T_{18}(1,3)$.

Theorem 3 (See [24]). *If $T_n(1,3)$ be a graph of the Toeplitz network with $n \ge 5$, then* $\dim(T_n(1,3)) = 3$.

In the next theorem, we will find $\mathrm{edim}(T_n(1,3))$.

Theorem 4. *Let $T_n(1,3)$ be the Toeplitz network. Then,* $\mathrm{edim}(T_n(1,3)) = 3$, *where* $n \ge 5$.

Proof. We have the following cases in order to compute $\mathrm{edim}(T_n(1,3))$.

Case (i): let $n = 3\rho$, $\rho \ge 2$, and $W_E = \{v_1, v_2, v_{n-1}\} \subset V(T_n(1,3))$; we will prove that W_E is an edge basis of $T_n(1,3)$. Now, representations of each edge of $T_n(1,3)$ are given by

$$\begin{aligned}
&r\left(v_{3\xi+1}v_{3\xi+2} \mid W_E\right) = (\xi, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{3\xi+2}v_{3\xi+3} \mid W_E\right) = (\xi + 1, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{3\xi+3}v_{3\xi+4} \mid W_E\right) = (\xi + 1, \xi + 1, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 2,\\
&r\left(v_{3\xi+1}v_{3\xi+4} \mid W_E\right) = (\xi, \xi + 1, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 2,\\
&r\left(v_{3\xi+2}v_{3\xi+5} \mid W_E\right) = (\xi + 1, \xi, \rho - \xi - 2), \quad \text{for } 0 \le \xi \le \rho - 2\\
&r\left(v_{3\xi+3}v_{3\xi+6} \mid W_E\right) = (\xi + 2, \xi + 1, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 2.
\end{aligned} \tag{3}$$

Since representations of every two edges are different, it shows that $\mathrm{edim}(T_n(1,3)) \le 3$.

Now, we will prove that the edge metric generator of cardinality two does not exist. Suppose contrarily that $\mathrm{edim}(T_n(1,3)) = 2$ and let $W_E = \{v_1, v_\beta\}$. Now, Table 3 provides conditions on α, β, and all edges (e_1, e_2) for which $r(e_1 \mid W_E) = r(e_2 \mid W_E)$.

Hence, there is no generator having two vertices which prove that $\mathrm{edim}(T_n(1,3)) = 3$ for $n = 3\rho$, $\rho \ge 2$.

Case (ii): let $n = 3\rho + 1$, $\rho \ge 2$, and $W_E = \{v_1, v_2, v_{n-2}\} \subset V(T_n(1,3))$; we will prove that W_E is an edge basis of $T_n(1,3)$. Now, representations of each edge of $T_n(1,3)$ are given by

$$\begin{aligned}
&r\left(v_{3\xi+1}v_{3\xi+2} \mid W_E\right) = (\xi, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{3\xi+2}v_{3\xi+3} \mid W_E\right) = (\xi + 1, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{3\xi+3}v_{3\xi+4} \mid W_E\right) = \begin{cases} (\xi + 1, \xi + 1, \rho - \xi - 1), & \text{if } 0 \le \xi \le \rho - 2,\\ (\rho, \rho, 1), & \text{if } \xi = \rho - 1,\end{cases}\\
&r\left(v_{3\xi+1}v_{3\xi+4} \mid W_E\right) = \begin{cases} (\xi, \xi + 1, \rho - \xi - 1), & \text{if } 0 \le \xi \le \rho - 2,\\ (\rho - 1, \rho, 1), & \text{if } \xi = \rho - 1,\end{cases}\\
&r\left(v_{3\xi+2}v_{3\xi+5} \mid W_E\right) = (\xi + 1, \xi, \rho - \xi - 2), \quad \text{for } 0 \le \xi \le \rho - 2,\\
&r\left(v_{3\xi+3}v_{3\xi+6} \mid W_E\right) = (\xi + 2, \xi + 1, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 2.
\end{aligned} \tag{4}$$

Since representations of every two edges are different, it shows that $\mathrm{edim}(T_n(1,3)) \le 3$.

Now, we will prove that the edge metric generator of cardinality two does not exist. Suppose contrarily that $\mathrm{edim}(T_n(1,3)) = 2$ and let $W_E = \{v_1, v_\beta\}$. Now, Table 4 provides conditions on α, β, and all edges (e_1, e_2) for which $r(e_1 \mid W_E) = r(e_2 \mid W_E)$.

Hence, there is no generator having two vertices which prove that $\mathrm{edim}(T_n(1,3)) = 3$ for $n = 3\rho + 1$, $\rho \ge 2$.

Table 3: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on β	(e_1, e_2)
$\beta = 3\alpha, 1 \le \alpha \le \rho$	(v_2v_3, v_3v_4)
$\beta = 3\alpha + 1, 1 \le \alpha \le \rho - 1$	(v_3v_4, v_4v_5)
$\beta = 3\alpha + 2, 1 \le \alpha \le \rho - 1$	(v_2v_3, v_3v_4)
$\beta = 2$	(v_3v_4, v_4v_5)

Table 4: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on β	(e_1, e_2)
$\beta = 3\alpha,\ 1 \le \alpha \le \rho$	(v_2v_3, v_3v_4)
$\beta = 3\alpha + 1, 1 \le \alpha \le \rho$	(v_3v_4, v_4v_5)
$\beta = 3\alpha + 2, 1 \le \alpha \le \rho - 1$	(v_2v_3, v_3v_4)
$\beta = 2$	(v_3v_4, v_4v_5)

Case (iii): let $n = 3\rho + 2$, $\rho \ge 1$ and $W_E = \{v_1, v_2, v_n\} \subset V(T_n(1,3))$; we will prove that W_E is an edge basis of $T_n(1,3)$. Now, representations of each edge of $T_n(1,3)$ are given by

$$\begin{aligned}
&r\left(v_{3\xi+1}v_{3\xi+2} \mid W_E\right) = (\xi, \xi, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho,\\
&r\left(v_{3\xi+2}v_{3\xi+3} \mid W_E\right) = (\xi + 1, \xi, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{3\xi+3}v_{3\xi+4} \mid W_E\right) = (\xi + 1, \xi + 1, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{3\xi+1}v_{3\xi+4} \mid W_E\right) = (\xi, \xi + 1, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{3\xi+2}v_{3\xi+5} \mid W_E\right) = (\xi + 1, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{3\xi+3}v_{3\xi+6} \mid W_E\right) = (\xi + 2, \xi + 1, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 2.
\end{aligned} \tag{5}$$

Since representations of every two edges are different, it shows that $edim(T_n(1,3)) \le 3$.

Now, we will prove that the edge metric generator of cardinality two does not exist. Suppose contrarily that $edim(T_n(1,3)) = 2$ and let $W_E = \{v_1, v_\beta\}$. Now, Table 5 provides conditions on α, β, and all edges (e_1, e_2) for which $r(e_1 \mid W_E) = r(e_2 \mid W_E)$.

Table 5: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on β	(e_1, e_2)
$\beta = 3\alpha, 1 \le \alpha \le \rho$	(v_2v_3, v_3v_4)
$\beta = 3\alpha + 1, 1 \le \alpha \le \rho$	(v_3v_4, v_4v_5)
$\beta = 3\alpha + 2, 1 \le \alpha \le \rho - 1$	(v_2v_3, v_3v_4)
$\beta = 2$	(v_3v_4, v_4v_5)

Hence, there is no generator having two vertices which prove that $edim(T_n(1,3)) = 3$ for $n = 3\rho + 2$, $\rho \ge 1$. □

4. Edge Metric Dimension of Toeplitz Networks $T_n(1,4)$

Now, we will determine $edim(T_n(1,4))$. It has $V(T_n(1,4)) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(T_n(1,4)) = \{v_\xi v_{\xi+1}\colon 1 \le \xi \le n-1\} \cup \{v_\xi v_{\xi+4}\colon 1 \le \xi \le n-4\}$. The Toeplitz network for $n = 20$ is shown in Figure 3. $T_n(1,4)$ has the following metric dimension.

Theorem 5 (See [24]). *If $T_n(1,4)$ be a graph of the Toeplitz network with $n \ge 6$, then* $\dim(T_n(1,4)) = 2$.

In next theorem, we will find $edim(T_n(1,4))$.

Theorem 6. *Let $T_n(1,4)$ be the Toeplitz network. Then* $edim(T_n(1,4)) = 3$, *where $n \ge 6$.*

Proof. We have the following cases in order to determine $edim(T_n(1,4))$.

Case (i): let $n = 4\rho$, $\rho \ge 2$, and $W_E = \{v_1, v_2, v_{n-2}\} \subset V(T_n(1,4))$, we will prove that W_E is an edge basis of $T_n(1,4)$. Now, representations of each edge of $T_n(1,4)$ are given by

$$\begin{aligned}
&r\left(v_{4\xi+4}v_{4\xi+8} \mid W_E\right) = \begin{cases} (\xi + 2, \xi + 2, \rho - \xi - 1), & \text{if } 0 \le \xi \le \rho - 3,\\ (\rho, \rho, 2), & \text{if } \xi = \rho - 2, \end{cases}\\
&r\left(v_{4\xi+1}v_{4\xi+2} \mid W_E\right) = (\xi, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{4\xi+2}v_{4\xi+3} \mid W_E\right) = (\xi + 1, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{4\xi+3}v_{4\xi+4} \mid W_E\right) = (\xi + 2, \xi + 1, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
&r\left(v_{4\xi+4}v_{4\xi+5} \mid W_E\right) = (\xi + 1, \xi + 2, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 2,\\
&r\left(v_{4\xi+1}v_{4\xi+5} \mid W_E\right) = (\xi, \xi + 1, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 2,\\
&r\left(v_{4\xi+2}v_{4\xi+6} \mid W_E\right) = (\xi + 1, \xi, \rho - \xi - 2), \quad \text{for } 0 \le \xi \le \rho - 2,\\
&r\left(v_{4\xi+3}v_{4\xi+7} \mid W_E\right) = (\xi + 2, \xi + 1, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 2.
\end{aligned} \tag{6}$$

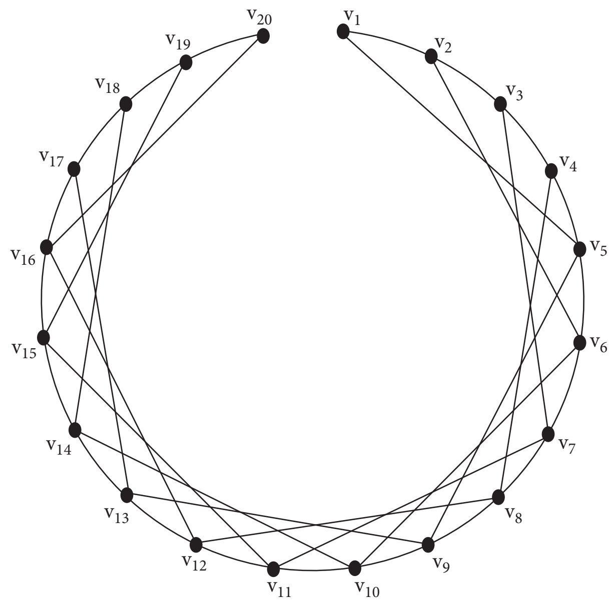

FIGURE 3: Toeplitz network $a + c$.

Since representations of every two edges are different, it shows that $edim(T_n(1,4)) \leq 3$.

Now, we will prove that the edge metric generator of cardinality two does not exist. Suppose contrarily that $edim(T_n(1,4)) = 2$ and let $W_E = \{v_1, v_\beta\}$. Now, Table 6 provides conditions on α, β and all edges (e_1, e_2) for which $r(e_1 \mid W_E) = r(e_2 \mid W_E)$.

Hence, there is no generator having two vertices which prove that $edim(T_n(1,4)) = 3$ for $n = 4\rho$, $\rho \geq 2$.

Case (ii): let $n = 4\rho + 1$, $\rho \geq 2$, and $W_E = \{v_1, v_4, v_n\} \subset V(T_n(1,4))$; we will prove that W_E is an edge basis of $T_n(1,4)$. Now, representations of each edge of $T_n(1,4)$ are given by

$$
\begin{aligned}
r\left(v_{4\xi+1}v_{4\xi+2} \mid W_E\right) &= \begin{cases} (0, 2, \rho), & \text{if } \xi = 0, \\ (\xi, \xi, \rho - \xi\rho), & \text{if } 1 \leq \xi \leq \rho - 1, \end{cases} \\
r\left(v_{4\xi+2}v_{4\xi+3} \mid W_E\right) &= (\xi + 1, \xi + 1, \rho - \xi + 1), \quad \text{for } 0 \leq \xi \leq \rho - 1, \\
r\left(v_{4\xi+3}v_{4\xi+4} \mid W_E\right) &= (\xi + 2, \xi, \rho - \xi), \quad \text{for } 0 \leq \xi \leq \rho - 1, \\
r\left(v_{4\xi+4}v_{4\xi+5} \mid W_E\right) &= (\xi + 1, \xi, \rho - \xi - 1), \quad \text{for } 0 \leq \xi \leq \rho - 1, \\
r\left(v_{4\xi+1}v_{4\xi+5} \mid W_E\right) &= \begin{cases} (0, 1, \rho - 1), & \text{if } \xi = 0, \\ (\xi, \xi, \rho - \xi - 1), & \text{if } 1 \leq \xi \leq \rho - 1, \end{cases} \\
r\left(v_{4\xi+2}v_{4\xi+6} \mid W_E\right) &= \begin{cases} (1, 2, \rho), & \text{if } \xi = 0, \\ (\xi + 1, \xi + 1, \rho - \xi), & \text{if } 1 \leq \xi \leq \rho - 2, \end{cases} \\
r\left(v_{4\xi+3}v_{4\xi+7} \mid W_E\right) &= (\xi + 2, \xi + 1, \rho - \xi), \quad \text{for } 0 \leq \xi \leq \rho - 2, \\
r\left(v_{4\xi+4}v_{4\xi+8} \mid W_E\right) &= (\xi + 2, \xi, \rho - \xi - 1), \quad \text{for } 0 \leq \xi \leq \xi - 2.
\end{aligned}
\tag{7}
$$

Since representations of every two edges are different, it shows that $edim(T_n(1,4)) \leq 3$.

Now, we will prove that the edge metric generator of cardinality two does not exist. Suppose contrarily that

Table 6: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on β	(e_1, e_2)
$\beta = 4\alpha, 1 \le \alpha \le \rho$	(v_2v_3, v_5v_6)
$\beta = 4\alpha + 1, 1 \le \alpha \le \rho - 1$	(v_4v_5, v_5v_6)
$\beta = 4\alpha + 2, 1 \le \alpha \le \rho - 1$	(v_2v_3, v_4v_5)
$\beta = 4\alpha + 3, 1 \le \alpha \le \rho - 1$	(v_2v_3, v_5v_6)
$\beta = 3$	(v_5v_6, v_5v_9)

Table 7: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on β	(e_1, e_2)
$\beta = 4\alpha, 1 \le \alpha \le \rho$	(v_2v_3, v_5v_6)
$\beta = 4\alpha + 1, 1 \le \alpha \le \rho$	(v_4v_5, v_5v_6)
$\beta = 4\alpha + 2, 1 \le \alpha \le \rho - 1$	(v_2v_3, v_4v_5)
$\beta = 4\alpha + 3, 1 \le \alpha \le \rho - 1$	(v_2v_3, v_5v_6)
$\beta = 2, 3$	(v_3v_4, v_3v_7)

$e\dim(T_n(1,4)) = 2$ and let $W_E = \{v_1, v_\beta\}$. Now, Table 7 provides conditions on α, β, and all edges (e_1, e_2) for which $r(e_1 \mid W_E) = r(e_2 \mid W_E)$.

Hence, there is no generator having two vertices which prove that $e\dim(T_n(1,4)) = 3$ for $n = 4\rho + 1$, $\rho \ge 2$.

Case (iii): let $n = 4\rho + 2$, $\rho \ge 1$, and $W_E = \{v_1, v_2, v_n\} \subset V(T_n(1,4))$; we will prove that W_E is an edge basis of $T_n(1,4)$. Now, representations of each edge of $T_n(1,4)$ are given by

$$
\begin{aligned}
r\left(v_{4\xi+1}v_{4\xi+2} \mid W_E\right) &= (\xi, \xi, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho,\\
r\left(v_{4\xi+2}v_{4\xi+3} \mid W_E\right) &= (\xi + 1, \xi, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+3}v_{4\xi+4} \mid W_E\right) &= (\xi + 2, \xi + 1, \rho - \xi + 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+4}v_{4\xi+5} \mid W_E\right) &= (\xi + 1, \xi + 2, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+1}v_{4\xi+5} \mid W_E\right) &= (\xi, \xi + 1, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+2}v_{4\xi+6} \mid W_E\right) &= (\xi + 1, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+3}v_{4\xi+7} \mid W_E\right) &= (\xi + 2, \xi + 1, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 2,\\
r\left(v_{4\xi+4}v_{4\xi+8} \mid W_E\right) &= (\xi + 2, \xi + 2, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 2.
\end{aligned}
\tag{8}
$$

Since representations of every two edges are different, it shows that $e\dim(T_n(1,4)) \le 3$.

Now, we will prove that the edge metric generator of cardinality two does not exist. Suppose contrarily that $e\dim(T_n(1,4)) = 2$ and let $W_E = \{v_1, v_\beta\}$. Now, Table 8 provides conditions on α, β, and all edges (e_1, e_2) for which $r(e_1 \mid W_E) = r(e_2 \mid W_E)$.

Hence, there is no generator having two vertices which prove that $e\dim(T_n(1,4)) = 3$ for $n = 4\rho + 2$, $\rho \ge 1$.

Case (iv): let $n = 4\rho + 3$, $\rho \ge 1$, and $W_E = \{v_1, v_2, v_{n-1}\} \subset V(T_n(1,4))$; we will prove that W_E is an edge basis of $T_n(1,4)$. Now, representations of each edge of $T_n(1,4)$ are given by

$$
\begin{aligned}
r\left(v_{4\xi+1}v_{4\xi+2} \mid W_E\right) &= (\xi, \xi, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho,\\
r\left(v_{4\xi+2}v_{4\xi+3} \mid W_E\right) &= (\xi + 1, \xi, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho,\\
r\left(v_{4\xi+3}v_{4\xi+4} \mid W_E\right) &= (\xi + 2, \xi + 1, \rho - \xi + 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+4}v_{4\xi+5} \mid W_E\right) &= (\xi + 1, \xi + 2, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+1}v_{4\xi+5} \mid W_E\right) &= (\xi, \xi + 1, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+2}v_{4\xi+6} \mid W_E\right) &= (\xi + 1, \xi, \rho - \xi - 1), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+3}v_{4\xi+7} \mid W_E\right) &= (\xi + 2, \xi + 1, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 1,\\
r\left(v_{4\xi+4}v_{4\xi+8} \mid W_E\right) &= (\xi + 2, \xi + 2, \rho - \xi), \quad \text{for } 0 \le \xi \le \rho - 2.
\end{aligned}
\tag{9}
$$

Table 8: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on β	(e_1, e_2)
$\beta = 4\alpha, 1 \leq \alpha \leq \rho$	(v_2v_3, v_5v_6)
$\beta = 4\alpha + 1, 1 \leq \alpha \leq \rho$	(v_4v_5, v_5v_6)
$\beta = 4\alpha + 2, 1 \leq \alpha \leq \rho$	(v_2v_3, v_4v_5)
$\beta = 4\alpha + 3, 1 \leq \alpha \leq \rho - 1$	(v_2v_3, v_5v_6)
$\beta = 2, 3$	(v_3v_4, v_3v_7)

Table 9: (e_1, e_2) for which $r(e_1|W_E) = r(e_2|W_E)$.

Conditions on β	(e_1, e_2)
$\beta = 4\alpha, 1 \leq \alpha \leq \rho$	(v_2v_3, v_5v_6)
$\beta = 4\alpha + 1, 1 \leq \alpha \leq \rho$	(v_4v_5, v_5v_6)
$\beta = 4\alpha + 2, 1 \leq \beta \leq \rho$	(v_2v_3, v_4v_5)
$\beta = 4\alpha + 3, 1 \leq \alpha \leq \rho$	(v_2v_3, v_5v_6)
$\beta = 2, 3$	(v_3v_4, v_3v_7)

Since representations of every two edges are different, it shows that $edim(T_n(1,4)) \leq 3$.

Now, we will prove that the edge metric generator of cardinality two does not exist. Suppose contrarily that $edim(T_n(1,4)) = 2$ and let $W_E = \{v_1, v_\beta\}$. Now, Table 9 provides conditions on α, β, and all edges (e_1, e_2) for which $r(e_1 | W_E) = r(e_2 | W_E)$.

Hence, there is no generator having two vertices which prove that $edim(T_n(1,4)) = 3$ for $n = 4\rho + 3$, $\rho \geq 1$. □

5. Upper Bounds for the Edge Metric Dimension of Toeplitz Networks $T_n(1,2,3)$

We shall compute the upper bound of $edim(T_n(1,2,3))$ in this section. It has $V(T_n(1,2,3)) = \{v_1, v_2, \ldots, v_n\}$ and $E(T_n(1,2,3)) = \{v_\xi v_{\xi+1}\colon 1 \leq \xi \leq n-1\} \cup \{v_\xi v_{\xi+2}\colon 1 \leq \xi \leq n-2\} \cup \{v_\xi v_{\xi+3}\colon 1 \leq \xi \leq n-3\}$. Figure 4 shows the Toeplitz network $T_{19}(1,2,3)$. The metric dimension of $T_n(1,2,3)$ is stated.

Theorem 7 (See [24]). *If $T_n(1,2,3)$ be a graph of the Toeplitz network with $n \geq 5$, then* $\dim(T_n(1,2,3)) = 3$.

In the next result, we will find the upper bound of $edim(T_n(1,2,3))$.

Theorem 8. *Let $T_n(1,2,3)$ be the Toeplitz networks. Then,* $edim(T_n(1,2,3)) \leq 6$, *where* $n \geq 5$.

Proof. We have the following cases to calculate the upper bound of $edim(T_n(1,2,3))$:

Case (i): let $n = 3\rho$, $\rho \geq 2$, and $W_E = \{v_1, v_2, v_3, v_{n-2}, v_{n-1}, v_n\} \subset V(T_n(1,2,3))$; we will prove that W_E is an edge basis of $T_n(1,2,3)$. Now, representations of each edge of $T_n(1,2,3)$ are given by

$$
\begin{aligned}
r\left(v_{3\xi+1}v_{3\xi+2} \,|\, W_E\right) &= \begin{cases} (0,0,1,\rho-1,\rho-1,\rho), & \text{if } \xi = 0, \\ (\xi,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi), & \text{if } 1 \leq \xi \leq \rho-1, \end{cases} \\
r\left(v_{3\xi+2}v_{3\xi+3} \,|\, W_E\right) &= \begin{cases} (\xi+1,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi-1), & \text{if } 0 \leq \xi \leq \rho-2, \\ (\rho,\rho-1,\rho-1,1,0,0), & \text{if } \xi = \rho-1, \end{cases} \\
r\left(v_{3\xi+3}v_{3\xi+4} \,|\, W_E\right) &= (\xi+1,\xi+1,\xi,\rho-\xi-2,\rho-\xi-1,\rho-\xi-1), \quad \text{for } 0 \leq \xi \leq \rho-2, \\
r\left(v_{3\xi+1}v_{3\xi+4} \,|\, W_E\right) &= \begin{cases} (0,1,1,\rho-2,\rho-1,\rho-1), & \text{if } \xi = 0, \\ (\xi,\xi,\xi,\rho-\xi-2,\rho-\xi-1,\rho-\xi-1), & \text{if } 1 \leq \xi \leq \rho-2, \end{cases} \\
r\left(v_{3\xi+2}v_{3\xi+5} \,|\, W_E\right) &= \begin{cases} (1,0,1,\rho-2,\rho-2,\rho-1), & \text{if } \xi = 0, \\ (\xi+1,\xi,\xi,\rho-\xi-2,\rho-\xi-2,\rho-\xi-1), & \text{if } 1 \leq \xi \leq \rho-3, \\ (\rho-1,\rho-2,\rho-2,1,0,1), & \text{if } \xi = \rho-2, \end{cases} \\
r\left(v_{3\xi+3}v_{3\xi+6} \,|\, W_E\right) &= \begin{cases} (\xi+1,\xi+1,\xi,\rho-\xi-2,\rho-\xi-2,\rho-\xi-2), & \text{if } 0 \leq \xi \leq \rho-3, \\ (\rho-1,\rho-1,\rho-2,1,1,0), & \text{if } \xi = \rho-2, \end{cases} \\
r &= \begin{cases} (0,1,0,\rho-1,\rho-1,\rho-1), & \text{if } \xi = 0, \\ (\xi,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi-1), & \text{if } 1 \leq \xi \leq \rho-2, \\ (\rho-1,\rho-1,\rho-1,0,1,0), & \text{if } \xi = \rho-1, \end{cases} \\
r\left(v_{3\xi+2}v_{3\xi+4} \,|\, W_E\right) &= \begin{cases} (1,0,1,\rho-2,\rho-1,\rho-1), & \text{if } \xi = 0, \\ (\xi+1,\xi,\xi,\rho-\xi-2,\rho-\xi-1,\rho-\xi-1), & \text{if } 1 \leq \xi \leq \rho-2, \end{cases} \\
r\left(v_{3\xi+3}v_{3\xi+5} \,|\, W_E\right) &= \begin{cases} (\xi+1,\xi+1,i,\rho-\xi-2,\rho-\xi-2,\rho-\xi-1), & \text{if } 0 \leq \xi \leq \rho-3, \\ (\rho-1,\rho-1,\rho-2,1,0,1), & \text{if } \xi = \rho-2. \end{cases}
\end{aligned} \tag{10}
$$

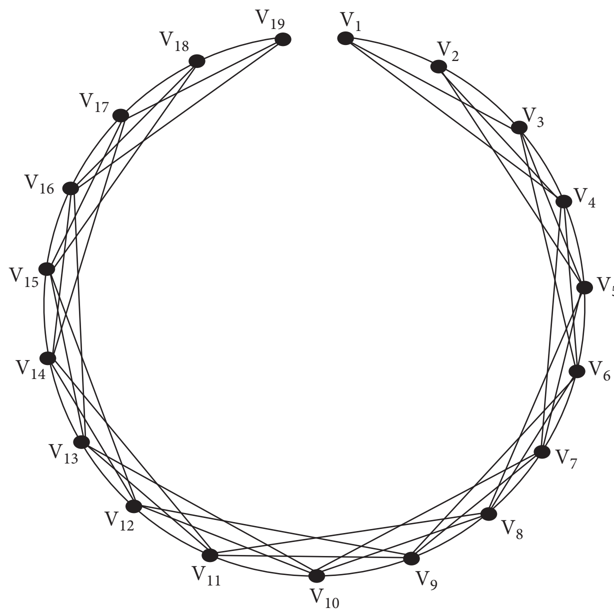

FIGURE 4: Toeplitz network $T_{19}(1,2,3)$.

Since representations of every two edges are different, it shows that $edim(T_n(1,2,3)) \leq 6$.

Case (ii): let $n = 3\rho + 1$, $\rho \geq 2$, and $W_E = \{v_1, v_2, v_3, v_{n-2}, v_{n-1}, v_n\} \subset V(T_n(1,2,3))$; we will prove that W_E is an edge metric generator of $T_n(1,2,3)$. Now, representations of each edge of $T_n(1,2,3)$ are given by

$$
\begin{aligned}
r(v_{3\xi+1}v_{3\xi+2} \mid W_E) &= \begin{cases} (0,0,1,\rho-1,\rho,\rho), & \text{if } \xi = 0, \\ (\xi,\xi,\xi,\rho-\xi-1,\rho-\xi,\rho-\xi), & \text{if } 1 \leq \xi \leq \rho-1, \end{cases} \\
r(v_{3\xi+2}v_{3\xi+3} \mid W_E) &= (\xi+1,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi), \quad \text{for } 0 \leq \xi \leq \rho-1, \\
r(v_{3\xi+3}v_{3\xi+4} \mid W_E) &= \begin{cases} (\xi+1,\xi+1,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi-1), & \text{if } 0 \leq \xi \leq \rho-2, \\ (\rho,\rho,\rho-1,1,0,0), & \text{if } \xi = \rho-1, \end{cases} \\
r(v_{3\xi+1}v_{3\xi+4} \mid W_E) &= \begin{cases} (0,1,1,\rho-1,\rho-1,\rho-1), & \text{if } \xi = 0, \\ (\xi,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi-1), & \text{if } 1 \leq \xi \leq \rho-2, \\ (\rho-1,\rho-1,\rho-1,1,1,0), & \text{if } \xi = \rho-1, \end{cases} \\
r(v_{3\xi+2}v_{3\xi+5} \mid W_E) &= \begin{cases} (1,0,1,\rho-2,\rho-1,\rho-1), & \text{if } \xi = 0, \\ (\xi+1,\xi,\xi,\rho-\xi-2,\rho-\xi-1,\rho-\xi-1), & \text{if } 1 \leq \xi \leq \rho-2, \end{cases} \\
r(v_{3\xi+3}v_{3\xi+6} \mid W_E) &= \begin{cases} (\xi+1,\xi+1,\xi,\rho-\xi-2,\rho-\xi-2,\rho-\xi-1), & \text{if } 0 \leq \xi \leq \rho-3, \\ (\rho-1,\rho-1,\rho-2,1,0,1), & \text{if } \xi = \rho-2, \end{cases} \\
r(v_{3\xi+1}v_{3\xi+3} \mid W_E) &= \begin{cases} (0,1,0,\rho-1,\rho-1,\rho), & \text{if } \xi = 0, \\ (\xi,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi), & \text{if } 1 \leq \xi \leq \rho-2, \\ (\rho-1,\rho-1,\rho-1,1,0,1), & \text{if } \xi = \rho-1, \end{cases} \\
r(v_{3\xi+2}v_{3\xi+4} \mid W_E) &= \begin{cases} (1,0,1,\rho-1,\rho-1,\rho-1), & \text{if } \xi = 0, \\ (\xi+1,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi-1), & \text{if } 1 \leq \xi \leq \rho-2, \\ (\rho,\rho-1,\rho-1,0,1,0), & \text{if } \xi = \rho-1, \end{cases} \\
r(v_{3\xi+3}v_{3\xi+5} \mid W_E) &= (\xi+1,\xi+1,\xi,\rho-\xi-2,\rho-\xi-1,\rho-\xi-1), \quad \text{for } 0 \leq \xi \leq \rho-2.
\end{aligned} \tag{11}
$$

Since representations of every two edges are different, it shows that $edim(T_n(1,2,3)) \le 6$.

Case (iii): let $n = 3\rho + 2$, $\rho \ge 1$, and $W_E = \{v_1, v_2, v_3, v_{n-2}, v_{n-1}, v_n\} \subset V(T_n(1,2,3))$; we will prove that W_E is an edge metric generator of $T_n(1,2,3)$. Now, representations of each edge of $T_n(1,2,3)$ are given by

$$
\begin{aligned}
r\left(v_{3\xi+1}v_{3\xi+2} \mid W_E\right) &= \begin{cases} (0,0,1,\rho,\rho,\rho), & \text{if } \xi = 0,\\ (\xi,\xi,\xi,\rho-\xi,\rho-\xi,\rho-\xi), & \text{if } 1 \le \xi \le \rho-1,\\ (\rho,\rho,\rho,1,0,0), & \text{if } \xi = \rho, \end{cases}\\
r\left(v_{3\xi+2}v_{3\xi+3} \mid W_E\right) &= (\xi+1,\xi,\xi,\rho-\xi-1,\rho-\xi,\rho-\xi), \quad \text{for } 0 \le \xi \le \rho-1,\\
r\left(v_{3\xi+3}v_{3\xi+4} \mid W_E\right) &= (\xi+1,\xi+1,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi), \quad \text{for } 0 \le \xi \le \rho-1,\\
r\left(v_{3\xi+1}v_{3\xi+4} \mid W_E\right) &= \begin{cases} (0,1,1,\rho-1,\rho-1,\rho), & \text{if } \xi = 0,\\ (\xi,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi), & \text{if } 1 \le \xi \le \rho-2,\\ (\rho-1,\rho-1,\rho-1,1,0,1), & \text{if } \xi = \rho-1, \end{cases}\\
r\left(v_{3\xi+2}v_{3\xi+5} \mid W_E\right) &= \begin{cases} (1,0,1,\rho-1,\rho-1,\rho-1), & \text{if } \xi = 0,\\ (\xi+1,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi-1), & \text{if } 1 \le \xi \le \rho-2,\\ (\rho,\rho-1,\rho-1,1,1,0), & \text{if } \xi = \rho-1, \end{cases}\\
r\left(v_{3\xi+3}v_{3\xi+6} \mid W_E\right) &= (\xi+1,\xi+1,\xi,\rho-\xi-2,\rho-\xi-1,\rho-\xi-1), \quad \text{for } 0 \le \xi \le \rho-2,\\
r\left(v_{3\xi+1}v_{3\xi+3} \mid W_E\right) &= \begin{cases} (0,1,0,\rho-1,\rho,\rho), & \text{if } \xi = 0,\\ (\xi,\xi,\xi,\rho-\xi-1,\rho-\xi,\rho-\xi), & \text{if } 1 \le \xi \le \rho-1, \end{cases}\\
r\left(v_{3\xi+2}v_{3\xi+4} \mid W_E\right) &= \begin{cases} (1,0,1,\rho-1,\rho-1,\rho), & \text{if } \xi = 0,\\ (\xi+1,\xi,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi), & \text{if } 1 \le \xi \le \rho-2,\\ (\rho,\rho-1,\rho-1,1,0,1), & \text{if } \xi = \rho-1, \end{cases}\\
r\left(v_{3\xi+3}v_{3\xi+5} \mid W_E\right) &= \begin{cases} (\xi+1,\xi+1,\xi,\rho-\xi-1,\rho-\xi-1,\rho-\xi-1), & \text{if } 0 \le \xi \le \rho-2,\\ (\rho,\rho,\rho-1,0,1,0), & \text{if } \xi = \rho-1. \end{cases}
\end{aligned} \tag{12}
$$

Since representations of every two edges are different, it shows that $edim(T_n(1,2,3)) \le 6$. □

6. Conclusion

In this article, we have calculated the exact value of edge metric dimension of Toeplitz networks $T_n(1,2)$, $T_n(1,3)$, and $T_n(1,4)$ and the upper bound of the Toeplitz network $T_n(1,2,3)$. We conclude that the edge metric dimension of these Toeplitz networks is constant and does not depend on the number of vertices of the graph. Here, we end with the following open problem.

Open problem 1. Calculate $edim(T_n(1,2,t))$ for $n \ge t + 3$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Zohaib Zahid, Muhammad Ahsan, and Imran Siddique conceptualized the study. Dalal Alrowaili, Sohail Zafar, and Imran Siddique developed methodology. Muhammad Ahsan, Imran Siddique, and Dalal Alrowaili wrote the original draft. Sohail Zafar and Zohaib Zahid wrote and reviewed the study. Zohaib Zahid supervised the study. All authors have read and agreed to the published version of the manuscript.

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References

[1] N. Deo and M. S. Krishnamoorthy, "Toeplitz networks and their properties," *IEEE Transactions on Circuits and Systems*, vol. 36, no. 8, pp. 1089–1092, 1989.

[2] P. J. Slater, "Leaves of trees," *Congruent Number*, vol. 14, pp. 549–559, 1975.

[3] P. J. Slater, "Dominating and reference sets in graphs," *Journal of Mathematics Physics Science*, vol. 22, pp. 445–455, 1988.
[4] F. Haray and R. A. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, vol. 2, pp. 191–195, 1976.
[5] R. A. Melter and I. Tomescu, "Metric bases in digital geometry," *Computer Vision, Graphics, and Image Processing*, vol. 25, no. 1, pp. 113–121, 1984.
[6] A. Sebo and E. Tannier, "On metric generators of graphs," *Mathematics of Operating Research*, vol. 29, pp. 383–393, 2004.
[7] J. Cáceres, C. Hernando, M. Mora et al., "On the metric dimension of cartesian products of graphs," *SIAM Journal on Discrete Mathematics*, vol. 21, no. 2, pp. 423–441, 2007.
[8] G. Chartrand, C. Poisson, and P. Zhang, "Resolvability and the upper dimension of graphs," *Computers & Mathematics with Applications*, vol. 39, no. 12, pp. 19–28, 2000.
[9] S. Khuller, B. Raghavachari, and A. Rosenfeld, "Landmarks in graphs," *Discrete Applied Mathematics*, vol. 70, no. 3, pp. 217–229, 1996.
[10] M. Salman, I. Javaid, and M. A. Chaudhry, "Resolvability in circulant graphs," *Acta Mathematica Sinica, English Series*, vol. 28, no. 9, pp. 1851–1864, 2012.
[11] A. Kelenc, N. Tratnik, and I. G. Yero, "Uniquely identifying the edges of a graph: the edge metric dimension," *Discrete Applied Mathematics*, vol. 251, pp. 204–220, 2018.
[12] N. Zubrilina, "On the edge dimension of a graph," *Discrete Mathematics*, vol. 341, no. 7, pp. 2083–2088, 2018.
[13] V. Filipovi, A. Kartelj, and J. Kratica, "Edge metric dimension of some generalized petersen graphs," *Results in Mathematics*, vol. 74, pp. 1–15, 2019.
[14] Z. S. Mufti, M. F. Nadeem, A. Ahmad, and Z. Ahmad, "Computation of edge metric dimension of barcycentric subdivision of cayley graphs," *Italian Journal of Pure and Applied Mathematics*, vol. 44, pp. 714–722, 2020.
[15] M. Ahsan, Z. Zahid, and S. Zafar, "Edge metric dimension of some classes of circulant graphs," *Analele Universitatii "Ovidius" Constanta—Seria Matematica*, vol. 28, no. 3, pp. 15–37, 2020.
[16] J. Fang, I. Ahmed, A. Mehboob, K. Nazar, and H. Ahmad, "Irregularity of block shift networks and hierarchical hypercube networks," *Journal of Chemistry*, vol. 2019, Article ID 1042308, 12 pages, 2019.
[17] L. Chen, A. Mehboob, H. Ahmad, W. Nazeer, M. Hussain, and M. R. Farahani, *Hosoya and Harary Polynomials of TOX(n), RTOX(n), TSL(n) and RTSL(n), Discrete Dynamics in Nature and Society*, vol. 2019, Article ID 8696982, 18 pages, 2019.
[18] B. Yang, M. Rafiullah, H. M. A. Siddiqui, and S. Ahmad, "On resolvability parameters of some wheel-related graphs," *Journal of Chemistry*, vol. 2019, Article ID 9259032, 9 pages, 2019.
[19] C. Wei, M. Salman, S. Shahzaib, M. U. Rehman, and J. Fang, "Classes of planar graphs with constant edge metric dimension," *Complexity*, vol. 2021, Article ID 5599274, 10 pages, 2021.
[20] B. Deng, M. F. Nadeem, and M. Azeem, "On the edge metric dimension of different families of mobius networks," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6623208, 9 pages, 2021.
[21] A. Ahmad, S. Husain, M. Azeem, K. Elahi, and M. K. Siddiqui, "Computation of edge resolvability of benzenoid tripod structure," *Journal of Mathematics*, vol. 2021, Article ID 9336540, 8 pages, 2021.
[22] M. Ahsan, Z. Zahid, S. Zafar, A. Rafiq, M. S. Sindhu, and M. Umar, "Computing the edge metric dimension of convex polytopes related graphs," *Journal of Mathematics Computer Science*, vol. 22, pp. 174–188, 2021.
[23] B. H. Xing, S. K. Sharma, V. K. Bhat, H. Raza, and J. B. Liu, "The vertex-edge resolvability of some wheel-related graphs," *Journal of Mathematics*, vol. 2021, Article ID 1859714, 16 pages, 2021.
[24] J. B. Liu, M. F. Nadeem, H. M. A. Siddiqui, and W. Nazir, "Computing metric dimension of certain families of toeplitz graphs," *IEEE Access*, vol. 7, pp. 126734–126741, 2019.
[25] K. Chau and S. Gosselin, "The metric dimension of circulant graphs and their cartesian products," *Opuscula Mathematica*, vol. 37, no. 4, pp. 509–534, 2017.