Research Article

Product Antimagic Labeling of Caterpillars

Shengze Wang and Yuping Gao

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

Correspondence should be addressed to Yuping Gao; gaoyup@lzu.edu.cn

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Let G be a graph with m edges. A product antimagic labeling of G is a bijection from the edge set $E(G)$ to the set $\{1, 2, \dots, m\}$ such that the vertex-products are pairwise distinct, where the vertex-product of a vertex v is the product of labels on the incident edges of v . A graph is called product antimagic if it admits a product antimagic labeling. In this paper, we will show that caterpillars with at least three edges are product antimagic by an $O(m \log m)$ algorithm.

1. Introduction

Let $b > a$ be two integers. We use $[a, b]$ to denote the set $\{a, a+1, \dots, b\}$ and simply write $[1, a]$ as $[a]$. All graphs considered in this paper are simple and finite. Let G be a graph with m edges. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. For two vertex sets $X, Y \subset V(G)$, the set of edges with one end in X and the other end in Y is denoted by $E(X, Y)$. An antimagic labeling of G is a bijection τ from $E(G)$ to $[m]$ such that for any two distinct vertices u and v in G , the sum of labels on the edges incident with u differs from that of v . A graph is said to be antimagic if it admits an antimagic labeling. The concept of antimagic labeling was proposed by Hartsfield and Ringel in 1990 [1]. In the same paper, they conjectured that every connected graph other than K_2 is antimagic. This topic was investigated by many researchers; for instance, see [2–6]. Recently, Lozano et al. [7] proved that caterpillars are antimagic, where a caterpillar is a tree with at least three vertices such that the removal of its leaves produces a path.

In 2000, Figueroa-Centeno et al. [8] introduced multiplicative variation of antimagic labeling. A product antimagic labeling of a graph G with m edges is a bijection φ from $E(G)$ to $[m]$ such that the vertex-products are pairwise distinct, where the vertex-product $p(v)$ of a vertex v is the product of labels on the incident edges of v . A graph G is called product antimagic if there is a product antimagic labeling of G . In [8], the authors proved that paths with at

least four vertices and 2-regular graphs are product antimagic. Furthermore, they proposed the following conjecture.

Conjecture 1. *A connected graph with at least three edges is product antimagic.*

Kaplan et al. [9] proved that the following graphs are product antimagic: the disjoint union of cycles and paths where each path has at least three edges; connected graphs with n vertices and m edges where $m \geq 4n \ln n$; graphs G where each component has at least two edges and the minimum degree of G is at least $8\sqrt{\ln|E|\ln(\ln|E|)}$; and all complete k -partite graphs except K_2 and $K_{1,2}$. In [10], Pikhurko characterizes all large graphs that are product antimagic. More precisely, it is shown that there is an integer n_0 such that a graph with $n \geq n_0$ vertices is product antimagic if and only if it does not belong to any of the following four classes: graphs that have at least one isolated edge; graphs that have at least two isolated vertices; unions of vertex-disjoint of copies of $K_{1,2}$; graphs consisting of one isolated vertex; and graphs obtained by subdividing some edges of the star $K_{1,k+\ell}$.

In this paper, we prove that Conjecture 1 is affirmative for caterpillars with at least three edges.

Theorem 1. *Every caterpillar with at least three edges is product antimagic.*

There are some other variations of antimagic labeling, such as antimagic orientation; see [11–14] for some results of trees. For more types of labelings, refer to the survey of Joseph [15]. The remainder of this paper is organized as below. In the next section, we prove Theorem 1. In Section 3, we write the labeling procedure of Theorem 1 as an algorithm.

2. Proof of Theorem 1

Let T be a caterpillar. A leaf of T is a vertex of degree one in T , a spine of T is a longest path of T , and a leg of T is an edge that does not belong to the spine of T .

Proof. of Theorem 1. Let T be a caterpillar with $m (\geq 3)$ edges. Since it is proved that paths with at least three edges are product antimagic, we may assume that T is not a path. Let $P = v_0 v_1, \dots, v_\ell$ be a spine of T . Let $U = \{v_{h_1}, v_{h_2}, \dots, v_{h_t}\} \subset V(P)$ be the set of vertices of degree at least three in T , where $h_1 < h_2 < \dots < h_t$. Define

$$X = \{x \in V(T) \setminus V(P) \mid x \text{ is adjacent to a vertex in } U\}. \quad (1)$$

In fact, a vertex in X is a leaf incident with a leg in T . Note that $V(P) \cup X = V(T)$. Let M be a maximum matching in $E(U, X)$. Therefore, M is a matching of size $|U|$ that saturates all vertices in U . We define a product antimagic labeling of T in three steps in the following, and see Figure 1 for example.

Step 1. Label the edges in $E(P)$. If ℓ is odd, starting from the edge $v_0 v_1$, we label the edges in $E(P)$ consecutively with $2, m - (\ell - 1)/2 + 1, 3, m - (\ell - 1)/2 + 2, \dots, \ell - 1/2 + 1, m, (\ell - 1)/2 + 2$, respectively.

If ℓ is even, starting from the edge $v_0 v_1$, we label the edges in $E(P)$ consecutively with $m - \ell/2 + 1, 2, m - \ell/2 + 2, 3, \dots, m - 1, \ell/2, m, \ell/2 + 1$, respectively.

The current labeling is denoted by φ_1 , and the vertex-product of a vertex x in $V(T)$ is denoted by $p_1(x)$. It can be seen that

$$\varphi_1(v_i v_{i+1}) = \begin{cases} \frac{i}{2} + 2, & i \equiv 0 \pmod{2}, \\ m - \frac{\ell - 1}{2} + \frac{i + 1}{2}, & i \equiv 1 \pmod{2}, \end{cases} \quad (2)$$

where $i \in [0, \ell - 1]$, in which ℓ is odd.

And

$$\varphi_1(v_i v_{i+1}) = \begin{cases} m - \frac{\ell}{2} + \frac{i}{2} + 1, & i \equiv 0 \pmod{2}, \\ \frac{i + 1}{2} + 1, & i \equiv 1 \pmod{2}, \end{cases} \quad (3)$$

where $i \in [0, \ell - 1]$, in which ℓ is even.

Claim 1. For any two distinct vertices, $v_i, v_j \in V(P)$ and $p_1(v_i) \neq p_1(v_j)$.

Proof. By the definition of vertex-product,

$$p_1(v_i) = \begin{cases} \varphi_1(v_0 v_1), & i = 0, \\ \varphi_1(v_{i-1} v_i) \cdot \varphi_1(v_i v_{i+1}), & i \in [1, \ell - 1], \\ \varphi_1(v_{\ell-1} v_\ell), & i = \ell. \end{cases} \quad (4)$$

By equations (2) and (3), it is easy to verify that

$$p_1(v_1) < p_1(v_2) < \dots < p_1(v_{\ell-1}). \quad (5)$$

Furthermore, $p_1(v_0) < p_1(v_1)$, $p_1(v_\ell) < p_1(v_{\ell-1})$, and $p_1(v_0) \neq p_1(v_\ell)$. So, for any two distinct vertices, $v_i, v_j \in V(P)$ and $p_1(v_i) \neq p_1(v_j)$. $\square$

Step 2. Label edges in $E(U, X) \setminus E(M)$ if it is not an empty set.

If ℓ is odd, label the edges in $E(U, X) \setminus E(M)$ using numbers in $[1] \cup [(\ell - 1)/2 + 3, m - (\ell - 1)/2 - t]$ one by one arbitrarily (we reserve numbers in $[m - (\ell - 1)/2 - t + 1, m - (\ell - 1)/2]$ for edges in M).

If ℓ is even, label the edges in $E(U, X) \setminus E(M)$ using numbers in $[1] \cup [\ell/2 + 2, m - \ell/2 - t]$ one by one arbitrarily (we reserve numbers in $[m - \ell/2 - t + 1, m - \ell/2]$ for edges in M).

Denote the current labeling by φ_2 and the partial vertex-product of a vertex $x \in V(T)$ by $p_2(x)$.

Step 3. Label edges in M .

Case 1. $E(U, X) \setminus E(M) \neq \emptyset$.

Assume that $p_2(v_{i_1}) \leq p_2(v_{i_2}) \leq \dots \leq p_2(v_{i_t})$, where $\{i_1, i_2, \dots, i_t\} = \{h_1, h_2, \dots, h_t\}$. Label edges in M incident with $v_{i_1}, v_{i_2}, \dots, v_{i_t}$ with $m - \lfloor \ell/2 \rfloor - t + 1, \dots, m - \lfloor \ell/2 \rfloor$, respectively, in this order. The resulting labeling is denoted by φ_3 , and the current vertex-product of a vertex $x \in V(T)$ is denoted by $p_3(x)$.

Claim 2. φ_3 is a product antimagic labeling of T .

Proof. By the way of assigning labels to edges in M , we have

$$p_3(v_{i_1}) < p_3(v_{i_2}) < \dots < p_3(v_{i_t}). \quad (6)$$

It is easy to see that φ_3 is a bijection from $E(T)$ to $[m]$. We show that φ_3 is a product antimagic labeling of T in the following. Let V_1 and V_2 be the sets of vertices of degree 1 and 2, respectively. Then, $V(T) = V_1 \cup V_2 \cup U$. For each vertex $x \in V_1 \setminus V(P)$, we know that $p_3(x) \leq m - \lfloor \ell/2 \rfloor$ by Step 3. Combining with Claim 1 and the labeling steps, it follows that no vertex in V_1 receives the same vertex-product as other vertices in T . Also, by Claim 1 and equation (5), it suffices to prove that for any vertex $u \in V_2$ and any vertex $v \in U$, $p_3(u) \neq p_3(v)$.

If ℓ is odd, then

$$p_3(u) \leq m \cdot \left(\frac{\ell - 1}{2} + 2 \right), \quad (7)$$

$$p_3(v) \geq 2 \cdot \left(m - \frac{\ell - 1}{2} + 1 \right) \cdot \left(\frac{\ell - 1}{2} + 3 \right),$$

FIGURE 1: Examples of product antimagic labelings of caterpillars: (a) a caterpillar with the spine containing odd number of edges; (b) a caterpillar with the spine containing even number of edges.

Input: A caterpillar T with m edges
Output: A product antimagic labeling of T
Step 1: Label the edges in path P
 (1) $2 \rightarrow \varphi(v_0v_1)$
 (2) $m - \lfloor \ell/2 \rfloor + 1 \rightarrow \varphi(v_1v_2)$
 if ℓ is even **then**
 (3) Exchange $\varphi(v_0v_1)$ and $\varphi(v_1v_2)$
 (4) **for** $i = 2$ to $\ell - 1$ **do**
 (5) $\varphi(v_{i-2}v_{i-1}) + 1 \rightarrow \varphi(v_i v_{i+1})$
Step 2: Label the edges in the legs except one incident with each vertex in U
 (6) **for all** $v \in U$ **do**
 (7) **for all** legs e incident with v except one **do**
 (8) a random label from $([1] \cup [\lfloor \ell/2 \rfloor + 2, m - \lfloor \ell/2 \rfloor]) \setminus \varphi(E(T)) \rightarrow \varphi(e)$
Step 3: Label the remaining edges
 (9) Sort the vertices in U as $u_1, \dots, u_t$ s.t. $p(u_i) \leq p(u_{i+1})$ for all $i \in [1, t-1]$
 (10) Sort the labels in $([1] \cup [\lfloor \ell/2 \rfloor + 2, m - \lfloor \ell/2 \rfloor]) \setminus \varphi(E(T))$ as $l_1, \dots, l_t$ in increasing order
 (11) **for** $i = 1$ to t **do**
 (12) $l_i \rightarrow \varphi(e_i^*)$, where $\varphi(e_i^*)$ is the unlabeled leg incident with u_i

ALGORITHM 1: Product antimagic labeling of a caterpillar.

since $(\ell-1)/2 + 2 < (\ell-1)/2 + 3$ and $m < 2 \cdot (m - (\ell-1)/2 + 1)$ if and only if $\ell < m + 3$. It follows that $p_3(u) < p_3(v)$.

If ℓ is even, then

$$p_3(u) \leq m \cdot \left(\frac{\ell}{2} + 1\right), \quad (8)$$

$$p_3(v) \geq \left(m - \frac{\ell}{2} + 1\right) \cdot 2 \cdot \left(\frac{\ell}{2} + 2\right),$$

since $\ell/2 + 1 < \ell/2 + 2$ and $m < 2 \cdot (m - \ell/2 + 1)$ if and only if $\ell < m + 2$. It follows that $p_3(u) < p_3(v)$.

By the above discussion, φ_3 is a product antimagic labeling of T . $\square$

Case 2. $E(U, X) = E(M)$.

In this case, Step 2 does not exist. We assign the remaining labels $1, \lfloor \ell/2 \rfloor + 2, \lfloor \ell/2 \rfloor + 3, \dots, m - \lfloor \ell/2 \rfloor$ to the edges in $E(M)$ incident with $v_{h_1}, v_{h_2}, \dots, v_{h_t}$ one by one, respectively. The resulting labeling is denoted by φ' , and the vertex-product of a vertex $x \in V(T)$ is denoted by $p'(x)$.

Claim 3. φ' is a product antimagic labeling of T .

Proof. By equation (5) in the proof of Claim 1 and the labeling φ_1 , $p_1(v_1) < p_1(v_2) < \dots < p_1(v_{\ell-1}) \leq m \cdot (\lfloor \ell/2 \rfloor$

$+ 1)$. By the labeling of edges in $E(M)$, we know that $p'(v_{h_1}) = p_1(v_{h_1})$ and $2 \cdot (m - \lfloor \ell/2 \rfloor + 1) \cdot (\lfloor \ell/2 \rfloor + 2) \leq p'(v_{h_2}) < p'(v_{h_3}) < \dots < p'(v_{h_t})$. It can be seen that $\lfloor \ell/2 \rfloor + 1 < \lfloor \ell/2 \rfloor + 2$ and $m < 2 \cdot (m - \lfloor \ell/2 \rfloor + 1)$. So, $m \cdot (\lfloor \ell/2 \rfloor + 1) < 2 \cdot (m - \lfloor \ell/2 \rfloor + 1) \cdot (\lfloor \ell/2 \rfloor + 2)$. It follows that no two vertices in $V(T)$ receive the same vertex-product and φ' is a product antimagic labeling of T . $\square$

3. An Algorithm

In this section, we will write the steps of labeling a caterpillar as Algorithm 1. The notation follows from the last section.

Finally, we show that Algorithm 1 runs in time $O(m \log m)$. Assignments in Step 1 of the algorithm can be done in constant time. Therefore, the cost of Step 1 is the result of the linear loop at lines 5 and 6 and the assignments at line 7, that is, $O(m)$. Note that if each vertex in U is of degree three, Step 2 does not run. Otherwise, Step 2 visits at most m edges and assigns a random label to each of them, but labels can be chosen increasingly from the unused labels in $[1] \cup [\lfloor \ell/2 \rfloor + 2, m - \lfloor \ell/2 \rfloor]$, thus giving a cost $O(m)$. Step 3 requires time $O(m \log m)$ due to the fact that the partial vertex-products must be sorted (line 10). The total cost of the algorithm is, then, $O(m \log m)$. Since T is a tree, the vertex number $n = m + 1$, so the cost can be also expressed as $O(n \log n)$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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