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Research Article

An Inertial Iterative Algorithm to Find Common Solution of a Split Generalized Equilibrium and a Variational Inequality Problem in Hilbert Spaces

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In this paper, we introduce and study an iterative algorithm via inertial and viscosity techniques to find a common solution of a split generalized equilibrium and a variational inequality problem in Hilbert spaces. Further, we prove that the sequence generated by the proposed theorem converges strongly to the common solution of our problem. Furthermore, we list some consequences of our established algorithm. Finally, we construct a numerical example to demonstrate the applicability of the theorem. We emphasize that the result accounted in the manuscript unifies and extends various results in this field of study.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The variational inequality problem (in short, VIP) is to find $x^* \in C$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1}$$

where B: $C \longrightarrow H_1$ is a nonlinear mapping. The solution set of VIP (1) is denoted by Ω. It is introduced by Hartman and Stampacchia [1].

In 1994, Blum and Oettli [2] introduced and studied the following equilibrium problem (in short, EP): find $x^* \in C$ such that

$$G_1(x^*, y) \geq 0, \quad \forall y \in C, \tag{2}$$

where G_1: $C \times C \longrightarrow \mathbb{R}$ is a bifunction. The solution set of EP (3) is denoted by Sol (EP(3)).

In the last two decades, EP (2) has been generalized and extensively studied in many directions due to its importance; see, for example [3–7], for the literature on the existence and iterative approximation of solution of the various generalizations of EP (2).

Censor et al. [8] introduced the split feasibility problem (in short, S_PFP) in finite-dimensional Hilbert spaces for modelling of inverse problems that arise from phase retrievals and in medical image restoration as

$$\text{find } x^* \in C \text{ such that } Bx^* \in Q, \tag{3}$$

where B: $H_1 \longrightarrow H_2$ is a bounded linear operator.

In this paper, we consider the following split generalized equilibrium problem (in short, S_PGEP):

Let G_1, b_1: $C \times C \longrightarrow \mathbb{R}$ and G_2, b_2: $Q \times Q \longrightarrow \mathbb{R}$ be nonlinear mappings, and B: $H_1 \longrightarrow H_2$ be a bounded linear operator, then S_PGEP is to find $x^* \in C$ such that

$$G_1(x^*, x) + b_1(x, x^*) - b_1(x^*, x^*) \geq 0, \quad \forall x \in C, \tag{4}$$

and such that

$$y^* = Bx^* \in Q \text{ solves } G_2(y^*, y) + b_2(y, y^*) - \phi(y^*, y^*) \geq 0, \quad \forall y \in Q. \tag{5}$$

If we take $b_1, b_2 \equiv 0$, then S_PGEP becomes split equilibrium problem (in short, S_PEP) as

$$G_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{6}$$

and such that

$$y^* = Bx^* \in Q \text{ solves } G_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{7}$$

When looked separately, (4) is the generalized equilibrium problem (GEP) and we denote its solution set by Sol (GEP(4)). The S_PGEP(4) and (5) constitute a pair of generalized equilibrium problems which have to be solved so that the image $y^* = Bx^*$ under a given bounded linear operator B of the solution x^* of the GEP(4) in H_1 is the solution of another GEP (4) in another space H_2. We denote the solution set of GEP (5) by Sol (GEP (5)). The solution set of S_PGEP (4) and (5) is denoted by $\Gamma = \{p \in \mathrm{Sol}(\mathrm{GEP}(4))\colon Bp \in \mathrm{Sol}(\mathrm{GEP}(5))\}$.

S_PGEP (4) and (5) generalize multiple-sets split feasibility problem. It also includes as special case, the split variational inequality problem, which is the generalization of split zero problems and split feasibility problems, see for details [9–12].

In 2008, Mainge [13] introduced the following inertial Krasnosel'skiǐ–Mann algorithm by combining Krasnosel'skiǐ–Mann algorithm and the inertial extrapolation:

$$\left\{ \begin{array}{l} t_n = x_n + \theta_n (x_n - x_{n-1}), \\ x_{n+1} = (1 - \eta_n) t_n + \eta_n T t_n, \end{array} \right\}, \tag{8}$$

for each $n \geq 1$. He proved that the sequence $\{x_n\}$ generated by algorithm (8) converges weakly to a fixed point of T under some conditions. Recently, Bot et al. [14] studied the convergence analysis of the inertial Krasnosel'skiǐ–Mann algorithm for approximating a fixed point of nonexpansive mapping T by getting rid of some conditions used in the main result of Mainge [13]. Recently, Dong et al. [15, 16] introduced the inertial hybrid algorithm and established a strong convergence theorem for approximating a fixed point of nonexpansive mapping T in the setting of Hilbert space. For further study of some generalization of iterative algorithm (8), see for instance [17, 18]. Very recently, Monairah et al. [19] introduced and studied a hybrid iterative algorithm to approximate a common solution of generalized equilibrium problem, variational inequality problem, and fixed point problem in the framework of a 2 uniformly convex and uniformly smooth real Banach space. The inertial method has been studied by many researchers. The results and other related ones analyzed the convergence properties of inertial type algorithms and demonstrated their performance numerically on some imaging and data analysis problems, see for details [20–23].

Motivated by the work given in [6, 13, 24], we propose an iterative algorithm via inertial and viscosity techniques to find a common solution of a split generalized equilibrium and a variational inequality problem in Hilbert spaces. We obtained the strong convergence for the proposed algorithm. Further, we give some consequences of the main result. Finally, we discuss a numerical example to demonstrate the applicability of the iterative algorithm. The method and result presented in this paper generalize and unify the previously known related methods and results. Our result can extend several iterative methods given in the literature.

2. Preliminaries

In this section, we collect some concepts and results which are required for the presentation of the work. Let symbols $\longrightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

For every point $x \in H_1$, there exists a unique nearest point to x in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{9}$$

The mapping P_C is called the metric projection of H_1 onto C. It is well known that P_C is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{10}$$

Moreover, $P_C x$ is characterized by the fact that $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall y \in C. \tag{11}$$

This implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H_1, \forall y \in C. \tag{12}$$

In a real Hilbert space H_1, it is well known that

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in H_1 \text{ and } \lambda \in [0, 1], \tag{13}$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H_1. \tag{14}$$

Definition 1 (see [25]). A multivalued mapping $M\colon H_1 \longrightarrow 2^{H_1}$ is called monotone if for all $x, y \in H_1$, $u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v \rangle \geq 0. \tag{15}$$

Definition 2 (see [25]). A multivalued monotone mapping $M\colon H_1 \longrightarrow 2^{H_1}$ is maximal if the graph (M), the graph of M, is not properly contained in the graph of any other monotone mapping.

Remark 1. It is known that a multivalued monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in \mathrm{Graph}(M)$ implies that $u \in Mx$.

Lemma 1 (see [26]). *Let $\{x_n\}$ and $\{u_n\}$ be bounded sequences in a Banach space E and let β_n be a sequence in $(0, 1)$ with* $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1 -$

$\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,*

$$\lim_{n\longrightarrow\infty}\|y_n - x_n\| = 0. \tag{16}$$

Lemma 2 (see [27]). *Let* $\{b_n\}$ *be a sequence of nonnegative real numbers such that there exists a subsequence* $\{b_{n_i}\}$ *of* $\{b_n\}$ *such that* $b_{n_i} < b_{n_{i+1}}$, $\forall i \in \mathbb{N}$. *Then, there exists a nondecreasing sequence* $\{m_j\}$ *of* $\mathbb{N}$ *such that* $\lim_{j\longrightarrow\infty} m_j = \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $j \in \mathbb{N}$

$$\begin{aligned} b_{m_j} &\leq b_{m_{j+1}}, \\ b_j &\leq b_{m_j}. \end{aligned} \tag{17}$$

In fact, m_j is the largest number n in the set $\{1, 2, 3, \ldots, j\}$ such that $b_n < b_{n+1}$.

Lemma 3 (see [28]). *Assume that D is a strongly positive self-adjoint bounded linear operator on a Hilbert space* H_1 *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|D\|^{-1}$. *Then,* $\|I - \rho D\| \leq 1 - \rho\overline{\gamma}$.

Lemma 4 (see [29]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0, \tag{18}$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\longrightarrow\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < +\infty$.

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Assumption 1. Let $G_1: C \times C \longrightarrow \mathbb{R}$ and $b_1: C \times C \longrightarrow \mathbb{R}$ be bimappings satisfying the following conditions:

(1) $G_1(x,x) = 0$, $\forall x \in C$;

(2) G_1 is monotone, i.e.,

$$G_1(x,y) + G_1(y,x) \leq 0, \quad \forall x, y \in C. \tag{19}$$

(3) For each $y \in C$, $x \longrightarrow G_1(x,y)$ is weakly upper semicontinuous;

(4) For each $x \in C$, $y \longrightarrow G_1(x,y)$ is convex and lower semicontinuous;

(5) $b_1(.,.)$ is weakly continuous and $b_1(.,y)$ is convex;

(6) b_1 is skew-symmetric, i.e.,

$$\begin{aligned} b_1(x,x) - b_1(x,y) + b_1(y,y) - b_1(y,x) &\geq 0, \\ \forall x, y &\in C. \end{aligned} \tag{20}$$

Now, we define $T_r^{(G_1,b_1)}: H_1 \longrightarrow C$ as follows:

$$\begin{aligned} T_r^{(G_1,b_1)}(z) = \Big\{x \in C&: G_1(x,y) + b_1(y,x) - b_1(x,x) \\ &+ \frac{1}{r}\langle y - x, x - z\rangle \geq 0,\ \forall y \in C\Big\}, \end{aligned} \tag{21}$$

where r is a positive real number.

Lemma 5 (see [30]). *Let C be a nonempty closed convex subset of Hilbert space* H_1. *Let* $G_1, b_1: C \times C \longrightarrow \mathbb{R}$ *be nonlinear mappings satisfying Assumption 1. Assume that for each* $z \in H_1$ *and for each* $x \in C$, *there exists a bounded subset* $D_x \subseteq C$ *and* $z_x \in C$ *such that for any* $y \in C, D_x$,

$$G_1(y,z_x) + b_1(z_x,y) - b_1(y,y) + \frac{1}{r}\langle z_x - y, y - z\rangle < 0. \tag{22}$$

Let the mapping $T_r^{(G_1,b_1)}$ *be defined by (21). Then, the following conclusions hold:*

(i) $T_r^{(G_1,\phi_1)}(z)$ *is nonempty for each* $z \in H_1$;

(ii) $T_r^{(G_1,\phi_1)}$ *is single-valued;*

(iii) $T_r^{(G_1,\phi_1)}$ *is a firmly nonexpansive mapping, i.e., for all* $z_1, z_2 \in H_1$,

$$\begin{aligned} &\left\|T_r^{(G_1,\phi_1)}(z_1) - T_r^{(G_1,\phi_1)}(z_2)\right\|^2 \\ &\leq \langle T_r^{(G_1,\phi_1)}(z_1) - T_r^{(G_1,\phi_1)}(z_2), z_1 - z_2\rangle. \end{aligned} \tag{23}$$

(iv) $Fix(T_r^{(G_1,\phi_1)}) = Sol(GEP(4))$;

(v) $Sol(GEP(4))$ *is closed and convex.*

Further, assume that $G_2: Q \times Q \longrightarrow \mathbb{R}$ *and* $b_2: Q \times Q \longrightarrow \mathbb{R}$ *satisfy Assumption 1. For* $s > 0$ *and for all* $u \in H_2$, *define a mapping* $T_s^{(G_2,b_2)}: H_2 \longrightarrow Q$ *as follows:*

$$\begin{aligned} T_s^{(G_2,b_2)}(u) = \Big\{v \in Q&: G_2(v,w) + b_2(w,v) - b_2(v,v) \\ &+ \frac{1}{s}\langle w - v, v - u\rangle \geq 0, \forall w \in Q\Big\}. \end{aligned} \tag{24}$$

Then, we easily observe that $T_s^{(G_2,b_2)}$ *is nonempty, single-valued, firmly nonexpansive,* $fix(T_s^{(G_2,b_2)}) = Sol(GEP(5))$, *and* $Sol(GEP(5))$ *is closed and convex.*

Lemma 6 (see [30]). *Let* G_1 *and* b_1 *satisfy Assumption 1 and let the mapping* $T_r^{(G_1,b_1)}$ *be defined by (21). Let* $x_1, x_2 \in H_1$ *and* $r_1, r_2 > 0$, *then*

$$\left\|T_{r_2}^{(G_1,b_1)}(x_2) - T_{r_1}^{(G_1,b_1)}(x_1)\right\| \leq \|x_2 - x_1\| + \frac{|r_2 - r_1|}{r_2}\left\|T_{r_2}^{(G_1,b_1)}(x_2) - x_2\right\|. \tag{25}$$

3. Main Result

In this section, we prove a strong convergence theorem based on the proposed iterative algorithm to approximate a common solution of S_PGEP (4), (5), and VIP (1) (Algorithm 1).

Theorem 1. *Let C and Q be two nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $B: H_1 \longrightarrow H_2$ be a bounded linear operator. Assume that $G_1: C\times C \longrightarrow \mathbb{R}$, $G_2: Q\times Q \longrightarrow \mathbb{R}$, $b_1: C\times C \longrightarrow \mathbb{R}$, and $b_2: Q\times Q \longrightarrow \mathbb{R}$ are nonlinear mappings satisfying Assumption 1 and G_2 is upper semicontinuous in the first argument. Assume that $\Theta = \Gamma\cap\Omega \neq \varnothing$. Let $g: H_1 \longrightarrow H_1$ be a contraction mapping with constant $\alpha\in(0,1)$ and $D: C \longrightarrow H_1$ be a τ-inverse strongly monotone mapping. Let $\{x_n\}$ be generated by Algorithm 1 and satisfy the following conditions:*

(i) $\lim_{n\longrightarrow\infty}\eta_n = 0, \sum_{n=0}^{\infty}\eta_n = \infty$;

(ii) $\lim_{n\longrightarrow\infty}(\theta_n/\eta_n)\|x_n - x_{n-1}\| = 0$;

(iii) $\{\theta_n\}\subset[0,\theta]$, *for some* $\theta>0$ *and* $\delta_n\subset(0,2\tau)$;

(iv) $\liminf_{n\longrightarrow\infty} r_n > 0$ *and* $\lim_{n\longrightarrow\infty}|r_{n+1}-r_n| = 0$;

(v) $\{\lambda_n\}\subset\mathbb{R}$, *such that* $a\le\lambda_n\le b<(1/L)$, *where* $L=\|B\|^2$.

Then, the sequence $\{x_n\}$ converges strongly to some $q\in\Theta$, where $q = P_\Theta(g)q$.

Proof. We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Let $q\in\Theta=\Gamma\cap\Omega$, then $q = T_{r_n}^{(G_1,b_1)}q$ and $Bq = T_{r_n}^{(G_2,b_2)}(Bq)$. Applying the similar steps used in Theorem 1 [6], we obtain

$$\|u_n - q\|^2 \le \|t_n - q\|^2 - \lambda_n(1-\lambda_n L)\left\|\left(T_{r_n}^{(G_2,b_2)} - I\right)Bt_n\right\|^2. \tag{26}$$

Thus,

$$\|u_n - q\| \le \|t_n - q\|. \tag{27}$$

We estimate

$$\begin{aligned}\|t_n - q\| &= \|x_n - \theta_n(x_{n-1}-x_n) - q\|\\ &\le \|x_n - q\| + \theta_n\|x_n - x_{n-1}\|\\ &= \|x_n - q\| + \left(\frac{\theta_n}{\eta_n}\right)\eta_n\|x_n - x_{n-1}\|.\end{aligned} \tag{28}$$

From condition (ii), $\exists N_1>0$ such that

$$\frac{\theta_n}{\eta_n}\|x_n - x_{n-1}\| \le N_1, \quad \forall n\ge 1. \tag{29}$$

By (27)–(29), we have

$$\|u_n - q\| \le \|t_n - q\| \le \|x_n - q\| + \eta_n N_1. \tag{30}$$

Since the mapping $I-\delta_n B$ is nonexpansive, therefore

$$\begin{aligned}\|v_n - q\| &= \|P_C(I-\delta_n D)u_n - q\|\\ &\le \|(I-\lambda_n D)u_n - (I-\delta_n D)p\| \le \|u_n - q\|.\end{aligned} \tag{31}$$

We estimate

$$\begin{aligned}\|x_{n+1}-q\| &= \|\eta_n g(x_n) + (1-\eta_n)v_n - q\|\\ &= \|\eta_n(g(x_n)-q) + (1-\eta_n)(v_n-q)\|\\ &\le \eta_n\|g(x_n)-q\| + (1-\eta_n)\|v_n-q\|\\ &\le \eta_n\|g(x_n)-g(q)\| + \eta_n\|g(q)-q\| + (1-\eta_n)\|v_n-q\|\\ &\le \eta_n\alpha\|x_n-q\| + \eta_n\|g(q)-q\| + (1-\eta_n)\|v_n-q\|.\end{aligned} \tag{32}$$

Using (30) and (31) in the above inequality, we have

$$\begin{aligned}\|x_{n+1}-q\| &\le (1-(1-\alpha)\eta_n)\|x_n-q\| + \eta_n N_1 + \eta_n\|g(q)-q\|\\ &= (1-(1-\alpha)\eta_n)\|x_n-q\| + \eta_n N_1 + (1-\alpha)\eta_n\frac{N_1+\|g(q)-q\|}{1-\alpha}\\ &\le \max\|x_n-q\|,\frac{N_1+\|g(q)-q\|}{1-\alpha} \le \cdots \le \max\|x_n-q\|,\frac{N_1+\|g(q)-q\|}{1-\alpha}.\end{aligned} \tag{33}$$

Initialization: choose $x_0, x_1 \in H_1$ to be arbitrary.
Iterative Steps: given the current iterate x_n, compute:
Step 1. Compute $t_n = x_n - \theta_n (x_{n-1} - x_n)$
Step 2. Compute $u_n = T_{r_n}^{(G_1,b_1)} (t_n + \lambda_n B^* (T_{r_n}^{(G_2,b_2)} - I)Bt_n)$
Step 3. Compute $v_n = P_C(u_n - \delta_n D u_n)$
and calculate the next iterate x_{n+1} as follows:
$x_{n+1} = \eta_n g(x_n) + (1 - \eta_n) v_n$
Set $n := n + 1$ *and go to* **Step 1**.

Algorithm 1: Iterative algorithm.

Thus, $\{x_n\}$ is bounded. Also, $\{u_n\}$, $\{v_n\}$, and $\{t_n\}$ are bounded.

Step 2. We show that $(1-\eta_n)\lambda_n(1-\lambda_n L)\|(T_{r_n}^{(G_2,b_2)} - I)Bt_n\| \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \eta_n N_4$, for some $N_4 > 0$. We estimate

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \eta_n \|g(x_n) - q\|^2 + (1-\eta_n)\|v_n - q\|^2 \\
&\leq \eta_n \left(\|g(x_n) - g(q)\| + \|g(q) - q\|\right)^2 + (1-\eta_n)\|v_n - q\|^2 \\
&\leq \eta_n \left(\alpha\|x_n - q\| + \|g(q) - q\|\right)^2 + (1-\eta_n)\|v_n - q\|^2 \\
&\leq \eta_n \left(\|x_n - q\| + \|g(q) - q\|\right)^2 + (1-\eta_n)\|v_n - q\|^2 \\
&= \eta_n \|x_n - q\|^2 + \eta_n \left(2\|x_n - q\|\|g(q) - q\| + \|g(q) - q\|^2\right) + (1-\eta_n)\|v_n - q\|^2 \\
&\leq \eta_n \|x_n - q\|^2 + (1-\eta_n)\|u_n - q\|^2 + \eta_n N_2, \quad \text{for some } N_2 > 0.
\end{aligned} \tag{34}$$

Using (26) in the above inequality, we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \eta_n \|x_n - q\|^2 + (1-\eta_n)\|t_n - q\|^2 \\
&\quad - (1-\eta_n)\lambda_n (1-\lambda_n L)\left\|\left(T_{r_n}^{(G_2,b_2)} - I\right)Bt_n\right\|^2 + \eta_n N_2.
\end{aligned} \tag{35}$$

From (30), we obtain

$$\begin{aligned}
\|t_n - q\|^2 &\leq \left(\|x_n - q\| + \eta_n N_1\right)^2 \\
&= \|x_n - q\|^2 + \eta_n\left(2N_1\|x_n - q\| + \eta_n N_1^2\right) \\
&\leq \|x_n - q\|^2 + \eta_n N_3, \quad \text{for some } N_3 > 0.
\end{aligned} \tag{36}$$

By (35) and (36), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \eta_n \|x_n - q\|^2 + (1-\eta_n)\|x_n - q\|^2 + \eta_n N_3 \\
&\quad - (1-\eta_n)\lambda_n (1-\lambda_n L)\left\|\left(T_{r_n}^{(G_2,b_2)} - I\right)Bt_n\right\|^2 + \eta_n N_2 \\
&= \|x_n - q\|^2 + \eta_n N_3 - (1-\eta_n)\lambda_n (1-\lambda_n L)\left\|\left(T_{r_n}^{(G_2,b_2)} - I\right)Bt_n\right\|^2 + \eta_n N_2,
\end{aligned} \tag{37}$$

which yields that

$$\begin{aligned}
&(1-\eta_n)\lambda_n (1-\lambda_n L)\left\|\left(T_{r_n}^{(G_2,b_2)} - I\right)Bt_n\right\|^2 \\
&\leq \|x_{n+1} - q\|^2 - \|x_n - q\|^2 + \eta_n N_4,
\end{aligned} \tag{38}$$

where $N_4 = N_2 + N_3$.

Step 3. We show that

$$\begin{aligned}
(1-\eta_n)\|u_n - t_n\|^2 &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \eta_n N_4 \\
&\quad + 2(1-\eta_n)\lambda_n \|u_n - q\|\left\|B^*\left(T_{r_n}^{(G_2,b_2)} - I\right)Bt_n\right\|.
\end{aligned} \tag{39}$$

Using the concept of firmly nonexpansive of $T_{r_n}^{(G_2,b_2)}$, we have

$$\begin{aligned}
\|u_n - q\|^2 &= \left\| T_{r_n}^{(G_1,\phi_1)}\left(t_n + \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\right) - q\right\|^2 \\
&= \left\| T_{r_n}^{(G_1,\phi_1)}\left(t_n + \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n - T_{r_n}^{(G_1,\phi_1)} q\right\|^2 \\
&\le \langle u_n - q, t_n + \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n - q\rangle \\
&= \frac{1}{2}\|u_n - q\|^2 + \frac{1}{2}\left\| t_n + \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n - q\right\|^2 \\
&\quad - \frac{1}{2}\left\| u_n - q - t_n - \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n + q\right\|^2 \\
&= \frac{1}{2}\|u_n - q\|^2 + \frac{1}{2}\left\| t_n - q + \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\right\|^2 \\
&\quad - \frac{1}{2}\left\| u_n - t_n - \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\right\|^2 \\
&= \frac{1}{2}\|u_n - q\|^2 + \frac{1}{2}\|t_n - q\|^2 + \frac{1}{2}\lambda_n^2\left\| B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\right\|^2 \\
&\quad + \langle t_n - q, \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\rangle - \frac{1}{2}\|u_n - t_n\|^2 \\
&\quad - \frac{1}{2}\lambda_n^2\left\| B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\right\|^2 + \langle u_n - t_n, \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\rangle \\
&= \frac{1}{2}\|u_n - q\|^2 + \frac{1}{2}\|t_n - q\|^2 - \frac{1}{2}\|u_n - t_n\|^2 + \langle u_n - q, \lambda_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\rangle,
\end{aligned} \tag{40}$$

which implies that

$$\begin{aligned}
\|u_n - q\|^2 &\le \|t_n - q\|^2 - \|u_n - t_n\|^2 + 2\langle u_n - q, \mu_n B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\rangle \\
&\le \|t_n - q\|^2 - \|u_n - t_n\|^2 + 2\mu_n\|u_n - q\|\left\| B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\right\|.
\end{aligned} \tag{41}$$

Using (41) in (35), we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \eta_n\|x_n - q\|^2 + (1-\eta_n)\|t_n - q\|^2 - (1-\eta_n)\|u_n - t_n\|^2 \\
&\quad + 2\mu_n(1-\eta_n)\|u_n - q\|\left\| B^*\left(T_{r_n}^{(G_2,\phi_2)} - I\right)Bt_n\right\| + \eta_n N_2.
\end{aligned} \tag{42}$$

Using (36), in the above inequality,

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \eta_n\|x_n-q\|^2+(1-\eta_n)\|x_n-q\|^2+\eta_n N_3-(1-\eta_n)\|u_n-t_n\|^2\\ &\quad+2\mu_n(1-\eta_n)\|u_n-q\|\left\|B^*\left(T_{r_n}^{(G_2,\phi_2)}-I\right)Bt_n\right\|\\ &\le \|x_n-q\|^2-(1-\eta_n)\|u_n-t_n\|^2\\ &\quad+2\mu_n(1-\eta_n)\|u_n-q\|\left\|B^*\left(T_{r_n}^{(G_2,\phi_2)}-I\right)Bt_n\right\|+\eta_n N_4,\end{aligned}\tag{43}$$

which implies

$$\begin{aligned}(1-\eta_n)\|u_n-t_n\|^2 &\le \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad+2\mu_n(1-\eta_n)\|u_n-q\|\\ &\quad\left\|B^*\left(T_{r_n}^{(G_2,\phi_2)}-I\right)Bt_n\right\|+\eta_n N_4.\end{aligned}\tag{44}$$

Step 4. We show that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le (1-(1-\alpha)\eta_n)\|x_n-q\|^2+(1-\alpha)\eta_n\\ &\quad\times\left[\frac{2}{1-\alpha}\langle g(q)-q,x_{n+1}-q\rangle+\frac{\theta_n}{\eta_n}\|x_n-x_{n-1}\|\frac{N}{1-\alpha}\right],\end{aligned}\tag{45}$$

for some $N>0$.

We estimate

$$\begin{aligned}\|t_n-q\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-q\|^2\\ &= \|x_n-q\|^2+2\theta_n\langle x_n-q,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\\ &\le \|x_n-q\|^2+2\theta_n\|x_n-q\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2\\ &= \|x_n-q\|^2+\theta_n\|x_n-x_{n-1}\|\left(2\|x_n-q\|+\theta_n\|x_n-x_{n-1}\|\right)\\ &\le \|x_n-q\|^2+\theta_n\|x_n-x_{n-1}\|N,\quad \text{for some } N>0.\end{aligned}\tag{46}$$

Using (14), we calculate

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\eta_n g(x_n)+(1-\eta_n)u_n-q\|^2\\ &= \|\eta_n(g(x_n)-g(q))+(1-\eta_n)(u_n-q)+\eta_n(g(q)-q)\|^2\\ &\le \|\eta_n(g(x_n)-g(q))+(1-\eta_n)(u_n-q)\|^2+2\eta_n\langle g(q)-q,x_{n+1}-q\rangle\\ &\le \eta_n\|g(x_n)-g(q)\|^2+(1-\eta_n)\|u_n-q\|^2+2\eta_n\langle g(q)-q,x_{n+1}-q\rangle\\ &\le \eta_n\alpha\|x_n-q\|^2+(1-\eta_n)\|u_n-q\|^2+2\eta_n\langle g(q)-q,x_{n+1}-q\rangle\\ &\le \eta_n\alpha\|x_n-q\|^2+(1-\eta_n)\|t_n-q\|^2+2\eta_n\langle g(q)-q,x_{n+1}-q\rangle.\end{aligned}\tag{47}$$

From (46) and (47), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le (1-(1-\alpha)\eta_n)\|x_n-q\|^2+\theta_n\|x_n-x_{n-1}\|N+2\eta_n\langle g(q)-q,x_{n+1}-q\rangle\\ &=(1-(1-\alpha)\eta_n)\|x_n-q\|^2+(1-\alpha)\eta_n\\ &\quad\times\left[\frac{2}{1-\alpha}\langle g(q)-q,x_{n+1}-q\rangle+\frac{\theta_n}{\eta_n}\|x_n-x_{n-1}\|\frac{N}{1-\alpha}\right].\end{aligned}\tag{48}$$

Step 5. We show that $\lim_{n\longrightarrow\infty}\|x_n-q\|=0$.

To show it, we have the two cases as follows:

Case 1. There exists $m\in\mathbb{N}$ such that $\|x_{n+1}-q\|^2\le\|x_n-q\|^2, \forall n\ge m$. This shows that $\lim_{n\longrightarrow\infty}\|x_n-q\|$ exists, and by step 2, we have

$$\lim_{n\longrightarrow\infty}\left\|\left(T_{r_n}^{(G_2,b_2)}-I\right)Bt_n\right\|=0.\tag{49}$$

Thanks to step 3 and (49), we obtain

$$\lim_{n\longrightarrow\infty}\|u_n-t_n\|=0.\tag{50}$$

Since $\|x_{n+1}-u_n\|=\eta_n\|u_n-g(x_n)\|$, therefore

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-v_n\|=0.\tag{51}$$

Now,

$$\|x_n-t_n\|=\theta_n\|x_n-x_{n-1}\|=\frac{\theta_n}{\eta_n}\|x_n-x_{n-1}\|\eta_n\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\tag{52}$$

Next, prove that $\lim_{n\longrightarrow\infty}\|u_n-v_n\|=0$.

By (47), we have

$$\begin{aligned}\|x_{n+1}-q\|^2&\le\eta_n\alpha\|x_n-q\|^2+(1-\eta_n)\|v_n-q\|^2+2\eta_n\langle g(q)-q,x_{n+1}-q\rangle\\ &=\eta_n\alpha\|x_n-q\|^2+(1-\eta_n)\|v_n-q\|^2+2\eta_n\langle\zeta,x_{n+1}-q\rangle\\ &\le\eta_n\alpha\|x_n-q\|^2+(1-\eta_n)\|v_n-q\|^2+2\eta_n\varrho^2.\end{aligned}\tag{53}$$

We set $\zeta=f(q)-q$ and let $\varrho>0$ be a suitable constant with $\varrho\ge\sup_n\{\|\zeta\|,\|x_n-q\|\}$ in the above inequality. Thus,

$$\begin{aligned}\|x_{n+1}-q\|^2&\le(1-\eta_n)\left\{\|P_C(u_n-\lambda_nBu_n)-P_C(q-\lambda_nBq)\|^2\right\}+\eta_n\|x_n-q\|^2+2\eta_n\varrho^2\\ &\le(1-\eta_n)\left\{\|u_n-q\|^2+\lambda_n(\lambda_n-2\gamma)\|Bu_n-Bq\|^2\right\}+\eta_n\|x_n-q\|^2+2\eta_n\varrho^2\\ &\le(1-\eta_n)\left\{\|x_n-\tilde{x}\|^2+\lambda_n(\lambda_n-2\gamma)\|Bu_n-Bq\|^2\right\}+\eta_n\|x_n-\tilde{x}\|^2+2\eta_n\varrho^2\\ &\le(1-\eta_n)\lambda_n(\lambda_n-2\gamma)\|Bu_n-Bq\|^2+\|x_n-q\|^2+2\eta_n\varrho^2.\end{aligned}\tag{54}$$

This implies

$$\begin{aligned}&(1-\eta_n)\lambda_n(2\gamma-\lambda_n)\|Bu_n-Bq\|^2\\&\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+2\eta_n\varrho^2.\end{aligned}\tag{55}$$

Thus,

$$\lim_{n\longrightarrow\infty}\|Bu_n-Bq\|=0.\tag{56}$$

We compute

$$\begin{aligned}\|v_n-q\|^2&=\|P_C(u_n-\lambda_nBu_n)-P_C(q-\lambda_nBq)\|^2\\&\le\langle v_n-q,(u_n-\lambda_nBu_n)-(q-\lambda_nBq)\rangle\\&\le\frac{1}{2}\left\{\|v_n-q\|^2+\|(u_n-\lambda_nBu_n)-(q-\lambda_nBq)\|^2-\|(v_n-u_n)+\lambda_n(Bu_n-Bq)\|^2\right\}\\&\le\frac{1}{2}\left\{\|v_n-q\|^2+\|u_n-q\|^2-\|(v_n-u_n)+\lambda_n(Bu_n-Bq)\|^2\right\}\\&\le\|u_n-q\|^2-\|v_n-u_n\|^2-\lambda_n^2\|Bu_n-Bq\|^2+2\lambda_n\langle v_n-u_n,Bv_n-Bq\rangle\\&\le\|u_n-q\|^2-\|v_n-u_n\|^2+2\lambda_n\|v_n-u_n\|\|Bu_n-Aq\|\\&\le\|x_n-q\|^2-\|v_n-u_n\|^2+2\lambda_n\|v_n-u_n\|\|Bu_n-Aq\|.\end{aligned}\tag{57}$$

By (53), we get

$$\begin{aligned}\|x_{n+1}-q\|^2&\le(1-\lambda_n)\|v_n-q\|^2+\lambda_n\|x_n-q\|^2+2\eta_n\varrho^2\\&\le(1-\lambda_n)\left\{\|x_n-q\|^2-\|v_n-u_n\|^2+2\lambda_n\|v_n-u_n\|\|Bu_n-Bq\|\right\}+\lambda_n\|x_n-q\|^2+2\eta_n\varrho^2,\end{aligned}\tag{58}$$

which implies

$$(1-\lambda_n)\|v_n-u_n\|^2\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+2(1-\lambda_n)\lambda_n\|v_n-u_n\|\|Bu_n-Bq\|+2\eta_n\varrho^2.\tag{59}$$

Using (56) and the given conditions, we get

$$\lim_{n\longrightarrow\infty}\|v_n-u_n\|=0.\tag{60}$$

From (50)–(52) and (60), we have

$$\|x_{n+1}-x_n\|\le\|x_{n+1}-u_n\|+\|u_n-t_n\|+\|t_n-x_n\|\longrightarrow 0,\quad\text{as } n\longrightarrow\infty.\tag{61}$$

We prove that $\limsup_{n\longrightarrow\infty}\langle(g-I)q,x_n-q\rangle\le 0$.

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which converges weakly to some $p\in C$. Without loss of generality, we can assume that $u_{n_i}\rightharpoonup p$ such that

$$\limsup_{n\longrightarrow\infty}\langle(g-I)q,u_n-q\rangle=\lim_{i\longrightarrow\infty}\langle(g-I)q,u_{n_i}-q\rangle.\tag{62}$$

We prove that $p\in\Gamma\cap\Omega$.

Since $u_n=T_{r_n}^{(G_1,b_1)}d_n$ where $d_n:=t_n+\lambda_nB^*(T_{r_n}^{(G_2,b_2)}-I)Bt_n$, we have

$$G_1(u_n,u)+b_1(u,u_n)-b_1(u_n,u_n)+\frac{1}{r_n}\langle u-u_n,u_n-d_n\rangle\ge 0,\quad\forall u\in C,\tag{63}$$

which implies that

$$b_1(u,u_n) - b_1(u_n,u_n) + \frac{1}{r_n}\langle u - u_n, u_n - d_n\rangle \geq G_1(u,u_n),$$
$$\forall u \in C \,(\text{using monotonocity of } G_1). \tag{64}$$

Hence,

$$b_1\left(u,u_{n_k}\right) - b_1\left(u_{n_k},u_{n_k}\right) + \langle u - u_{n_k}, \frac{u_{n_k} - d_{n_k}}{r_{n_k}}\rangle \geq G_1\left(u,u_{n_k}\right),$$
$$\forall u \in C. \tag{65}$$

Let $u_t = (1-t)w + tu$, for all $t \in (0,1]$. Since $u \in C$ and $w \in C$, we get $u_t \in C$ and from (65), we have

$$0 \leq G_1\left(u_t,u_{n_k}\right) - b_1\left(u_t,u_{n_k}\right) + b_1\left(u_{n_k},u_{n_k}\right)$$
$$- \langle u_t - u_{n_k}, \frac{u_{n_k} - t_{n_k}}{r_{n_k}} + \lambda_n B^*\left(\frac{\left(T_{r_{n_k}}^{(G_2,b_2)} - I\right)Bt_{n_k}}{r_{n_k}}\right)\rangle. \tag{66}$$

Since B^* is bounded linear, it follows from (49), (50), (60), and $\liminf r_n > 0$ that $((u_{n_k} - t_{n_k})/r_{n_k}) \longrightarrow 0$ and $B^*((T_{r_{n_k}}^{(G_2,b_2)} - I)Bt_{n_k}/r_{n_k}) \longrightarrow 0$ and so

$$b_1(u_t,p) - b_1(p,p) \leq G_1(u_t,p). \tag{67}$$

Now, for $t > 0$,

$$\begin{aligned} 0 &= G_1(u_t,u_t) \\ &= tG_1(u_t,u_t) + (1-t)G_1(u_t,p) \\ &\geq tG_1(u_t,u_t) + (1-t)[b_1(u_t,p) - b_1(p,p)] \\ &\geq tG_1(u_t,u_t) + (1-t)t[b_1(u,p) - b_1(p,p)] \\ &\geq G_1(u_t,u_t) + (1-t)[b_1(u,p) - b_1(p,p)]. \end{aligned} \tag{68}$$

Letting $t \longrightarrow 0$, we have

$$G_1(p,u) + b_1(u,p) - b_1(p,p) \geq 0, \quad \forall u \in C. \tag{69}$$

This implies that $p \in \mathrm{Sol}(\mathrm{GEP}(4))$.

Next, we show that $Bp \in \mathrm{Sol}(\mathrm{GEP}(5))$. Since $\|u_n - t_n\| \longrightarrow 0, u_n \rightharpoonup p$ as $n \longrightarrow \infty$ and $\{t_n\}$ is bounded, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \rightharpoonup p$ and since B is a bounded linear operator so that $Bt_{n_k} \rightharpoonup Bp$.

Now, setting $v_{n_k} = Bt_{n_k} - T_{r_{n_k}}^{(G_2,b_2)}Bt_{n_k}$, it follows that from (49), $\lim_{k\longrightarrow\infty} v_{n_k} = 0$ and $Bt_{n_k} - v_{n_k} = T_{r_{n_k}}^{(G_2,b_2)}Bt_{n_k}$. Therefore, from Lemma 5, we have

$$G_2\left(Bt_{n_k} - v_{n_k}, z\right) + b_1\left(z,u_{n_k}\right) - b_1\left(u_{n_k},u_{n_k}\right)$$
$$+ \frac{1}{r_{n_k}}\langle z - \left(Bt_{n_k} - v_{n_k}\right), \left(Bt_{n_k} - v_{n_k}\right) - Bt_{n_k}\rangle \geq 0, \quad \forall z \in Q. \tag{70}$$

Since G_2 is upper semicontinuous in the first argument, taking limit superior to the above inequality as $k \longrightarrow \infty$ and using condition, we obtain

$$G_2(Bp,z) + b_1\left(z,u_{n_k}\right) - b_1\left(u_{n_k},u_{n_k}\right) \geq 0, \quad \forall z \in Q, \tag{71}$$

which means that $Bp \in \mathrm{Sol}(\mathrm{GEP}(5))$ and hence $p \in \Gamma$.

Next, we prove $p \in \Omega$. Since $\lim_{n\longrightarrow\infty}\|u_n - v_n\| = 0$ and $\lim_{n\longrightarrow\infty}\|u_n - t_n\| = 0$, there exist subsequences $\{u_{n_i}\}$ and $\{v_{n_i}\}$ of $\{u_n\}$ and $\{v_n\}$, respectively such that $u_{n_i} \rightharpoonup p$ and $v_{n_i} \rightharpoonup p$.

Define the mapping M as

$$M(z) = \begin{cases} D(z) + N_C(p), & \text{if } p \in C, \\ \varnothing, & \text{if } p \notin C, \end{cases} \tag{72}$$

where $N_C(p) := \{v \in H_1 : \langle p - u, v\rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $p \in H_1$. In this case, the mapping M is maximal monotone and hence $0 \in Mp$ mapping if and only if $p \in \mathrm{Sol}(\mathrm{VIP}(1))$. Let $(p,v) \in \mathrm{graph}(M)$. Then, we have $v \in Mp = Dp + N_C(p)$ and hence $v - Dp \in N_C(p)$. So, we have $\langle p - u, v - Dp\rangle \geq 0$, for all $u \in C$. On the other hand, from $v_n = P_C(u_n - \delta_n Du_n)$ and $p \in C$, we have

$$\langle (u_n - \delta_n Du_n) - v_n, v_n - p\rangle \geq 0. \tag{73}$$

This implies that

$$\langle z - v_n, \frac{v_n - u_n}{\delta_n} + Du_n\rangle \geq 0. \tag{74}$$

Since $\langle z - u, v - Dz\rangle \geq 0$, for all $z \in C$ and $v_{n_i} \in C$, using monotonicity of D, we have

$$\begin{aligned}\langle z - v_{n_i}, v\rangle &\geq \langle z - v_{n_i}, Dp\rangle \\ &\geq \langle z - v_{n_i}, Dz\rangle - \left\langle z - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\delta_n} + Du_{n_i}\right\rangle \\ &= \langle z - v_{n_i}, Dz - Du_{n_i}\rangle + \langle z - v_{n_i}, Dv_{n_i} - Du_{n_i}\rangle - \left\langle z - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\delta_n}\right\rangle \\ &\geq \langle z - v_{n_i}, Dv_{n_i} - Du_{n_i}\rangle \langle \left\langle z - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\delta_n}\right\rangle.\end{aligned} \tag{75}$$

Since D is continuous therefore on taking limit $i \longrightarrow \infty$, we have $\langle z - p, v\rangle \geq 0$. Since M is maximal monotone, we have $p \in M^{-1}(0)$ and hence $p \in \Omega$. Thus, $p \in \Gamma \cap \Omega$.

Since $q = P_\Theta(g)q$, therefore from (62),

$$\limsup_{n\longrightarrow\infty} \langle (g - I)q, u_n - q\rangle = \lim_{i\longrightarrow\infty} \langle (g - I)q, p - q\rangle \leq 0. \tag{76}$$

Using Lemma 4, (76), and the given conditions in step 4, we get $x_n \longrightarrow q$, where $q = P_\Theta(g)q$.

Case 2. There exists a subsequence $\{\|x_{n_i} - q\|^2\}$ of $\{\|x_n - q\|^2\}$ such that $\|x_{n_i} - q\|^2 < \|x_{n_{i+1}} - q\|^2, \forall i \in \mathbb{N}$. Thus, by Lemma 2, ∃ is a nondecreasing sequence m_j of $\mathbb{N}$ such that $\lim_{j\longrightarrow\infty} m_j = \infty$ and

$$\begin{aligned}\left\|x_{m_j} - q\right\|^2 &\leq \left\|x_{m_{j+1}} - q\right\|^2, \\ \left\|x_j - q\right\|^2 &\leq \left\|x_{m_j} - q\right\|^2.\end{aligned} \tag{77}$$

By step 2, we get

$$\begin{aligned}&\left(1 - \eta_{m_j}\right)\lambda_{m_j}\left(1 - \lambda_{m_j}L\right)\left\|\left(T_{r_{m_j}}^{(G_2,b_2)} - I\right)Bt_{m_j}\right)\right\| \\ &\leq \left\|x_{m_j} - q\right\|^2 - \left\|x_{m_{j+1}} - q\right\|^2 + \alpha_{m_j}N_4 \leq \alpha_{m_j}N_4.\end{aligned} \tag{78}$$

Thus,

$$\lim_{n\longrightarrow\infty}\left\|\left(T_{r_{m_j}}^{(G_2,b_2)} - I\right)Bt_{m_j}\right)\right\| = 0. \tag{79}$$

By step 3, we obtain

$$\begin{aligned}\left(1 - \eta_{m_j}\right)\left\|u_{m_j} - t_{m_j}\right\|^2 &\leq \left\|x_{m_j} - q\right\|^2 - \left\|x_{m_{j+1}} - q\right\|^2 + \eta_{m_j}N_4 \\ &\quad + 2\left(1 - \eta_{m_j}\right)\lambda_{m_j}\left\|u_{m_j} - q\right\|\left\|B^*\left(T_{r_{m_j}}^{(G_2,b_2)} - I\right)Bt_{m_j}\right\| \\ &\leq \eta_{m_j}N_4 + 2\left(1 - \eta_{m_j}\right)\lambda_{m_j}\left\|u_{m_j} - q\right\|\left\|B^*\left(T_{r_{m_j}}^{(G_2,b_2)} - I\right)Bt_{m_j}\right\|.\end{aligned} \tag{80}$$

Hence,

$$\lim_{n\longrightarrow\infty}\left\|u_{m_j} - t_{m_j}\right\| = 0. \tag{81}$$

By the similar steps of case 1, we get

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\left\|x_{m_{j+1}} - x_{m_j}\right\| = 0, \\ &\limsup_{j\longrightarrow\infty}\langle (g - I)q, x_{m_{j+1}} - q\rangle \leq 0.\end{aligned} \tag{82}$$

By step 4, we have

$$\begin{aligned}\left\|x_{m_{j+1}} - q\right\|^2 &\leq \left(1 - (1 - \alpha)\eta_{m_j}\right)\left\|x_{m_j} - q\right\|^2 + (1 - \alpha)\eta_{m_j} \\ &\quad \times \left[\frac{2}{1 - \alpha}\langle g(q) - q, x_{m_{j+1}} - q\rangle + \frac{\sigma_{m_j}}{\eta_{m_j}}\left\|x_{m_j} - x_{m_{j-1}}\right\|\frac{N}{1 - \alpha}\right].\end{aligned} \tag{83}$$

By (77) and (83), we have

$$\left\|x_{m_{j+1}}-q\right\|^2 \leq \left(1-(1-\alpha)\eta_{m_{j+1}}\right)\left\|x_{m_j}-q\right\|^2+(1-\alpha)\eta_{m_j} \times\left[\frac{2}{1-\alpha}\langle g(q)-q, x_{m_{j+1}}-q\rangle+\frac{\sigma_{m_j}}{\eta_{m_j}}\left\|x_{m_j}-x_{m_{j-1}}\right\|\frac{N}{1-\alpha}\right]. \quad (84)$$

This implies

$$\left\|x_{m_{j+1}}-q\right\|^2 \leq \frac{2}{1-\alpha}\langle g(q)-q, x_{m_{j+1}}-q\rangle+\frac{\sigma_{m_j}}{\eta_{m_j}} \left\|x_{m_j}-x_{m_{j-1}}\right\|\frac{N}{1-\alpha}. \quad (85)$$

Thus,

$$\limsup_{j\longrightarrow\infty}\left\|x_{m_{j+1}}-q\right\|^2 \leq 0. \quad (86)$$

By (77) and (86), $x_j \longrightarrow q$. This completes the proof. □

Moreover, we have the following consequences. If we take $\theta_n = 0$, then Theorem 1 reduced to the following result without inertial as follows:

Corollary 1. *Let C and Q be two nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let B: $H_1 \longrightarrow H_2$ be a bounded linear operator. Assume that G_1: $C\times C \longrightarrow \mathbb{R}$, G_2: $Q\times Q \longrightarrow \mathbb{R}$, b_1: $C\times C\longrightarrow \mathbb{R}$, and b_2: $Q\times Q \longrightarrow \mathbb{R}$ are nonlinear mappings satisfying Assumption 1 and G_2 is upper semicontinuous in the first argument. Assume that $\Theta = \Gamma\cap\Omega \neq \varnothing$. Let g: $H_1 \longrightarrow H_1$ be a contraction mapping with constant $\alpha \in (0,1)$ and D: $C \longrightarrow H_1$ be a τ-inverse strongly monotone mapping. Let $\{x_n\}$ be generated by*

$$\left.\begin{array}{l} x_1 \in H_1 \\ u_n = T_{r_n}^{(G_1,b_1)}\left(t_n+\lambda_n B^*\left(T_{r_n}^{(G_2,b_2)}-I\right)Bt_n\right) \\ v_n = P_C(u_n-\delta_n Du_n) \\ x_{n+1} = \eta_n g(x_n)+(1-\eta_n)v_n \end{array}\right\}, \quad (87)$$

where the control sequence satisfies the following conditions:

(i) $\lim_{n\longrightarrow\infty}\eta_n = 0, \sum_{n=0}^{\infty}\eta_n = \infty$;
(ii) $\liminf_{n\longrightarrow\infty} r_n > 0$ *an* $\lim_{n\longrightarrow\infty}|r_{n+1}-r_n| = 0$;
(iii) $\{\lambda_n\}\subset\mathbb{R}$ *such that* $a\leq\lambda_n\leq b<(1/L)$, *where* $L = \|B\|^2$.

Then, the sequence $\{x_n\}$ converges strongly to some $q \in \Theta$, where $q = P_\Theta(g)q$.

Further, if we take $b_1, b_2 \equiv 0$, then Theorem 1 reduced to the following result as follows:

Corollary 2. *Let C and Q be two nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let B: $H_1 \longrightarrow H_2$ be a bounded linear operator. Assume that G_1: $C\times C \longrightarrow \mathbb{R}$ and G_2: $Q\times Q\longrightarrow\mathbb{R}$ are nonlinear mappings satisfying Assumption 1 (1)–(4) and G_2 is upper semicontinuous in first argument. Assume that $\Theta = \Upsilon\cap\Omega\neq\varnothing$, where Υ denotes the solution set of S_pEP (6) and (7). Let g: $H_1\longrightarrow H_1$ be a contraction mapping with constant $\alpha\in(0,1)$ and D: $C\longrightarrow H_1$ be a τ-inverse strongly monotone mapping. Let $\{x_n\}$ be generated by*

$$\left.\begin{array}{l} x_1 \in H_1 \\ u_n = T_{r_n}^{(G_1)}\left(t_n+\lambda_n B^*\left(T_{r_n}^{(G_2)}-I\right)Bt_n\right) \\ v_n = P_C(u_n-\delta_n Du_n) \\ x_{n+1} = \eta_n g(x_n)+(1-\eta_n)v_n \end{array}\right\}, \quad (88)$$

where the control sequence satisfies the following conditions:

(i) $\lim_{n\longrightarrow\infty}\eta_n = 0, \sum_{n=0}^{\infty}\eta_n = \infty$;
(ii) $\lim_{n\longrightarrow\infty}(\theta_n/\eta_n)\|x_n-x_{n-1}\| = 0$;
(iii) $\{\theta_n\}\subset[0,\theta]$, *for some* $\theta>0$ *and* $\delta_n\subset(0,2\tau)$;
(iv) $\liminf_{n\longrightarrow\infty} r_n>0$ *and* $\lim_{n\longrightarrow\infty}|r_{n+1}-r_n| = 0$;
(v) $\{\lambda_n\}\subset\mathbb{R}$ *such that* $a\leq\lambda_n\leq b<(1/L)$, *where* $L = \|B\|^2$.

Then, the sequence $\{x_n\}$ converges strongly to some $q\in\Theta$, where $q = P_\Theta(g)q$.

4. Numerical Illustration

Finally, to supporting our main theorem, we now give an example in infinitely dimensional spaces $L_2[0,1]$ such that $\|\cdot\|$ is L_2-norm defined by $\|x\| = \sqrt{\int_0^1|x(t)|^2 dt}$ where $x(t)\in L_2[0,1]$.

Example 1. Let $H_1 = H_2 = L_2[0,1]$ and $C = Q = \{x(t)\in L_2[0,1]: \int_0^1 tx(t)dt\leq 1\}$. Define mappings as follows:

(i) bounded linear operator B: $H_1\longrightarrow H_2$ by $Bx(t) = 3x(t), \forall x(t)\in L_2[0,1]$;
(ii) contraction mapping g: $H_1\longrightarrow H_1$ by $g(x(t)) = \alpha x(t)$ where $\alpha\in[0,1)$;
(iii) nonlinear mappings G_1: $C\times C\longrightarrow\mathbb{R}$, G_2: $Q\times Q\longrightarrow\mathbb{R}$ by

Table 1: Numerical results of $\bar{\theta}_n$.

$\bar{\theta}_n$	0	0.5	$(1/n)$	$(1/n^2)$	$\min\{(1/n^2\|x_n - x_{n-1}\|), 0.5\}$
No. of iters	15	4	14	10	4
CPU time (s)	4.106713	1.862612	3.764422	2.922132	2.023393

Table 2: Numerical results of r_n.

r_n	0.001	0.01	0.1	1	10
No. of iters	15	14	4	3	5
CPU time (s)	4.169121	3.927234	1.914940	1.678531	2.077360

Table 3: Numerical results of α_n.

η_n	$(1/(n+1))$	$(1/(10n+1))$	$(1/(20n+1))$	$(1/(100n+1))$	$(1/\sqrt{n})$
No. of ites.	3	8	8	8	6
CPU time (s)	1.678531	2.710736	2.728162	2.700151	3.312465

$$G_1(x(t), y(t)) = G_2(x(t), y(t)) = \langle y(t) - x(t), x(t)\rangle, \quad \forall x(t), y(t) \in L_2[0,1], \tag{89}$$

(iv) nonlinear mappings $b_1: C \times C \longrightarrow \mathbb{R}$ and $b_2: Q \times Q \longrightarrow \mathbb{R}$ by

$$b_1(x(t), y(t)) = b_2(x(t), y(t)) = \langle y(t), y(t)\rangle, \quad \forall x(t), y(t) \in L_2[0,1]. \tag{90}$$

(v) τ-inverse strongly monotone mapping $D: C \longrightarrow H_1$ by

$$Dx(t) = B^*\left(I - P_Q\right)Bx(t), \quad \forall x(t) \in C. \tag{91}$$

It is obvious that G_1, G_2, b_1, b_2 satisfy Assumption 1 and G_2 is upper semicontinuous in the first argument by the definition of the inner product $\langle .,.\rangle$. On the other hand, we consider

$$\begin{aligned} 0 &\le G_1(x,y) + b_1(y,x) - b_1(x,x) + \frac{1}{r}\langle y - x, x - z\rangle \\ &= \langle y - x, x\rangle + \frac{1}{r}\langle y - x, x - z\rangle \\ &= \langle y - x, (1+r)x - z\rangle. \end{aligned} \tag{92}$$

This implies that $T_r^{(G_1,b_1)}(z(t)) = P_C(z(t)/(1+r))$, $\forall z(t) \in L_2[0,1]$ where

$$P_C(x(t)) = \begin{cases} \dfrac{1 - \langle t, x(t)\rangle}{\|t\|^2} t + x(t), & \text{if } \langle t, x(t)\rangle > 1, \\ x(t), & \text{if } \langle t, x(t)\rangle \le 1. \end{cases} \tag{93}$$

Similarly, we have $T_r^{(G_2,b_2)}(z(t)) = P_C(z(t)/(1+r))$, $\forall z(t) \in L_2[0,1]$. For the experiments in this section, we use the Cauchy error $\|x_{n+1} - x_n\|^2 < 10^{-5}$ for the stopping criterion. We split considering all of the performances of our algorithm in five cases.

Case I: we start computation by comparing of the algorithm with different parameters $\bar{\theta}_n$ where

$$\theta_n = \begin{cases} \bar{\theta}_n, & \text{if } n \le N, x_n \ne x_{n-1}, \\ \dfrac{\eta_n}{n\left\|x_n - x_{n-1}\right\|}, & \text{if } n > N, x_n \ne x_{n-1}, \\ \eta_n, & \text{otherwise}, \end{cases} \tag{94}$$

where N is the number of iteration that we want to stop. We choose $r_n = 0.1, \lambda_n = \delta_n = 0.1, \eta_n = (1/(n+1))$, and $\alpha = 0.1$ and initializations $x_0 = \sin(t)$ and $x_1 = (\sin(t)/2)$. Then, the results are presented as follows:

Case II: we compare the performance of the algorithm with different parameters r_n by setting $\bar{\theta}_n = 0.5$, $\lambda_n = \delta_n = 0.1$, $\eta_n = (1/(n+1))$, and $\alpha = 0.1$ and initializations $x_0 = \sin(t)$ and $x_1 = (\sin(t)/2)$. Then, the results are presented as follows:

Case III: we compare the performance of the algorithm with different parameters η_n by setting $\bar{\theta}_n = 0.5$, $r_n = 1$,

Table 4: Numerical results of λ_n and μ_n.

$\lambda_n = \delta_n$	1	0.1	0.01	0.001	0.0001
No. of iters	3	3	3	3	3
CPU time (s)	1.667450	1.672732	1.698085	1.663979	1.678559

Table 5: Numerical results of α.

α	0.001	0.1	0.5	0.9	0.999
No. of iters	7	3	8	7	7
CPU time (s)	2.512896	1.663979	2.691693	2.505802	2.631734

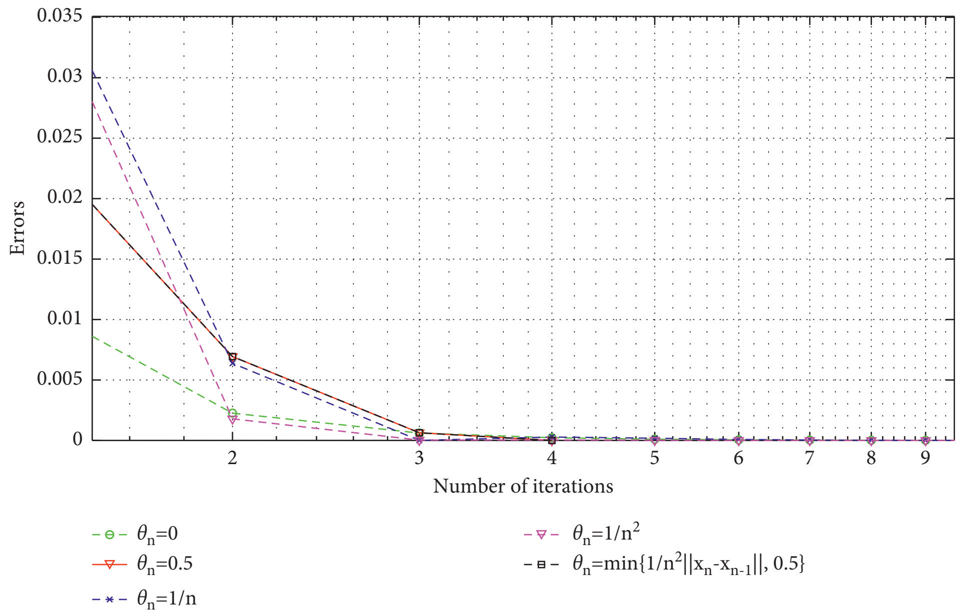

Figure 1: The Cauchy error plotting number of iterations for different parameters $\overline{\theta}_n$.

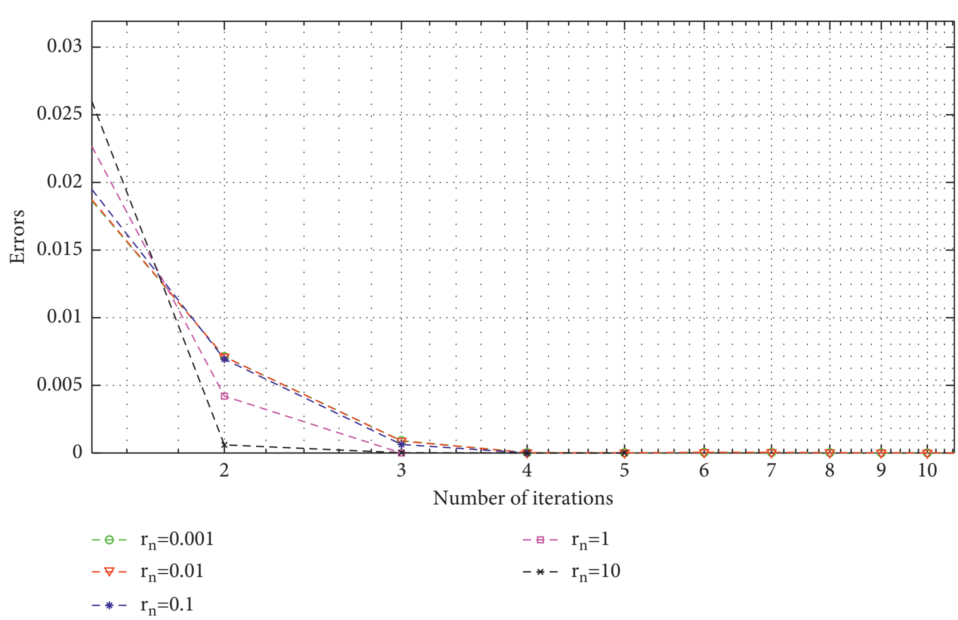

Figure 2: The Cauchy error plotting number of iterations for different parameters r_n.

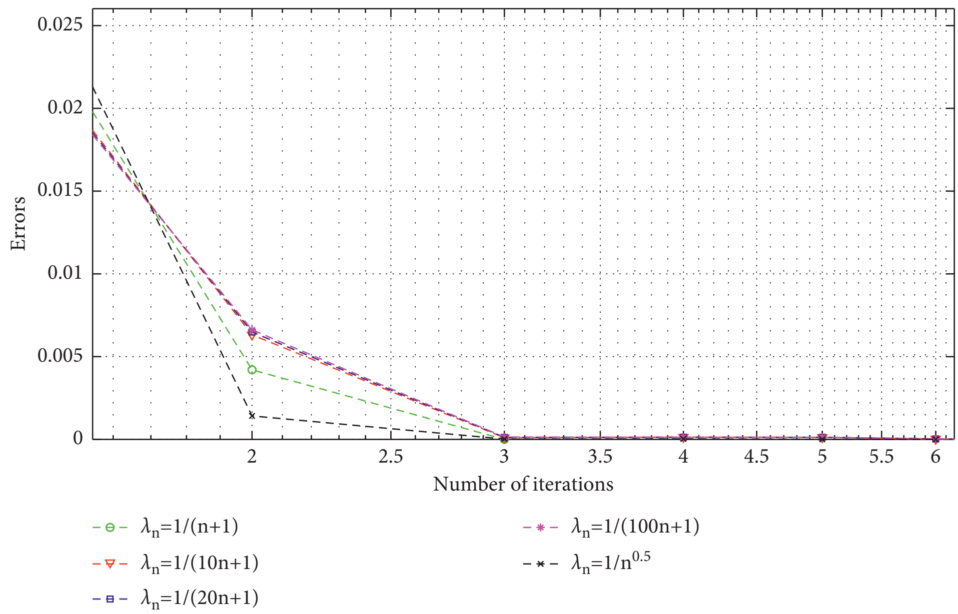

FIGURE 3: The Cauchy error plotting number of iterations for different parameters η_n.

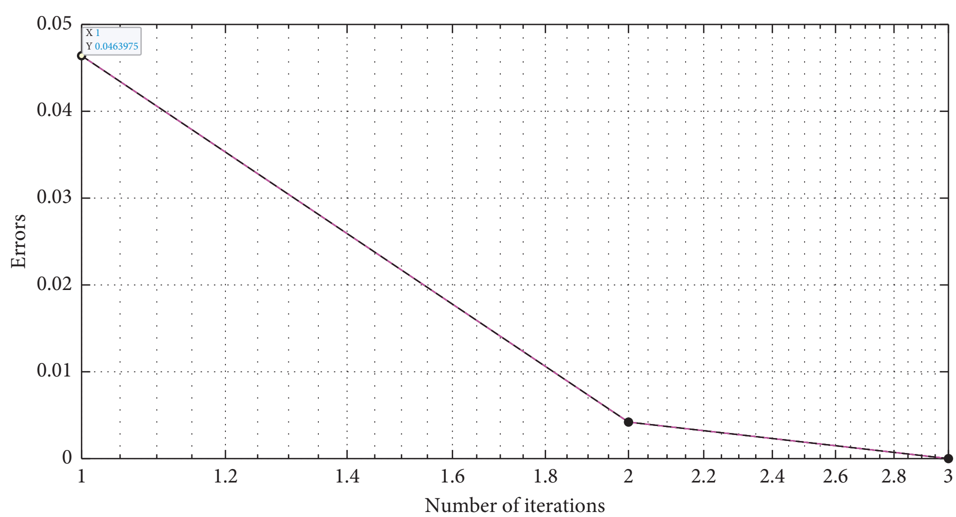

FIGURE 4: The Cauchy error plotting number of iterations for different parameters λ_n and δ_n.

$\lambda_n = \delta_n = 0.1$, and $\alpha = 0.1$ and initializations $x_0 = \sin(t)$ and $x_1 = (\sin(t)/2)$. Then, the results are presented as follows:

Case IV: we compare the performance of the algorithm with different parameters λ_n and δ_n by setting $\overline{\theta}_n = 0.5$, $r_n = 1$, $\eta_n = (1/(n+1))$, and $\alpha = 0.1$ and initializations $x_0 = \sin(t)$ and $x_1 = (\sin(t)/2)$. Then, the results are presented as follows:

Case V: we compare the performance of the algorithm with different parameters α by setting $\overline{\theta}_n = 0.5$, $r_n = 1$, $\eta_n = (1/(n+1))$, and $\lambda_n = \delta_n = 0.001$ and initializations $x_0 = \sin(t)$ and $x_1 = (\sin(t)/2)$. Then, the results are presented as follows:

From Tables 1–5 and Figures 1–5, we noticed that in all the above 5 cases, selecting $\overline{\theta}_n = 0.5$, $r_n = 1$, $\eta_n = (1/(n+1))$, $\lambda_n = \delta_n = 0.001$, and $\alpha = 0.1$ yields the best results.

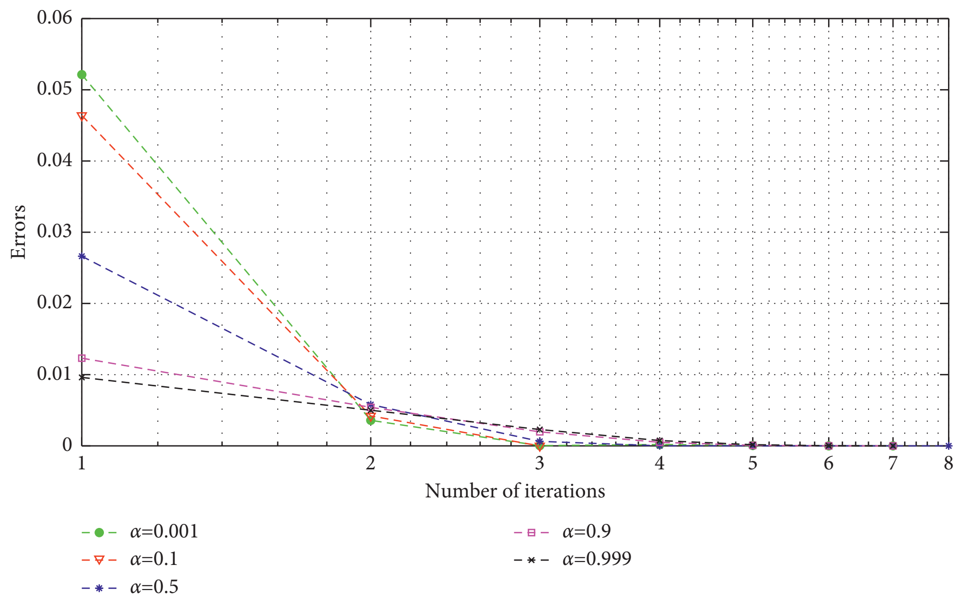

FIGURE 5: The Cauchy error plotting number of iterations for different parameters α.

5. Conclusion

In this paper, we developed an iterative algorithm via inertial and viscosity techniques to find a common solution of a split generalized equilibrium and a variational inequality problem in Hilbert spaces. Further, we study the convergence analysis of our main result and point out some consequences. Finally, we constructed a numerical example to demonstrate the applicability of theorem and compared the performance of algorithm by taking different parameters.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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